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PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH.

PROCEEDINGS

OF

THE ROYAL SOCIETY

OF

EDINBURGH

THE SCRIPPS INSTITUTION
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PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH.

VOL. XLV. 1924–25.

I.—Lord Kelvin: A Centenary Tribute. By The President, Sir J. Alfred Ewing, K.C.B., F.R.S.

(Read October 27, 1924.)

In this the centenary year of Lord Kelvin's birth it is fitting that the Society should call to remembrance one who was for long its most distinguished Fellow, who first became its President at the age of thirty-nine, and was repeatedly re-elected to the office, which he held for twenty-one years in all, and who used the Society as a medium for the publication of many of his most brilliant discoveries. In the long list of his published papers there are at least one hundred and twenty items communicated to the Royal Society of Edinburgh. No other contributor has done so much to give to our *Proceedings* and *Transactions* a world-wide and lasting fame. It was to this Society that he brought, in 1849, his account of Carnot's Theory, which marks the beginning of his study of Thermodynamics, and it was in our *Transactions* that he published his epoch-making series of papers on the "Dynamical Theory of Heat," from 1851 to 1854. It was here in 1852 that he propounded the doctrine of the Dissipation of Energy. It was here that his investigations of underground temperature and the secular cooling of the Earth appeared in 1860 and 1862. It was here in 1865 that he "briefly refuted" the doctrine of Uniformity in Geology. Here, too, were published his long series of papers on Vortex Motion and Vortex Atoms, from 1867 to 1881, and much of his work on the molecular constitution of matter. Here he first showed, in germ, his mariner's compass, in 1874. One might cite other communications scarcely less noteworthy, among them a group of papers on deep-water waves, written when he was over eighty, and his last work of all, on the "Problem of a Spherical Gaseous Nebula," which was published posthumously in 1908.

Thus for nearly sixty years his ardent spirit of discovery and his loyalty to this Society combined to enrich our *Proceedings* and *Transactions* with contributions which take a permanent place in the development of scientific thought.

At the request of the Royal Philosophical Society of Glasgow I recently attempted to record some personal recollections of Kelvin and impressions of parts of his work, in the form of a centenary address. As it seemed possible that these notes might interest our Fellows, it was suggested that they might be read here to-day, on the occasion of my first presence in the chair which he so long and so greatly filled.

⁂ *For the Address which followed, see the "Proceedings of the Royal Philosophical Society of Glasgow," October* 8, 1924.

(*Issued separately December* 18, 1924.)

II.—The Irreducible System of Concomitants of Two Double Binary (2, 1) Forms. By **W. Saddler,** M.A., Lecturer in the University of St Andrews. *Communicated by* Professor H. W. TURNBULL, M.A.

(MS. received July 10, 1924. Read November 24, 1924.)

INTRODUCTION.

LITTLE is known of the details of systems of concomitants belonging to double binary forms. The cases of the single ground form of orders (1, 1), (2, 1), (2, 2) respectively, together with the simultaneous system of any number of (1, 1) forms, are the only four cases which have been published. The following pages establish the simultaneous system of two (2, 1) forms.

This system is fundamental for the geometrical treatment of two twisted cubics lying upon a quadric surface and having four common points.

§ 1. If $F(x, \xi) = \sum\limits_{r,s}\binom{m}{r}\binom{n}{s}a_{rs}x^{m-r}\xi^{n-s}$ denotes the double form of order (m, n), then the symbolic way of writing the coefficient a_{rs} is $a_1^{m-r}a_2^r\alpha_1^{n-s}\alpha_2^s$. By introducing equivalent symbols $a, b, c, \ldots$ so that $f = a_x^m\alpha_\xi^n = b_x^m\beta_\xi^n \ldots$ is a set of ways of representing the (m, n) form F, we have a means at our disposal of constructing invariants of F analogous to the invariants of the ordinary binary form. The invariancy which is contemplated holds for *independent* linear transformations of the variables x and ξ.

A transvectant of indices (ρ, ρ') for two double binary forms $a_x^m\alpha_\xi^n$ and $a'^p_x\alpha'^q_\xi$ is defined as $(aa')^\rho(\alpha\alpha')^{\rho'}a_x^{m-\rho}a'^{p-\rho}_x\alpha_\xi^{n-\rho'}\alpha'^{q-\rho'}_\xi$ and is also expressed as $(a_x^m\alpha_\xi^n, a'^p_x\alpha'^q_\xi)_{\rho,\rho'}$.

It is well known* that if S_1, S_2 be any two finite and complete systems of forms, then the system S formed by taking transvectants of powers and products of S_1 with powers and products of S_2 is both finite and complete. If S_1 and S_2 be the complete systems for any two double binary forms f_1 and f_2, then S, which contains both f_1 and f_2, is the complete system of concomitants for the forms taken separately.

---

* H. W. Turnbull, *Proc. Edin. Math. Soc.*, vol. xli, p. 116; *Proc. Roy. Soc. Edin.*, vol. xliii, p. 33, 1923. Kasner, *Trans. Amer. Math. Soc.*, vol i, 1900. Peano, *Battaglini*, vol. xx, 1882.

§ 2. *The two double binary* (2, 1) *forms.*

Let

$$f = a_x^2 a_\xi = b_x^2 \beta_\xi = \ldots$$
$$f' = a'_x{}^2 a'_\xi = b'_x{}^2 \beta'_\xi = \ldots$$

The irreducible system of these are *

$$\begin{array}{ll}
f = a_x^2 a_\xi & f' = a'_x{}^2 a'_\xi \\
\mathrm{D} = \mathrm{D}_x^2 = (ff)_{1,1} = (ab)(\alpha\beta) a_x b_x & \mathrm{D}' = \text{etc.} \\
\Delta = \Delta_\xi^2 = (ff)_{2,0} = (ab)^2 \alpha_\xi \beta_\xi & \Delta' \\
\mathrm{P} = p_x^2\ {}_\xi = (f\mathrm{D})_{1,0} = (a\mathrm{D}) a_x \mathrm{D}_x \alpha_\xi & \mathrm{P}' \\
\qquad = -(ab)^2 (a\gamma) a_x^2 \beta_\xi & \\
\qquad = (f\Delta)_{0,1} = a_x^2 (\alpha\Delta) \Delta_\xi & \\
\mathrm{R} = (f\mathrm{P})_{2,1} = (\mathrm{DD})_{2,0} & \mathrm{R}' \\
\qquad = (\Delta\Delta)_{0,2} &
\end{array}$$

One relation holds between the concomitants of each set, namely

$$\mathrm{P}^2 = -\tfrac{1}{2}(\Delta \mathrm{D}^2 + \mathrm{R} f^2)\text{; similarly for } \mathrm{P}'.$$

We require to combine the two systems for f and f' by transvection and find the irreducible system so formed. We therefore consider all types included in

$$(f^\alpha \mathrm{D}^\beta \Delta^\gamma \mathrm{P}^\delta,\ f'^{\alpha'} \mathrm{D}'^{\beta'} \Delta'^{\gamma'} \mathrm{P}'^{\delta'})_{\rho,\rho'} = (u, v)_{\rho,\rho'} \text{ (say)}$$

wherein the invariants R and R′ are omitted since they will give rise to no new forms.

$(u, v)_{\rho\rho'}$ will be reducible if, when $u = u_1 u_2$; $v = v_1 v_2$; $\rho = \sigma + \tau$; $\rho' = \sigma' + \tau'$, it is possible to construct $(u_1 v_1)_{\sigma\sigma'}$ and $(u_2, v_2)_{\tau,\tau'}$. (A)

All the forms at most are quadratic in x or ξ; we therefore may classify the transvectants as follows:

$$\left.\begin{array}{l}\text{I. } (\theta, \phi') \ldots \theta = f, \mathrm{D}, \Delta, \mathrm{P} \\ \qquad\qquad \phi = f', \mathrm{D}', \Delta', \mathrm{P}'\end{array}\right\} \text{omitting the cases } (\mathrm{D}, \Delta') \ldots \text{and } (\mathrm{D}', \Delta) \ldots$$

II. $(\theta_1\theta_2, \phi') \ldots$ (*a*) $\phi' = \Delta'$, $\theta_1\theta_2 = ff, fp$, or pp

(*b*) $\phi' = f'$, $\theta_1\theta_2 = \mathrm{D}\Delta$

(*c*) $\phi' = p'$, $\theta_1\theta_2 = \mathrm{D}\Delta$.

III. $(\theta_1\theta_2, \phi'_1\phi'_2) \ldots \theta_1\theta_2 = \mathrm{D}\Delta$

$\phi'_1\phi'_2 = f'f', f'p', p'p'$.

IV. Higher products $\theta_1\theta_2\theta_3$ are reduced by syzygy or by splitting up into parts $\theta_1\theta_2 . \theta_3$, the parts combining with the forms in v to split up into the product of two transvectants (A).

---

* Peano, *loc. cit.*

Identities.—The identities used in the reductions are the well-known binary cases

$$(bc)a_x+(ca)b_x+(ab)c_x=0$$

and

$$2(ab)(ac)b_xc_x=(ab)^2c_x{}^2+(ac)^2b_x{}^2-(bc)^2a_x{}^2.$$

§ 3. There are in all 37 invariants (in the broad sense)—included in this of course are those of the two forms f and f':—

8 invariants.
6 linear in ξ.
6 quadratic in x.
6 quadratic in ξ.
4 quartics in x.
6 2, 1 forms.
1 2, 2 form.

There are no (1, 1) forms, however.

§ 4. I.—(1–37).

1. $(ff')_{1,0}=(aa')a_xa'_x\alpha_\xi\alpha'_\xi=\mathrm{E}_x{}^2\epsilon_\xi{}^2.$

2. $(ff')_{0,1}=(\alpha\alpha')a_x{}^2a'_x{}^2=\mathrm{F}_x{}^4.$

3. $(ff')_{1,1}=(aa')(\alpha\alpha')a_xa'_x=\mathrm{D}_{12}{}^2{}_x.$

4. $(ff')_{2,0}=(aa')^2\alpha_\xi\alpha'_\xi=\Delta_{12}{}^2{}_\xi.$

5. $(f,f')_{2,1}=(aa')^2(\alpha\alpha')=\mathrm{A}.$

6. $(f\mathrm{D}')_{1,0}=(a\mathrm{D}')a_x\mathrm{D}'_x\alpha_\xi$
$=-(aa')^2(\alpha'\beta')b'_x{}^2\alpha_\xi=\mathrm{G}_x{}^2\gamma_\xi.$

7. $(f'\mathrm{D})_{1,0}=(a'\mathrm{D})a'_x\mathrm{D}_x\alpha'_\xi=\mathrm{G}'_x{}^2\gamma'_\xi.$

8. $(f,\mathrm{D}')_{2,0}=(a\mathrm{D}')^2\alpha_\xi=-2(aa')(ab')(a'b')(\alpha\alpha')\beta'_\xi$
$=-2\{\mathrm{D}_{12},f'\}_{2,0}=\mathrm{K}_\xi.$

9. $(f',\mathrm{D})_{2,0}=(a'\mathrm{D})^2\alpha'_\xi=-2\{\mathrm{D}_{12},f\}_{2,0}=\mathrm{K}'_\xi.$

10. $(f\Delta')_{0,1}=(\alpha\Delta')a_x{}^2\Delta'_\xi$
$=(a'b')^2(\alpha\alpha')a_x{}^2\beta'_\xi=\mathrm{H}_x{}^2\eta_\xi.$

11. $(f'\Delta)_{0,1}=(\alpha'\Delta)a'_x{}^2\Delta_\xi=\mathrm{H}'_x{}^2\eta'_\xi.$

12. $(\mathrm{D},\mathrm{D}')_{1,0}=(\mathrm{DD}')\mathrm{D}_x\mathrm{D}'_x$
$=(ab)(\alpha\beta)(a\mathrm{D}')b_x\mathrm{D}'_x$
$=(a\mathrm{D}')^2(\alpha\beta)b_x{}^2=\mathrm{U}_x{}^2.$

13. $(\mathrm{D},\mathrm{D}')_{2,0}=(\mathrm{DD}')^2=(ab)(\alpha\beta)(a\mathrm{D}')(b\mathrm{D}')=\mathrm{R}_2.$

14. $(\Delta\Delta')_{0,1}=(ab)^2(\alpha\Delta')\beta_\xi\Delta'_\xi$
$=(ab)^2(a'b')^2(\alpha\alpha')\beta_\xi\beta'_\xi=u_\xi{}^2.$

15. $(\Delta\Delta')_{0,2}=(ab)^2(\alpha\Delta')(\beta\Delta')$
$=(ab)^2(a'b')^2(\alpha\alpha')(\beta\beta')=\mathrm{R}_3.$

16. $(f,\mathrm{P}')_{1,0}=(a_x{}^2\alpha_\xi,\ p'_x{}^2\pi'_\xi)_{1,0}$

is the Jacobian of a Jacobian and therefore reducible.

$=\tfrac{1}{2}\mathrm{K}f'-\tfrac{1}{2}\mathrm{D}'\Delta_{12}.$

17. $(f'\mathrm{P})_{1,0}=\tfrac{1}{2}\mathrm{K}'f-\tfrac{1}{2}\mathrm{D}\Delta_{12}.$

18. $$(fP')_{2.0}=(aa')(aD')(a'D')\alpha_\xi\alpha'_\xi$$
$$=(aa')^2(\alpha'\Delta')\alpha_\xi\Delta'_\xi=\tau_\xi^2.$$

19. $$(f'P)_{2.0}=(aa')^2(\alpha\Delta)\alpha'_\xi\Delta_\xi=\tau'^2_\xi.$$

20. $$(fP')_{0.1}=a_x^2a'^2_x(\alpha\Delta')(\alpha'\Delta')$$
$$=(a'D')(\alpha\alpha')a_x^2a'_xD'_x$$
$$=-(a'b')^2(\alpha'\gamma')(\alpha\beta')a_x^2c'^2_x=Q_x^4.$$

21. $$(f'P)_{0.1}=a_x^2a'^2_x(\alpha\Delta)(\alpha'\Delta)=Q'^4_x.$$

22. $$(fP')_{1.1}=(aa')(a'D')(\alpha\alpha')a_xD'_x=M_x^2.$$

23. $$(f'P)_{1.1}=(aD)(a'a)(\alpha'\alpha)D_xa'_x=M'^2_x.$$

24. $$(fP')_{2.1}=(aa')(aD')(a'D')(\alpha\alpha')$$
$$=(aa')^2(\alpha\Delta')(\alpha'\Delta')$$
$$=(D_{12},\ D')_{2.0}=(\Delta_{12},\ \Delta')_{0.2}=B.$$

25. $$(f'P)_{2.1}=(\Delta_{12},\ \Delta)=(D_{12},\ D)_{2.0}=B'.$$

26. $$(P\Delta')_{0.1}=((fD)_{1.0},\ \Delta')_{0.1}=(aD)(\alpha\Delta')a_xD_x\Delta'_\xi$$
$$=(aD)(\alpha\alpha')(a'b')^2a_xD_x\beta'_\xi.$$

[This reduction is the basis of many of the other reductions and will be shown in detail; factors containing x or ξ will be omitted.]

$$=(aD)(\alpha\alpha')(a'b')(Db')\ \ldots+(aD)(\alpha\alpha')(a'b')(a'D)\ \ldots$$
$$=(aD)(a'D)(a'b')(\alpha\alpha')\ \ldots+(a'D)(Db')(ab')(\alpha\alpha')\ \ldots$$
$$+(aa')(\alpha\alpha')(Db')^2\ \ldots$$
$$=-(aD)(a'D)(aa')(\alpha\alpha')\ \ldots+(aD)(a'D)(ab')(\alpha\alpha')\ \ldots$$
$$+(a'D)(Db')(ab')(\alpha\alpha')\ \ldots+D_{12}.K'$$
$$=D_{12}.K'-B'f'+(ab')^2(a'D)(\alpha\alpha')\ \ldots$$

Now

$$(ab')^2(a'D)(\alpha\alpha')\ \ldots=\{(ab')^2(\alpha\alpha')a'^2_x\beta'_\xi,\ D_x^2\}_{1.0}$$

and

$$(ab')^2(\alpha\alpha')a'^2_x\beta'_\xi=(D'f)_{1.0}+Af'.$$

Thus

$$(ab')^2(a'D)(\alpha\alpha')a'_xD_x\beta_\xi=AG'+\tfrac{1}{2}R_2f.$$

So

$$(P\Delta')_{0.1}=D_{12}K'-f'B+AG'+\tfrac{1}{2}R_2f.$$

Similarly

27. $$(P'\Delta)_{0.1}=D_{12}K+\tfrac{1}{2}R_2f'+AG-B'f.$$

28. $$(PD')_{1.0}=((fD)_{1.0},\ D')_{1.0}$$
$$=\tfrac{1}{2}K.D-\tfrac{1}{2}R_2f.$$

29. $$(P'D)_{1.0}=\tfrac{1}{2}K'D'-\tfrac{1}{2}R_2f'.$$

30. $$(PD')_{2.0}=-(ab)^2(\alpha\gamma)(cD')^2\beta_\xi=\rho_\xi.$$

31. $$(P'D)_{2.0}=-(a'b')^2(\alpha'\gamma')(c'D)^2\beta'_\xi=\rho'_\xi.$$

32. $$(PP')_{1.0}=((fD)_{1.0},\ P')_{1.0}$$

is the Jacobian of a Jacobian and therefore reducible—

$$=\tfrac{1}{2}\tau . \mathrm{D}-\tfrac{1}{2}\rho' f,$$

also

$$\begin{aligned}&=-((f'\mathrm{D}')_{1.0}, \mathrm{P})_{1.0}\\&=-\tfrac{1}{2}\tau'\mathrm{D}'+\tfrac{1}{2}\rho f'.\end{aligned}$$

Thus

33. $$\rho f'+\rho' f=\tau . \mathrm{D}+\tau'\mathrm{D}'.$$

34. $$\begin{aligned}(\mathrm{PP}')_{2.0}&=(a_x^2(\alpha\Delta)\Delta_\xi,\ a'^2_x(\alpha'\Delta')\Delta'_\xi)_{2.0}\\&=(aa')^2(\alpha\Delta)(\alpha'\Delta')\Delta_\xi\Delta'_\xi\\&=(aa')^2(\alpha\Delta)(\Delta\Delta')\alpha'_\xi\Delta'_\xi-(aa')^2(\alpha\Delta)(\Delta\alpha')\Delta'^2_\xi,\end{aligned}$$

but

$$(\Delta\Delta')(\alpha\Delta)a_x^2\Delta'_\xi=\mathrm{D}_{12}\mathrm{K}'+\tfrac{1}{2}\mathrm{R}_2 f+\mathrm{AG}'-\mathrm{B}'f.$$

Thus

$$\begin{aligned}(aa')^2(\alpha\Delta)(\Delta\Delta')\alpha'_\xi\Delta'_\xi&=(\mathrm{D}_{12}\mathrm{K}',\ a'^2_x\alpha'_\xi)_{2.0}+\text{reducible terms}\\&=-\tfrac{1}{2}\mathrm{K}.\mathrm{K}'-\Delta'\mathrm{B}'+\tfrac{1}{2}\mathrm{R}_2\Delta_{12},\end{aligned}$$

giving

$$(\mathrm{PP}')_{2.0}=-\tfrac{1}{2}\mathrm{KK}'+\tfrac{1}{2}\mathrm{R}_2\Delta_{12}.$$

35. $$\begin{aligned}(\mathrm{PP}')_{01}&=a_x^2(\alpha\Delta)(\alpha'\Delta')(\Delta\Delta')a'^2_x\\&=\text{term in }(a_x^2(\alpha\Delta)(\Delta\Delta')\Delta'_\xi,\ a'^2_x\alpha'_\xi)_{0.1},\end{aligned}$$

which as before is reducible.

36. $$\begin{aligned}(\mathrm{PP}')_{1.1}&=(a_x^2(\alpha\Delta)\Delta_\xi,\ a'^2_x(\alpha'\Delta')\Delta'_\xi)_{1.1}\\&=(aa')(\Delta\Delta')(\alpha\Delta)(\alpha'\Delta')a_x a'_x\\&=-\{a_x^2(\Delta\Delta')(\alpha\Delta)\Delta'_\xi,\ a'^2_x\alpha'_\xi\}_{1.1},\end{aligned}$$

and by the use of (26) this is equivalent to

$$(\mathrm{D}_{12}\mathrm{K}', f')_{1.1},$$

which itself reduces thus:

$$\begin{aligned}&2\{(a'\mathrm{D})^2\alpha'_\xi . (ab')(\alpha\beta')a_x b'_x,\ c'^2_x\gamma'_\xi\}_{1.1}\\&=(a'\mathrm{D})^2(ab')(ac')(\alpha\beta')(\alpha'\gamma')b'_x c'_x+(a'\mathrm{D})^2(ab')(b'c')(\alpha\beta')(\alpha'\gamma')a_x c'_x\\&=2(a'\mathrm{D})^2(ab')(ac')(\alpha\beta')(\alpha'\gamma')b'_x c'_x-(a'\mathrm{D})(\alpha'\gamma')c'^2_x . (ab')^2(\alpha\beta'),\end{aligned}$$

and

$$(a'\mathrm{D})^2(ab')(ac')(\alpha\beta')(\alpha'\gamma')b'_x c'_x$$

is a term in

$$\{(a'\mathrm{D})^2(\alpha'\gamma')c'^2_x b'^2_x\beta'_\xi,\ a_x^2\alpha_\xi\}_{2.1}$$

and is therefore reducible.

37. $$\begin{aligned}(\mathrm{P}, \mathrm{P}')_{2.1}&=(aa')^2(\alpha\Delta)(\alpha'\Delta')(\Delta\Delta')\\&=-\{a_x^2(\Delta\Delta')(\alpha\Delta)\Delta'_\xi,\ a'^2_x\alpha'_\xi\}_{2.1}\\&=-\{\tau_\xi^2\Delta_\xi^2\}_{0.2}\\&=(\tau'_\xi{}^2,\ \Delta'_\xi{}^2)_{0.2}=\mathrm{R}_4.\end{aligned}$$

II.—(38–51).

38. $$(f.f,\ \Delta')_{0.2} = a_x^2 b_x^2 (\alpha\Delta')(\beta\Delta') = X_x^4.$$

39. $$(ff',\ \Delta)_{0.2} = Y_x^4.$$

Now

$$D_{12}^2 = A.F + \tfrac{1}{2}D.D' - \tfrac{1}{2}X - \tfrac{1}{2}Y \ (\S\ 5c).$$

Hence only one of the quartics is included in the system.

40. $$\begin{aligned}(fP,\ \Delta')_{0.2} &= (a'b')^2(\alpha a')(\Delta\beta')(\beta\Delta)a_x^2 b_x^2 \\ &= (\alpha\Delta')(\beta\Delta)(\Delta\Delta')a_x^2 b_x^2 \\ &= -\{(\beta\Delta)(\Delta\Delta')b_x^2\Delta'_\xi,\ a_x^2 a_\xi\}_{0.1},\end{aligned}$$

which is reducible by (26).

41. $$(fP',\ \Delta)_{0.2}$$

is reducible.

42. $$(PP,\ \Delta')_{0.2}$$

is reducible, since

$$P^2 = -\tfrac{1}{2}(\Delta D^2 + Rf^2).$$

43. $$(P'P',\ \Delta')_{0.2}$$

is reducible in a similar manner.

44. $$\begin{aligned}(D\Delta, f')_{1.1} &= (Da')(\Delta a')D_x a'_x \Delta_\xi \\ &= -\tfrac{1}{2}D.K - B'f + \tfrac{1}{2}R.f',\end{aligned}$$

and is thus reducible. Similarly,

45. $$(D'\Delta', f)_{1.1}.$$

46. $$\begin{aligned}(D\Delta, f')_{2.1} &= (ab)^2(Da')^2(aa')\beta_\xi \\ &= -\{(Da')^2 a'_\xi,\ \Delta_\xi^2\}_{0.1} = \sigma_\xi\end{aligned}$$

47. $$(D'\Delta', f)_{2.1} = (a'b')^2(D'a)^2(a'a)\beta'_\xi = \sigma_\xi.$$

48. $$(D\Delta,\ P')_{1.1}$$

has a term common to

$$((f\Delta,\ P')_{02}, f)_{1.1},$$

so is reducible. So

49. $$(D'\Delta',\ P)_{1.1}.$$

50. $$\begin{aligned}(D\Delta,\ P')_{2.1} &= (ab)(a\beta)(aa')(ba')(\Delta\Delta')(a'\Delta')\Delta_\xi \\ &= \{(\Delta\Delta')(a'\Delta')a'^2_x\Delta_\xi,\ (ab)(a\beta)a_x b_x\}_{2.1},\end{aligned}$$

which is reducible by (26).

Similarly

51. $$(D'\Delta',\ P)_{2.1}.$$

III.—(52–58).

52. $$(D\Delta, f'f')_{2,2} = \tfrac{1}{3}(a'D)^2(\Delta\alpha')(\Delta\beta')b'^2_x + \tfrac{2}{3}(a'D)(b'D)(\Delta\alpha')(\Delta\beta')a'_x b'_x$$
$$= (a'D)(b'D)(\Delta\alpha')(\Delta\beta')a'_x b'_x + \tfrac{1}{6}(a'b')^2(\Delta\alpha')(\Delta\beta').D^2_x.$$

Now

$$(a'D)(b'D)(\Delta\alpha')(\Delta\beta')a'_x b'_x$$
$$= 2\{(Da')(\Delta\alpha')D_x a'_x \Delta_\xi,\ b'^2_x \beta'_\xi\}_{1,1} + \text{reducible part},$$

and

$$(Da')(\Delta\alpha')D_x a'_x \Delta_\xi = -\tfrac{1}{2}D.K - (aa')(\alpha a')(Da)(Da').f + \tfrac{1}{2}R.f,$$

while

$$(D.K, f')_{1,1} = -\tfrac{1}{2}(Da')(Db')(a'b')(\alpha'\beta').D + \tfrac{1}{2}D'.R.$$

Thus is reducible.

Similarly

53. $$(D'D', ff)_{2,2}.$$

54. $$(D\Delta, f'P')_{2,2}$$

is reducible.

$$= \tfrac{1}{6}(Da')^2(\Delta\alpha')(\Delta\pi')p'^2_x + \tfrac{1}{6}(Dp')^2(\Delta\alpha')(\Delta\pi')a'^2_x + \tfrac{2}{3}(Da')(Dp')(\Delta\alpha')(\Delta\pi')a'_x p'_x$$
$$= (Da')(Dp')(\Delta\alpha')(\Delta\pi')a'_x p'_x + (a'p')^2(\Delta\alpha')(\Delta\pi').D^2_x$$
$$= \{(Da')(\Delta\alpha')D_x a'_x \Delta_\xi,\ p'^2_x \pi'_\xi\}_{1,1} + \text{reducible terms}.$$

Now

$$(Da')(\Delta\alpha')D_x a'_x \Delta_\xi = -\tfrac{1}{2}D.K + \text{reducible terms}$$
$$= \{D.K, p'\}_{1,1} + \ldots$$
$$= (Db')(Da')(\alpha'\Delta')(\beta'\Delta')(a'D_1)b'_x D_{1x} + \ldots$$
$$= \text{term in } \{(\alpha'\Delta')(\beta'\Delta')a'^2_x b'^2_x,\ D^2_x D^2_x\}_{8,0} + \ldots$$
$$= \text{term in } \{D'^2, D^2\}_{3,0} + \ldots$$
$$(D^2_{1x} \text{ is cogredient with } D^2_x) \ldots$$

and thus is reducible.

Similarly

55. $$(D'\Delta', fP)_{2,2}$$

is reducible.

56. $$(D\Delta, P'^2)_{2,2}$$

is reducible, since

$$P'^2 = -\tfrac{1}{2}(\Delta'D'^2 + R'f'^2).$$

So

57. $$(D'\Delta', P^2)_{2,2}$$

is reducible.

58. $$(f^2, P'^2)_{4,2}$$

is reducible, being a term in

$$(f, P')_{2,1}.(fP')_{2,1}.$$

IV.—(59–62).

59. $$(f^2,\ D'^2\Delta')_{4.2}$$

is reducible, since

$$D'^2\Delta' = -2P'^2 - R'f'^2.$$

So

60. $$(f'^2,\ D^2\Delta)_{4.2}$$

is reducible.

61. $$(P^2,\ D'^2\Delta')_{4.2}$$

is reducible by using the syzygy for P^2.

62. $$(P'^2,\ D^2\Delta)_{4.2}$$

is reducible by using the syzygy for P'^2.

§ 5. *Syzygies.*—Many syzygies arise (*a*) from the product of two Jacobians as for binary forms.* Thus

$$2(f, f')_{1.0} \cdot (\phi, \phi')_{1.0} = -(f, \phi)_{2.0} f'\phi' - (f'\phi')_{2.0} f\phi + (f, \phi')_{2.0} f'\phi + (f'\phi)_{2.0} f\phi'.$$

(*b*) From the square of a Jacobian,* *e.g.*

$$\{(ff')_{1.0}\}^2 = -\tfrac{1}{2}\{(f, f)_{2.0} f'^2 - 2(f, f')_{2.0} ff' + (f'f')_{2.0} f^2\}.$$

(*c*) The square of a "one-one" transvectant is also reducible. Thus

$$\begin{aligned}
D_{12}{}^2 &= (aa')(\alpha\alpha')(bb')(\beta\beta') \ldots \\
&= (ba')(bb')(\alpha\alpha')(\beta\beta') \ldots + (ab)(bb')(\alpha\alpha')(\beta\beta') \ldots \\
&= \tfrac{1}{2}(ba')^2(\alpha\alpha')(\beta\beta') \ldots + \tfrac{1}{2}(ab')^2(\alpha\alpha')(\beta\beta') \ldots - \tfrac{1}{2}X - \tfrac{1}{2}Y \\
&= (aa')^2(\alpha\beta')(\beta\alpha') \ldots - \tfrac{1}{2}X - \tfrac{1}{2}Y \\
&= A \,.\, F - (aa')^2(\alpha\beta)(\alpha'\beta') \ldots - \tfrac{1}{2}X - \tfrac{1}{2}Y \\
&= A \,.\, F - \tfrac{1}{2}(\alpha\beta)(\alpha'\beta')\{(aa')^2 b_x{}^2 - (ba')^2 a_x{}^2\} b'_x{}^2 - \tfrac{1}{2}X - \tfrac{1}{2}Y \\
&= A \,.\, F + \tfrac{1}{2}D \,.\, D' - \tfrac{1}{2}X - \tfrac{1}{2}Y.
\end{aligned}$$

Similarly for

$$\begin{aligned}
\{(fP')_{1.1}\}^2 &= M^2 \\
&= Q \,.\, B + \tfrac{1}{4}R'(DD' + Y - X) + \tfrac{1}{4}R_3 D'^2.
\end{aligned}$$

(*d*) Further syzygies result from the linear forms in ξ. Thus between

$$\begin{aligned}
\rho_\xi &= -(ab)^2(\alpha\gamma)(cD')^2\beta_\xi \\
K_\xi &= (aD')^2\alpha_\xi \\
K'_\xi &= (a'D)^2\alpha'_\xi
\end{aligned}$$

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* Grace and Young, *Algebra of Invariants*, pp. 78–80 and 163.

we have

$$(\rho K)K'_\xi + (KK')\rho_\xi + (K'\rho)K_\xi = 0$$

or

$$(\rho,\ K)_{0.1}K'_\xi + (K,\ K')_{0.1}\rho_\xi + (K',\ \rho)_{0.1}K_\xi = 0,$$

$$(\rho K)_{0.1} = \tfrac{1}{2}R_2{}^2 - \tfrac{1}{2}R\,.\,R',$$

$$(KK')_{0.1} = AR_2 - 2R_4,$$

$$(K'\rho)_{0.1} = B'R_2 - BR,$$

$$(R_2{}^2 - RR')K' + 2(AR_2 - 2R_4)\rho + 2(B'R_2 - BR)K = 0.$$

Thus

$$\begin{aligned}(\rho K)_{0.1} &= -(ab)^2(a\gamma)(cD')^2(\beta\delta)(dD'_1)^2\\ &= (ab)(bd)(aD'_1)(cD')^2(dD'_1)(a\gamma)(\beta\delta) + \text{part which vanishes}\\ &= (ab)(bd)(cD'_1)(aD')(cD')(dD'_1(a\gamma)(\beta\delta)\\ &\qquad\qquad - (ab)(bd)(ac)(D'D'_1)(dD'_1)(cD')(a\gamma)(\beta\delta)\\ &= \tfrac{1}{2}(ac)(aD')(cD')(a\gamma)\,.\,(bd)(bD'_1)dD'_1)(\beta\delta)\\ &\qquad\qquad - \tfrac{1}{2}(D'D'_1)^2(ab)(ac)(bd)(cd)(a\gamma)(\beta\delta)\\ &= \tfrac{1}{2}R_2{}^2 - \tfrac{1}{2}R\,.\,R'.\end{aligned}$$

§ 6. The following list of syzygies is by no means complete :

1. $2PP' = -\Delta_{12}DD' - R_2ff' + KDf' + K'fD'.$
2. $2EP = -\Delta Df' - K'f^2 + \Delta_{12}fD.$
3. $2EP' = -\Delta_{12}f'D' + Kf'^2 + \Delta'D'f.$
4. $2GP' = -\Delta_{12}D'^2 - R'ff' + Kf'D'.$
5. $2EG = -\Delta f'D' + Kff' + \Delta_{12}fD'.$
6. $2EG' = -\Delta_{12}f'D - K'ff' + \Delta'fD.$
7. $2EU = Kf'D + K'fD'.$
8. $2GU = -R'fD + KDD' + R_2fD'.$
9. $2G'U = -K'DD' - R_2f'D + Rf'D'.$
10. $2GG' = -DD'D_{12} - R_2ff'.$
11. $E^2 = -\tfrac{1}{2}(\Delta f'^2 - 2\Delta_{12}ff' + \Delta'f^2).$
12. $G^2 = -\tfrac{1}{2}(\Delta D'^2 - 2KfD' + R'f^2).$
13. $G'^2 = -\tfrac{1}{2}(\Delta'D^2 - 2K'f'D + Rf'^2).$
14. $U^2 = -\tfrac{1}{2}(RD'^2 - 2R_2DD' + R'D^2).$
15. $(H - G - Af')^2 = -\{(R_2 + R_3 - A^2)f'^2 + KD_{12}f' + \Delta'D_{12}{}^2).$
16. $(K'f - D\Delta_{12})^2 = -2\{\Delta'P^2 - 2\tau Pf' + \tfrac{1}{2}R\Delta f'^2).$
17. $(Kf' - D'\Delta_{12})^2 = -2\{\Delta P'^2 - 2\tau'P'f + \tfrac{1}{2}R'\Delta'f^2).$
18. $(KD - R_2f)^2 = -2\{R'P^2 - 2\rho PD' + \tfrac{1}{2}R\Delta D'^2).$
19. $(K'D' - R_2f')^2 = -2\{RP'^2 - 2\rho'P'D + \tfrac{1}{2}R'\Delta'D^2\}.$
20. $\tau D - \rho'f = \rho f - \tau'D'.$
21. $R_2R_3ff' - R_2fK'D' - R_3f'KD + 2R_2PP' = 0.$
22. $D'\Delta'D_{12}\Delta_{12} + 2\Delta_{12}PP' + ff'R_2\Delta_{12} - Kf'\Delta'D_{12} - K'fD'\Delta_{12} = 0.$
23. $Xf' = Qf + HF.$
24. $Yf = Qf' + H'F.$
25. $QQ' = \tfrac{1}{4}D^2D'^2 + \tfrac{1}{2}F\{ADD' + R_2N - 2D_{12}U).$

26. $$XY = 2QQ' - F^2R_3 - \tfrac{1}{4}D^2D'^2.$$
27. $$2BB' = R_2R_3 + \tfrac{1}{2}R_2^2 + \tfrac{1}{2}RR' - 2AR_4.$$
28. $$M^2 = QE + \tfrac{1}{2}R'(DD' - X + AF - D_{12}^2) + \tfrac{1}{4}R_3D'^2.$$
29. $$(R_2^2 - RR')K' + 2(AR_2 - 2R_4)\rho + 2(B'R_2 - BR)K = 0.$$
30. $$(R_2^2 - RR')K + 2(BR_2 - B'R')K' + 2(2R_4 - AR_2)\rho' = 0.$$
31. $$(RR' - R_2^2)\rho' + 2(B'R' - BR_2)\rho + (2R_2R_4 - RR'A)K = 0.$$
32. $$(BR - B'R_2)K' + \{R(R_2 + R_3 - A^2) - 2B'^2\}K + (2R_4 - AR_2)\sigma = 0.$$
33. $$(B'R' - BR_2)K + \{R'(R_2 + R_3 - A^2) - 2B^2\}K' + (AR_2 - 2R_4)\sigma' = 0.$$

These results (29–33) follow from—

$$(KK')_{0.1} = AR_2 - 2R_4,$$
$$(K\sigma)_{0.1} = (\rho K')_{0.1} = BR - B'R_2,$$
$$(K\rho')_{0.1} = (\sigma'K')_{0.1} = BR_2 - B'R',$$
$$(\rho'K')_{0.1} = \tfrac{1}{2}R_2^2 - \tfrac{1}{2}RR',$$
$$(\rho\rho')_{0.1} = R_2R_4 - \tfrac{1}{2}ARR',$$
$$(\sigma K')_{0.1} = R(R_2 + R_3) - A^2R - 2B'^2.$$

34. $$D'M' - DM = -UD_{12} + ADD'.$$

A Geometrical Interpretation of the covariants of the two forms f and f' will be given in a subsequent paper.

§ 7. *Tabulated List of the Irreducibles.*

Reference.	Degree.			Orders in x, ξ.
Peano	1	f; f'	$a_x^2\alpha_\xi$	(2, 1)
	2	D; D′	$(ab)(\alpha\beta)a_xb_x$	(2, 0)
	2	Δ; Δ′	$(ab)^2\alpha_\xi\beta_\xi$	(0, 2)
	3	P; P′	$-(ab)^2(\alpha\gamma)c_x^2\beta_\xi$	(2, 1)
	4	R; R′	$(ab)^2(cd)^2(\alpha\gamma)(\beta\delta) = (f, P)_{2.1}$	0
(1)	1, 1	E	$(aa')a_xa'_x\alpha_\xi\alpha'_\xi$	(2, 2)
(2)	1, 1	F	$(\alpha\alpha')a_x^2a'^2_x$	(4, 0)
(3)	1, 1	D_{12}	$(aa')(\alpha\alpha')a_xa'_x$	(2, 0)
(4)	1, 1	Δ_{12}	$(aa')^2\alpha_\xi\alpha'_\xi$	(0, 2)
(5)	1, 1	A	$(aa')^2(\alpha\alpha')$	0
(6 & 7)	2, 2	G; G′	$-(aa')^2(\alpha'\beta')b'^2_x\alpha_\xi$	(2, 1)
(8 & 9)	2, 2	K; K′	$-2(aa')(ab')(a'b')(\alpha\alpha')\beta'_\xi$	(0, 1)
(10 & 11)	1, 2	H; H′	$(a'b')^2(\alpha\alpha')a_x^2\beta'_\xi$	(2, 1)
(12)	2, 2	U	$(ab)(a'b')(aa')(\alpha\beta)(\alpha'\beta')b_xb'_x$	(2, 0)
(13)	2, 2	R_2	$(ab)(a'b')(aa')(bb')(\alpha\beta)(\alpha'\beta') = (D, D')_{2.0}$	0
(14)	2, 2	u	$(ab)^2(a'b')^2(\alpha\alpha')\beta_\xi\beta'_\xi$	(0, 2)
(15)	2, 2	R_3	$(ab)^2(a'b')^2(\alpha\alpha')(\beta\beta') = (\Delta, \Delta')_{0.2}$	0
(18 & 19)	1, 3	τ; τ'	$(aa')^2(b'c')^2(\alpha'\beta')\gamma'_\xi\alpha_\xi$	(0, 2)
(20 & 21)	1, 3	ϕ; ϕ'	$-(a'b')^2(\alpha'\gamma')(\alpha\beta')a_x^2c'^2_x$	(4, 0)

Reference.	Degree.			Orders in x, ξ.
(22 & 23)	1, 3	M ; M′	$(aa')(a'b')(b'c')(\beta'\gamma')(\alpha\alpha')a_x c'_x$	(2, 0)
(24 & 25)	1, 3	β ; β'	$(aa')^2(b'c')^2(\alpha\beta')(\alpha'\gamma') = (f, \mathrm{P}')_{2.1}$	0
(30 & 31)	3, 2	ρ ; ρ'	$-(ab)^2(ca')(cb')(a'b')(\alpha'\beta')(\alpha\gamma)\beta_\xi$	(0, 1)
(37)	3, 3	R_4	$(aa')^2(bc)^2(b'c')^2(\alpha\beta)(\alpha'\beta')(\gamma\gamma') = (pp')_{2.1}$	0
(38 & 39)	2, 2	X ; Y	$(a'b')^2(\alpha\alpha')(\beta\beta')a_x^2 b_x^2$	(4, 0) :— (only one necessary)
(46 & 47)	4, 1	σ ; σ'	$(ab)^2(cd)(ca')(da')(\gamma\delta)(\alpha\alpha')\beta_\xi$	(0, 1)

The degree and symbolic equivalent of the first covariant only have been given.

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III.—A Series Formula for the Roots of Algebraic and Transcendental Equations. By A. C. Aitken, M.A. (New Zealand). *Communicated by* Professor E. T. Whittaker, F.R.S.

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§ 1. The Formula.

The object of this paper is to communicate a general formula for elementary symmetric functions of any assigned degree of a given number of the roots, in ascending order of magnitude, of an algebraic or transcendental equation with complex coefficients. In virtue of the simple relations that exist between these symmetric functions, the formula gives a literal expression in terms of infinite series for all the roots of an equation. In practice it is generally desirable to transform the equation first into one with roots widely separated in value (*e.g.* by the root-squaring process) in order to make the convergence of the series sufficiently rapid for satisfactory numerical computation.

The general formula is as follows. If the equation

$$0 = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots \qquad . \quad . \quad . \quad (1.1)$$

has roots in ascending order of moduli $z_1, z_2, z_3, \dots$ it will be shown that

$$(-)^m \sum^{r} z_1 z_2 z_3 \dots z_m = \frac{a_{r-m}}{a_r} + \frac{a_{r+1}\begin{vmatrix} a_{r-m} & a_r \\ a_{r-m-1} & a_{r-1} \end{vmatrix}}{a_r\begin{vmatrix} a_r & a_{r+1} \\ a_{r-1} & a_r \end{vmatrix}}$$

$$+ \frac{\begin{vmatrix} a_{r+1} & a_{r+2} \\ a_r & a_{r+1} \end{vmatrix}\begin{vmatrix} a_{r-m} & a_r & a_{r+1} \\ a_{r-m-1} & a_{r-1} & a_r \\ a_{r-m-2} & a_{r-2} & a_{r-1} \end{vmatrix}}{\begin{vmatrix} a_r & a_{r+1} \\ a_{r-1} & a_r \end{vmatrix}\begin{vmatrix} a_r & a_{r+1} & a_{r+2} \\ a_{r-1} & a_r & a_{r+1} \\ a_{r-2} & a_{r-1} & a_r \end{vmatrix}} + \dots \qquad (1.2)$$

where the left-hand member is the elementary symmetric function of the mth degree of the r least roots $z_1, z_2, z_3, \dots z_r$. The series on the right will be denoted by S_r^m.

The series S_r^m diverges if the roots z_r and z_{r+1} have equal moduli; converges slowly if the ratio of $|z_{r+1}|$ to $|z_r|$ is not large, rapidly if the ratio is large. The series is really the expansion by a theorem due to

Schweins* of the quotient of two infinite determinants which differ in their first columns only, namely

$$\begin{vmatrix} a_{r-m} & a_{r+1} & a_{r+2} & \cdot & \cdot \\ a_{r-m-1} & a_r & a_{r+1} & \cdot & \cdot \\ a_{r-m-2} & a_{r-1} & a_r & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \div \begin{vmatrix} a_r & a_{r+1} & a_{r+2} & \cdot & \cdot \\ a_{r-1} & a_r & a_{r+1} & \cdot & \cdot \\ a_{r-2} & a_{r-1} & a_r & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \quad . \quad (1.3)$$

The form of this quotient suggests that the proof consists in the setting up of a system of s equations

$$\left.\begin{array}{l} 0 = a_{s+r-1}Q_0 + a_{s+r-2}Q_1 + a_{s+r-3}Q_2 + \ldots + a_r \quad Q_{s-1} + a_{r-m} \quad Q_{s+m-1} \\ 0 = a_{s+r-2}Q_0 + a_{s+r-3}Q_1 + a_{s+r-4}Q_2 + \ldots + a_{r-1} \quad Q_{s-1} + a_{r-m-1} \quad Q_{s+m-1} \\ 0 = a_{s+r-3}Q_0 + a_{s+r-4}Q_1 + a_{s+r-5}Q_2 + \ldots + a_{r-2} \quad Q_{s-1} + a_{r-m-2} \quad Q_{s+m-1} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ 0 = a_r \quad Q_0 + a_{r-1} \quad Q_1 + a_{r-2} \quad Q_2 + \ldots + a_{r-s+1}Q_{s-1} + a_{r-s-m+1}Q_{s+m-1} \end{array}\right\} \quad . \quad (1.4)$$

where the a's and the Q's vanish for negative suffixes, and in the obtaining of $(-)^m\overset{r}{\Sigma}z_1z_2z_3 \ldots z_m$ as the limit of the fraction $-Q_{s-1}/Q_{s+m-1}$ when s is indefinitely increased. This was the method used originally in proving the particular case where $r=m=1$, Whittaker's series † for the smallest root of an equation, viz.

$$z_1 = -\frac{a_0}{a_1} - \frac{a_0^2a_2}{a_1\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}} - \frac{a_0^3\begin{vmatrix} a_2 & a_3 \\ a_1 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}\begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}} - \ldots \quad (1.5)$$

from which the existence of the present formula was in the first instance conjectured.

§ 2. PROOF OF THE FORMULA.

(a) Suppose the equation (1.1) to be of degree n. It will be convenient to write $z^{-1}=u$, so that $u_1, u_2, u_3, \ldots$ are in descending order of value. Then if v be any number of modulus less than each of the moduli of $z_1, z_2, z_3, \ldots z_n$, we have

$$\frac{a_0}{a_0 + a_1v + a_2v^2 + \ldots + a_nv^n} = \frac{1}{(1-u_1v)(1-u_2v) \ldots (1-u_nv)}$$
$$= (1 + u_1v + u_1^2v^2 + \ldots)(1 + u_2v + u_2^2v^2 + \ldots) \ldots (1 + u_nv + u_n^2v^2 + \ldots)$$
$$= P_0 + P_1v + P_2v^2 + P_3v^3 + \ldots,$$

---

* F. Schweins, *Theorie der Differenzen und Differentiale* (1825); Sir Thomas Muir, *Theory of Determinants*, vol. i, part 1, pp. 171-172.

† *Proc. Edin. Math. Soc.*, vol. xxxvi (1918), p. 103; or Whittaker and Robinson, *Calculus of Observations*, p. 120.

where P_r denotes the sum of the homogeneous powers and products of u_1, $u_2, u_3, \ldots u_n$, taken r at a time, and $P_0 = 1$.

Therefore

$$a_0 = (a_0 + a_1 v + a_2 v^2 + \ldots + a_n v^n)(P_0 + P_1 v + P_2 v^2 + \ldots)$$

Equating coefficients of powers of v, we have the relations *

$$\left.\begin{array}{l}
0 = a_0 P_0 \\
0 = a_1 P_0 + a_0 P_1 \\
0 = a_2 P_0 + a_1 P_1 + a_0 P_2 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
0 = a_{s+r-1} P_0 + a_{s+r-2} P_1 + \ldots + a_1 P_{s+r-2} + a_0 P_{s+r-1} \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{array}\right\} \quad . \quad . \quad (2.1)$$

Take the equation of the above set whose right-hand member begins with the term $a_{s+r-1}P_0$. From it, by means of those $r-1$ preceding equations which begin on the right with $a_{s+r-2}P_0$, $a_{s+r-3}P_0, \ldots a_{s+1}P_0$, $a_{s-m+1}P_0$ respectively, we can eliminate the $r-1$ coefficients $a_0, a_1, a_2, \ldots a_{r-m-1}, a_{r-m+1}, a_{r-m+2}, \ldots a_{r-1}$, obtaining the new equation

$$0 = a_{s+r-1} \begin{vmatrix}
P_0 & P_s & P_{s+1} & \ldots P_{s+m-2} & P_{s+m} & \ldots P_{s+r-2} & P_{s+r-1} \\
0 & P_{s-1} & P_s & \ldots P_{s+m-3} & P_{s+m-1} & \ldots P_{s+r-3} & P_{s+r-2} \\
0 & P_{s-2} & P_{s-1} & \ldots P_{s+m-4} & P_{s+m-2} & \ldots P_{s+r-4} & P_{s+r-3} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & P_{s-r+2} & P_{s-r+3} & \ldots P_{s-r+m} & P_{s-r+m+2} & \ldots P_s & P_{s+1} \\
0 & P_{s-r-m+2} & P_{s-r-m+3} & \ldots P_{s-r} & P_{s-r+2} & \ldots P_{s-m} & P_{s-m+1}
\end{vmatrix} + \ldots,$$

which we shall write shortly as

$$0 = a_{s+r-1}Q_0 + a_{s+r-2}Q_1 + a_{s+r-3}Q_2 + \ldots + a_r Q_{s-1} + a_{r-m}Q_{s+m-1},$$

where the Q's are determinants of the rth order differing only in their first columns, the first column of Q_k consisting of $P_k, P_{k-1}, P_{k-2}, \ldots P_{k-r+2}, P_{k-r-m+2}$.

But evidently, since the equations (2.1) have the property of remaining valid when the suffixes of the a's are all diminished by the same integer, we may write down the following s consecutive equations:

$$\left.\begin{array}{l}
0 = a_{s+r-1}Q_0 + a_{s+r-2}Q_1 + \ldots + a_r Q_{s-1} + a_{r-m} Q_{s+m-1} \\
0 = a_{s+r-2}Q_0 + a_{s+r-3}Q_1 + \ldots + a_{r-1} Q_{s-1} + a_{r-m-1} Q_{s+m-1} \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
0 = a_r Q_0 + a_{r-1} Q_1 + \ldots + a_{r-s+1}Q_{s-1} + a_{r-s-m+1}Q_{s+m-1}
\end{array}\right\} \quad . \quad (2.11)$$

---

* First given by H. Wronski, *Introd. à la philos. des math.* (1811), Paris; *cf.* Sir Thomas Muir, *Theory of Determinants*, vol. ii, pp. 216–217.

These are in fact the equations (1.4), and we obtain

$$
-\frac{Q_{s-1}}{Q_{s+m-1}} = \frac{\begin{vmatrix} a_{r+m} & a_{r+1} & a_{r+2} & \cdots & a_{s+r-1} \\ a_{r-m-1} & a_r & a_{r+1} & \cdots & a_{s+r-2} \\ a_{r-m-2} & a_{r-1} & a_r & \cdots & a_{s+r-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{r-s-m+1} & a_{r-s+2} & a_{r-s+3} & \cdots & a_r \end{vmatrix}}{\begin{vmatrix} a_r & a_{r+1} & a_{r+2} & \cdots & a_{s+r-1} \\ a_{r-1} & a_r & a_{r+1} & \cdots & a_{s+r-2} \\ a_{r-2} & a_{r-1} & a_r & \cdots & a_{s+r-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{r-s+1} & a_{r-s+2} & a_{r-s+3} & \cdots & a_r \end{vmatrix}} \quad . \quad . \quad . \quad (2.12)
$$

Expanding by Schweins' Theorem and increasing s indefinitely, we have

$$
-\lim_{s\to\infty} \frac{Q_{s-1}}{Q_{s+m-1}} = \mathbf{S}_r^m \quad . \quad . \quad . \quad . \quad . \quad (2.13)
$$

(b) It is now necessary to show that

$$
-\lim_{s\to\infty} \frac{Q_{s-1}}{Q_{s+m-1}} = (-)^m \sum^{r} z_1 z_2 z_3 \ldots z_m.
$$

Consider P_s, the sum of the homogeneous powers and products, s at a time, of the n numbers $u_1, u_2, u_3, \ldots u_n$. Separating it into those terms that contain a particular u_h and those that do not, we have

$$
P_s - u_h P_{s-1} = P_s(\bar{u}_h) \quad . \quad . \quad . \quad . \quad . \quad (2.2)
$$

where $P_s(\bar{u}_h)$ denotes the sum of the homogeneous powers and products, s at a time, of the $n-1$ numbers that remain when u_h is removed from $u_1, u_2, u_3, \ldots u_n$.

Again, by a double application of (2.2),

$$
\begin{aligned}
P_s - (u_h + u_k)P_{s-1} + u_h u_k P_{s-2} &= P_s(\bar{u}_h) - u_k(P_{s-1} - u_h P_{s-2}) \\
&= P_s(\bar{u}_h) - u_k P_{s-1}(\bar{u}_h) \\
&= P_s(\bar{u}_h, \bar{u}_k) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.21)
\end{aligned}
$$

In fact, introducing the operator E^{-1}, which diminishes suffixes by unity, we have

$$
(1 - u_h E^{-1})(1 - u_k E^{-1}) \ldots (1 - u_p E^{-1})P_s = P_s(\bar{u}_h, \bar{u}_k, \ldots \bar{u}_p) \quad . \quad (2.22)
$$

We may now establish the property of P_s on which the theorem depends, namely that

$$
\lim_{s\to\infty} \frac{P_s}{P_{s-1}} = u_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.3)
$$

Assume first that this property holds for the $n-1$ numbers involved in $P_s(\bar{u}_h)$, a particular u_h, less than u_1, being removed from $u_1, u_2, \ldots u_n$, *i.e.* assume that

$$\lim_{s\to\infty} \frac{P_s(\bar{u}_h)}{P_{s-1}(\bar{u}_h)} = u_1.$$

Then

$$\begin{aligned}\lim_{s\to\infty} \frac{P_s}{P_{s-1}} &= \lim_{s\to\infty} \frac{P_{s-1}}{P_{s-2}} \\ &= \lim_{s\to\infty} \frac{P_s - u_h P_{s-1}}{P_{s-1} - u_h P_{s-2}} \\ &= \lim_{s\to\infty} \frac{P_s(\bar{u}_h)}{P_{s-1}(\bar{u}_h)} \\ &= u_1, \text{ by hypothesis.}\end{aligned}$$

Hence if the property holds for $n-1$ numbers of which u_1 is greatest, it holds for n.

But it obviously holds for $n=1$, whatever s is.

Hence it holds generally.

Thus as s is increased the equation

$$P_s - u_1 P_{s-1} = P_s(\bar{u}_1)$$

tends to the limiting form

$$P_s - u_1 P_{s-1} = 0.$$

Again, $P_s(\bar{u}_h, \bar{u}_k)$ is equivalent to each of $P_s(\bar{u}_k) - u_h P_{s-1}(\bar{u}_k)$ and $P_s(\bar{u}_h) - u_k P_{s-1}(\bar{u}_h)$. Thus in the limit it becomes negligible if u_h is the greatest of $u_1, u_2, \ldots u_n$, with u_k removed, and in addition if u_k is the greatest of $u_1, u_2, \ldots u_n$, with u_h removed, *i.e.* altogether if u_h and u_k are the two greatest numbers (not necessarily different) of $u_1, u_2, \ldots u_n$.

Proceeding in this way, we have that if $u_1, u_2, u_3, \ldots u_r$ are the r numerically greatest numbers of $u_1, u_2, \ldots u_n$, we may in the limit, as s is indefinitely increased, neglect the right-hand member of the equation

$$(1 - u_1 E^{-1})(1 - u_2 E^{-1}) \ldots (1 - u_r E^{-1}) P_s = P_s(\bar{u}_1, \bar{u}_2, \ldots \bar{u}_r)$$

and write, dispensing with symbolic operators, the set of r equations

$$\left.\begin{aligned}
0 &= P_{s+r-1} - \Sigma u_1 P_{s+r-2} + \Sigma u_1 u_2 P_{s+r-3} - \ldots + (-)^{r-m} \Sigma u_1 u_2 \ldots u_{r-m} P_{s+m-1} \\
&\qquad + \ldots + (-)^r u_1 u_2 \ldots u_r P_{s-1} \\
0 &= P_{s+r-2} - \Sigma u_1 P_{s+r-3} + \Sigma u_1 u_2 P_{s+r-4} - \ldots + (-)^{r-m} \Sigma u_1 u_2 \ldots u_{r-m} P_{s+m-2} \\
&\qquad + \ldots + (-)^r u_1 u_2 \ldots u_r P_{s-2} \\
0 &= P_{s+r-3} - \Sigma u_1 P_{s+r-4} + \Sigma u_1 u_2 P_{s+r-5} - \ldots + (-)^{r-m} \Sigma u_1 u_2 \ldots u_{r-m} P_{s+m-3} \\
&\qquad + \ldots + (-)^r u_1 u_2 \ldots u_r P_{s-3} \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
0 &= P_{s+1} - \Sigma u_1 P_s + \Sigma u_1 u_2 P_{s-1} - \ldots + (-)^{r-m} \Sigma u_1 u_2 \ldots u_{r-m} P_{s-r+m+1} \\
&\qquad + \ldots + (-)^r u_1 u_2 \ldots u_r P_{s-r+1} \\
0 &= P_{s-m+1} - \Sigma u_1 P_{s-m} + u_1 u_2 P_{s-m-1} - \ldots + (-)^{r-m} \Sigma u_1 u_2 \ldots u_{r-m} P_{s-r+1} \\
&\qquad + \ldots + (-)^r u_1 u_2 \ldots u_r P_{s-r-m+}
\end{aligned}\right\} \quad (2.4)$$

From these equations we have

$$(-)^m\frac{\overset{r}{\sum} u_1u_2u_3\,.\,.\,u_{r-m}}{u_1u_2u_3\,.\,.\,u_r} = -\lim\frac{\begin{vmatrix} P_{s-1} & P_s & P_{s+1} & \cdot\cdot\, P_{s+m-2} & P_{s+m} & \cdot\cdot\, P_{s+r-2} & P_{s+r-1} \\ P_{s-2} & P_{s-1} & P_s & \cdot\cdot\, P_{s+m-3} & P_{s+m-1} & \cdot\cdot\, P_{s+r-3} & P_{s+r-2} \\ P_{s-3} & P_{s-2} & P_{s-1} & \cdot\cdot\, P_{s+m-4} & P_{s+m-2} & \cdot\cdot\, P_{s+r-4} & P_{s+r-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ P_{s-r+1} & P_{s-r+2} & P_{s-r+3} & \cdot\cdot\, P_{s-r+m} & P_{s-r+m+2} & \cdot\cdot\, P_s & P_{s+1} \\ P_{s-r-m+1} & P_{s-r-m+2} & P_{s-r-m+3} & \cdot\cdot\, P_{s-r} & P_{s-r+2} & \cdot\cdot\, P_{s-m} & P_{s-m+1} \end{vmatrix}}{\begin{vmatrix} P_{s+m-1} & P_s & P_{s+1} & \cdot\cdot\, P_{s+m-2} & P_{s+m} & \cdot\cdot\, P_{s+r-2} & P_{s+r-1} \\ P_{s+m-2} & P_{s-1} & P_s & \cdot\cdot\, P_{s+m-3} & P_{s+m-1} & \cdot\cdot\, P_{s+r-3} & P_{s+r-2} \\ P_{s+m-3} & P_{s-2} & P_{s-1} & \cdot\cdot\, P_{s+m-4} & P_{s+m-2} & \cdot\cdot\, P_{s+r-4} & P_{s+r-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ P_{s-r+m+1} & P_{s-r+2} & P_{s-r+3} & \cdot\cdot\, P_{s-r+m} & P_{s-r+m+2} & \cdot\cdot\, P_s & P_{s+1} \\ P_{s-r+1} & P_{s-r-m+2} & P_{s-r-m+3} & \cdot\cdot\, P_{s-r} & P_{s-r+2} & \cdot\cdot\, P_{s-m} & P_{s-m+1} \end{vmatrix}}.$$

$$\text{i.e. } (-)^m\sum^r z_1z_2z_3\ldots z_m = -\lim\frac{Q_{s-1}}{Q_{s+m-1}}$$

$$= \mathbf{S}_r^m, \text{ by } (2.13) \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2.5)$$

Thus the theorem is established.

§ 3. THE ROOTS OF AN EQUATION IN TERMS OF THE SERIES.

It will now be shown that all the roots of the general equation with complex coefficients can be expressed in terms of the series here considered.

By separating $\overset{r}{\Sigma} z_1z_2z_3\ldots z_m$ into terms containing z_r and terms not containing z_r we have the relation

$$\mathbf{S}_r^m = \mathbf{S}_{r-1}^m - z_r\mathbf{S}_{r-1}^{m-1} \qquad . \quad . \quad . \quad . \quad . \quad . \qquad (3.1)$$

and the three series involved converge provided that $|z_{r+1}|$, $|z_r|$ and $|z_{r-1}|$ are all different. Thus any root of an equation different in modulus from the other roots is given in a variety of ways, but most simply when $m=1$ by

$$\mathbf{S}_r^1 - \mathbf{S}_{r-1}^1 = -z_r \qquad . \quad . \quad . \quad . \quad . \quad . \qquad (3.11)$$

The above formula is useful for the numerical computation of the different real roots of an equation with real coefficients.

Again, by successive applications of (3.1) we easily derive a relation analogous to that of (2.22),

$$(1-z_rE^{-1})(1-z_{r-1}E^{-1})\ldots(1-z_{r-p+1}E^{-1})\mathbf{S}_{r-p}^m = \mathbf{S}_r^m \qquad . \quad . \qquad (3.2)$$

where E^{-1} is an operator diminishing the upper suffix of $\mathbf{S}$ by unity.

Thus if two roots z_r, z_{r+1} have the same modulus, $\mathbf{S}_r^m$ diverges, but we have such relations as

$$\mathbf{S}_{r-1}^2 - (z_r + z_{r+1})\mathbf{S}_{r-1}^1 + z_r z_{r+1} = \mathbf{S}_{r+1}^2,$$
$$\mathbf{S}_{r-1}^1 - (z_r + z_{r+1}) = \mathbf{S}_{r+1}^1,$$

from which z_r and z_{r+1} may be determined as the roots of a quadratic equation. Alternatively they may be determined, except in the case $z_r = z_{r+1}$, by transforming the equation by a linear change of variable and applying (3.11).

More generally, if p roots have the same modulus, the equation may be transformed by a linear change of variable into one with roots of different moduli, except in the case of repeated roots. The roots of different moduli may then be obtained by the series as in (3.11).

Finally, if an equation has repeated roots, for example if $z_r, z_{r+1}, z_{r+2}, \ldots z_{r+p-1}$ coincide, all the series $\mathbf{S}_r^m, \mathbf{S}_{r+1}^m, \ldots \mathbf{S}_{r+p-2}^m$ diverge, but we have

$$\begin{aligned}\mathbf{S}_{r+p-1}^1 - \mathbf{S}_{r-1}^1 &= -(z_r + z_{r+1} + \ldots z_{r+p-1}) \\ &= -pz_r,\end{aligned}$$

and z_r is determined.

In theory, therefore, all the roots of the general equation with complex coefficients can be given in terms of the series $\mathbf{S}_r^m$.

§ 4. THE USE OF THE SERIES FOR NUMERICAL COMPUTATION.

Actual practice shows that the series is best used in conjunction with the root-squaring method of Dandelin, Lobachevsky, and Graeffe. Here its value lies in the fact that it provides the necessary correction terms to the method. When a number of root-squaring operations have been performed on an equation, we have therefore the alternatives, either to continue the process in the usual manner, or to arrest it there and calculate for the equation obtained the series $\mathbf{S}_r^1$, $\mathbf{S}_r^2$, etc., according as we observe * from the trend of the successive coefficients of the powers of the variable whether the roots in numerical order are different, closely adjacent or complex. If the roots of the equation arrived at are widely separated the series are rapidly convergent, and two terms of each are usually sufficient for accuracy.

Example.†

$$x^7 + 7x^6 + 21x^5 + 63x^4 + 147x^3 + 189x^2 + 119x - 451 = 0.$$

---

* *Cf.* Whittaker and Robinson, *Calculus of Observations*, pp. 113-118.

† *Ibid.*, p. 116.

Six root-squaring operations lead to the equation

$$a_7y^7+a_6y^6+a_5y^5+a_4y^4+a_3y^3+a_2y^2+a_1y^1+a_0=0,$$

where

$$a_0=7{\cdot}367136^{169},\ a_1=1{\cdot}252791^{172},\ a_2=-1{\cdot}274626^{148},$$
$$a_3=1{\cdot}436597^{126},\ a_4=-2{\cdot}018878^{98},\ a_5=8{\cdot}259879^{69},$$
$$a_6=3{\cdot}006442^{35},\ a_7=1,$$

where $7{\cdot}367136^{169}$ represents $7{\cdot}367136\times10^{169}$.

During the process the behaviour of the coefficients shows that in ascending numerical order there are, first a real root, then a pair of complex roots, then another pair of complex roots, and finally two real roots closely adjacent in value. With this knowledge we find the moduli of the roots from the relations

$$-y_1=S^1_1,$$
$$S^1_1-(y_3+y_2)=S^1_3,\ -(y_3+y_2)S^1_1+y_3y_2=S^2_3,$$
$$S^1_3-(y_5+y_4)=S^1_5,\ S^2_3-(y_5+y_4)S^1_3+y_5y_4=S^2_5,$$
$$S^1_5-(y_7+y_6)=S^1_7,\ S^2_5-(y_6+y_7)S^1_5+y_6y_7=S^2_7.$$

To two terms each, we find

$$S^1_1=\frac{a_0}{a_1}+\frac{a_2}{a_1}\frac{\begin{vmatrix}a_0&a_1\\0&a_0\end{vmatrix}}{\begin{vmatrix}a_1&a_2\\a_0&a_1\end{vmatrix}}=5{\cdot}880580^{-3}.$$

$$S^1_3=\frac{a_2}{a_3}+\frac{a_4}{a_3}\frac{\begin{vmatrix}a_2&a_3\\a_1&a_2\end{vmatrix}}{\begin{vmatrix}a_3&a_4\\a_2&a_3\end{vmatrix}}=-8{\cdot}871323^{21}.$$

$$S^1_5=\frac{a_4}{a_5}+\frac{a_6}{a_5}\frac{\begin{vmatrix}a_4&a_5\\a_3&a_4\end{vmatrix}}{\begin{vmatrix}a_5&a_6\\a_4&a_5\end{vmatrix}}=-2{\cdot}444196^{28}.$$

$$S^2_3=\frac{a_1}{a_3}+\frac{a_4}{a_3}\frac{\begin{vmatrix}a_1&a_3\\a_0&a_2\end{vmatrix}}{\begin{vmatrix}a_3&a_4\\a_2&a_3\end{vmatrix}}=8{\cdot}720557^{45}.$$

$$S^2_5=\frac{a_3}{a_5}+\frac{a_6}{a_5}\frac{\begin{vmatrix}a_3&a_5\\a_2&a_4\end{vmatrix}}{\begin{vmatrix}a_5&a_6\\a_4&a_5\end{vmatrix}}=1{\cdot}739245^{56}.$$

$$S^1_7=\frac{a_6}{a_7}=3{\cdot}006442^{35},\quad S^2_7=\frac{a_5}{a_7}=8{\cdot}259879^{69}.$$

Therefore

$$y_1 = x_1^{64} = 5{\cdot}88058^{-3}.$$

$$y_3 + y_2 = -8{\cdot}871323^{21}.$$

$$y_3 y_2 = r_1^{128} = 8{\cdot}720557^{45},$$

where $$r_1 = |\, y_3 \,| = |\, y_2 \,|.$$

$$y_5 + y_4 = -2{\cdot}444195^{28}.$$

$$y_5 y_4 = r_2^{128} = 1{\cdot}739243^{56},$$

where $$r_2 = |\, y_5 \,| = |\, y_4 \,|.$$

$$y_7 + y_6 = 3{\cdot}006442^{35}.$$

$$y_7 y_6 = 8{\cdot}259872^{69},$$

whence $$y_7 = x_7^{64} = 2{\cdot}700587^{35},$$

$$y_6 = x_6^{64} = 3{\cdot}058547^{34}.$$

By logarithms, $|x_1| = {\cdot}922884$, $r_1^2 = 5{\cdot}221810$, $r_2^2 = 7{\cdot}564073$, $|x_6| = 3{\cdot}458090$, $|x_7| = 3{\cdot}577802$, and the signs of x_1, x_6, x_7 are easily fixed by inspection as $+, -, -$. Also if $x_2 = p_1 + iq_1$, $x_4 = p_2 + iq_2$, we have, considering the coefficients of x^6 and of x,

$$-7 = x_1 + x_6 + x_7 + 2(p_1 + p_2),$$

$$119 = r_1^2 r_2^2 (x_1 x_6 + x_1 x_7 + x_6 x_7) + 2x_1 x_6 x_7 (p_1 r_2^2 + p_2 r_1^2).$$

Substituting in these simultaneous equations in p_1 and p_2 the values of x_1, x_6, x_7, r_1^2, r_2^2, and solving, we obtain

$$p_1 = -1{\cdot}127811, \quad q_1 = 1{\cdot}987423,$$

$$p_2 = {\cdot}684315, \quad q_2 = 2{\cdot}663792.$$

Thus the roots are ·922884, $-3{\cdot}458090$, $-3{\cdot}577802$, $-1{\cdot}127811 \pm 1{\cdot}987423i$, ${\cdot}684315 \pm 2{\cdot}663792i$.

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IV.—The Electrolysis of Salts of Alkyloxyacids.

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(MS. received Sept. 2, 1924. Read December 8, 1924.)

MANY attempts have been made to synthesise the glycols by electrolysis of salts of hydroxyacids, in a manner comparable with Kolbe's ethane synthesis. These have almost invariably resulted in failure. Miller and Hofer [*] electrolysed sodium glycollate, and found that formaldehyde was the chief organic product. Similarly potassium lactate gave acetaldehyde, and sodium α-hydroxybutyrate propionic aldehyde. To explain the formation of these aldehydes, Hofer and Moest [†] advanced the theory that the discharged anions interacted with discharged hydroxyl ions according to the equation :—

$$CH_2(OH)COO + OH \longrightarrow CH_2(OH)_2 + CO_2.$$

The intermediate compound, being incapable of existence, would appear as formaldehyde and water :—

$$CH_2(OH)_2 \longrightarrow CH_2O + H_2O.$$

The expedient of protecting the hydroxyl group by substituting for its hydrogen atom an alkyl group naturally suggested itself. Miller and Hofer electrolysed salts of methoxyacetic acid, but found that, as with glycollic acid, formaldehyde was the chief product; formic acid, methylal, and methyl alcohol appeared in small quantity. J. W. Walker [‡] obtained only formaldehyde on electrolysis of sodium ethoxyacetate.

The only successful synthesis of this kind was carried out by Hamonet.[§] In attempting to find a method of preparation of tetramethylene glycol, he electrolysed sodium β-amyloxypropionate, and obtained in good yield the diamyl ether $(C_5H_{11}O).CH_2.CH_2.CH_2.CH_2.(OC_5H_{11})$. He also investigated the behaviour of salts of other similar acids when subjected to electrolysis, such as amyloxyacetic α-amyloxypropionic, γ-amyloxybutyric. He found that when the alkyloxy- group was in the α- or γ-positions, "inextricably complex mixtures of an acetallic nature" were produced, and no synthesis took place.

This research was undertaken in order to investigate somewhat more closely the behaviour of salts of alkyloxyacids on electrolysis. In particular it was thought that it might be possible to ascend the series of dibasic hydroxacids (*i.e.* tartronic, tartaric, mucic, etc.) by electrolysis of the potassium ethyl salts of their alkyl derivatives, in a manner analogous to the synthesis of the normal dibasic acids.

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[*] *Ber.*, **27**, 461.

[†] *Ann.*, **323**, 322.

[‡] *Jour. Chem. Soc.*, **69**, 1278.

[§] *Bull. Soc. Chim.*, **33**, 513.

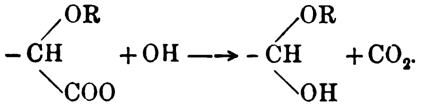
Accordingly the potassium ethyl salts of ethoxymalonic and symmetrical d-diethoxysuccinic acids have been prepared, and their aqueous solutions electrolysed. In addition, the electrolysis of salts of ethoxyacetic and amyloxyacetic acids has been repeated, and the resulting products have been investigated.

From none of these compounds has any appreciable amount of synthetic product been obtained. In only one case was the quantity sufficiently great to admit of detection: tartaric acid has been synthesised by the electrolysis of potassium ethyl ethoxymalonate and de-ethylation of the ethyl diethoxysuccinate so obtained.

The chief products of electrolysis are of an aldehydic nature. The replacement of the hydroxyl by an alkyloxy- group in the salt to be electrolysed has not prevented the formation of these compounds. This would suggest that their formation is not due to oxidation phenomena. A further objection to any oxidation theory is provided by the fact that ethyl formylethoxyacetate, EtOOC.CH(OEt).CHO, is the chief product of electrolysis of potassium ethyl diethoxysuccinate. This substance is very readily oxidised, and it is hardly likely that it would appear as the result of a process of oxidation by so energetic a reagent as nascent oxygen at a platinum surface.

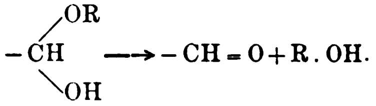
On the other hand, Hofer and Moest's explanation of the formation of formaldehyde from sodium glycollate is applicable to alkyloxy- as well as to hydroxy- compounds, alcohol instead of water splitting off from the intermediate compound. The process is represented by the following equations:—

(1) . . .



$$-\mathrm{CH}\genfrac{}{}{0pt}{}{\diagup\mathrm{OR}}{\diagdown\mathrm{COO}} + \mathrm{OH} \longrightarrow -\mathrm{CH}\genfrac{}{}{0pt}{}{\diagup\mathrm{OR}}{\diagdown\mathrm{OH}} + \mathrm{CO_2}.$$

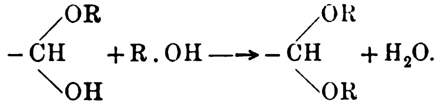
(2) . . .



$$-\mathrm{CH}\genfrac{}{}{0pt}{}{\diagup\mathrm{OR}}{\diagdown\mathrm{OH}} \longrightarrow -\mathrm{CH{=}O} + \mathrm{R.OH}.$$

That such an intermediate compound is indeed formed has been proved by the identification of the alcoholate of ethyl glyoxylate, EtOOC.CH(OH)(OEt), in the mixture resulting from the electrolysis of potassium ethyl ethoxymalonate. In this case the alcoholate is quite stable, showing no tendency to split off alcohol.

Further evidence of the formation of such an intermediate compound is afforded by the presence of acetals in the mixtures obtained by electrolysis of salts of ethoxyacetic, amyloxyacetic, and ethoxymalonic acids. It is difficult to see how these could be produced otherwise than by the action of the alcohol on the intermediate compound in the neighbourhood of the anode.



$$-\mathrm{CH}\begin{matrix}\diagup \mathrm{OR}\\ \diagdown \mathrm{OH}\end{matrix} + \mathrm{R.OH} \longrightarrow -\mathrm{CH}\begin{matrix}\diagup \mathrm{OR}\\ \diagdown \mathrm{OR}\end{matrix} + \mathrm{H_2O}.$$

As would be expected, oxidation products appear in every case, but only in relatively small quantities.

The behaviour of the ethoxyacetates has been shown to be quite analogous to that of the methoxyacetates.

The effect of replacing the ethyl group in ethoxyacetic acid by an amyl group is very marked, the amount of electrolytic oil being increased by about four times. This increase is due to the presence of amyl amyloxyacetate. That the amyl alcohol liberated at the anode simultaneously with the formaldehyde is in a peculiarly active state, is shown by the formation of the acetal. It is clear, therefore, that if any amyloxyacetic acid were liberated at the anode, the necessary conditions for ester formation would be present, and it is almost certain that this is the source of the ester. The acid can be assumed to result from the well-known reaction represented in the following equation :—

$$2CH_2(OC_5H_{11})COO + H_2O \longrightarrow 2CH_2(OC_5H_{11})COOH + (O).$$

EXPERIMENTAL.

ELECTROLYSIS OF POTASSIUM ETHYL ETHOXYMALONATE, $EtOOC.CH(OC_2H_5).COOK$.

Preparation of Potassium Ethyl Ethoxymalonate.—The di-ester was obtained by the decomposition of ethyl ethoxyoxalacetate, which results from the condensation of ethyl oxalate with ethyl ethoxyacetate. When the oxalester is heated, it breaks down into ethyl ethoxymalonate and carbon monoxide.* The ester was half-saponified in cold dilute alcoholic solution with the calculated quantity of potash.

Ethoxymalonamide.—This was obtained from the ester by the action of ammonia. Melting-point 203–204°. Found N = 19·26; $C_5H_{10}O_3N_2$ requires N = 19·17 per cent.

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* Wislicenus and Münzesheimer, *Ber.*, 31, 552.

Dissociation Constant of Ethoxymalonic Acid.—The acid used for this purpose had a m.p. of 124–126°. 0·0952 grm. required 25·66 c.c. of N/20 alkali for neutralisation. The calculated figure is 25·70 c.c. A value of 0·71 was found for the constant at 25° ($\mu\infty = 371$). The constant of the ethoxyacid is thus higher than that of the corresponding hydroxyacid (K for tartronic acid = 0·5). A similar increase due to the replacement of the hydroxyl hydrogen has been found by Ostwald in the case of glycollic acid, and by Findlay, Turner, and Owen* in the case of mandelic acid.

A solution of 20 grms. of the salt in 20 c.c. of water was electrolysed in a platinum crucible, which functioned as cathode. The anode consisted of a spiral of platinum wire. The crucible was cooled during electrolysis by immersion in running water, the temperature of the contents being kept below 25°.

A current of 3 amps. at 10 volts was passed for an hour, when the oil formed was separated off. The residual liquid was again electrolysed for an hour and finally extracted with ether. A yield of 4 grms. of oil was obtained from each operation.

Aqueous Residue.—During the last stages of the electrolysis potassium bicarbonate crystallised out from the liquid. It was also found to contain ethyl alcohol, oxalic acid, formaldehyde, and acetaldehyde

Oily Product.—After several distillations, four main fractions were isolated:—

1. 75–80°. This was small in quantity, and was identified as ethyl alcohol.

2. 130–145°. This fraction had strong reducing properties, giving a silver mirror readily with ammoniacal silver hydroxide, and reducing Fehling's solution on gentle warming. It coloured Schiff's reagent. With aniline acetate and acetic acid a white crystalline precipitate which turned bright yellow on standing was obtained. This reaction is characteristic of glyoxylic acid.

After further fractionation, it was obtained boiling between 137° and 140°. The alcoholate of ethyl glyoxylate has a b.p. of 136–138°. An analysis gave the following figures:—Found, C = 48·40, H = 7·83; $C_6H_{12}O_4$ requires C = 48·64, H = 8·11 per cent.

The phenylhydrazone was prepared and recrystallised twice, from aqueous alcohol, and from chloroform and ligroin. Its m.p. was found to be 128–129° (uncorr.). The phenylhydrazone of ethyl glyoxylate has a m.p. of 131°. Found, N = 14·72; $C_{10}H_{12}O_2N_2$ requires N = 14·58 per cent.

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* *Jour. Chem. Soc.*, **95**, 938.

3. 190–200°. This was the main product of the electrolysis, being the largest of the four fractions. It had a slight fruity odour. A determination of the molecular weight in benzene (Beckmann boiling-point method) gave a value of 180. When warmed with dilute hydrochloric acid, alcohol was liberated and an oily substance left which gave all the reactions of ethyl glyoxylate. The original substance gave no reactions characteristic of aldehydes.

By evaporating on the steam bath with concentrated hydrochloric acid, glyoxylic acid was obtained. The phenylhydrazone when recrystallised from water, appeared as short yellow needles of m.p. 136–137°, thus establishing the identity of the acid. The molecular weight of ethyl diethoxyacetate is 176, and its b.p. is 199°. The substance under investigation was identified with this by conversion into diethoxyacetamide. It was treated with concentrated ammonia and sufficient alcohol to keep it in solution, and allowed to stand over night. The amide, obtained on evaporation, was recrystallised twice from chloroform and petroleum ether. Its m.p. was 79–80°. This solvent gave two distinct crystalline forms in different operations: (1) The characteristic thin, transparent plates, flexible and greasy to the touch; * and (2) long needles. The melting-points of the two forms were identical. The substance is known to sublime in needles.

4. 150–165° at 15 mm. This fraction was small in quantity. In order to detect the presence of any ethyl diethoxysuccinate, which, if any synthesis had taken place, would be expected here, 2 grms. were heated with 20 grms. of hydroiodic acid (sp. gr. 1·82) in a sealed tube to 100° for eight hours. From this a syrupy substance was isolated. The following reactions indicated the presence of tartaric acid:—

When heated, it charred and gave an odour of burnt sugar. It gave the colour reactions when warmed with concentrated sulphuric acid and resorcinol (red), and pyrogallol (purple). A neutral solution gave with calcium chloride solution a precipitate soluble in acetic acid on warming; and with silver nitrate a precipitate which gave a metallic mirror when dissolved in dilute ammonia and warmed.

A solution of the substance prevented precipitation of copper hydroxide by sodium hydroxide.

A white crystalline precipitate was obtained with potassium acetate and alcohol. This was recrystallised from water, and when seen under the

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* Schreiber, *Jahrsb. der Chem.*, 1870, 642.

microscope appeared to be identical with crystals of potassium hydrogen tartrate. The amount was too small to admit of quantitative estimation.

ELECTROLYSIS OF POTASSIUM ETHYL DIETHOXYSUCCINATE, EtOOC . CH(OEt) . CH(OEt) . COOK.

Preparation of Potassium Ethyl Diethoxysuccinate.—The di-ester was obtained by the ethylation of ethyl tartrate by means of silver oxide and ethyl iodide. The procedure given by Purdie and Pitkeathly * was followed, save that the temperature of the reaction mixture was not allowed to exceed 30–40°. By this means the oxidising action of the silver oxide was almost entirely avoided, and the yield increased by about 20 per cent. to almost the theoretical value. The product, unlike that of Purdie and Pitkeathly, was readily purified by distillation, and boiled at 138–139° at 11 mm. Found, $C=54{\cdot}83$, $H=8{\cdot}51$; calculated for $C_{12}H_{22}O_6$ $C=54{\cdot}92$, $H=8{\cdot}46$ per cent. Optical rotation of ester, $[\alpha]_D^{18}=+93{\cdot}74°$. (P. & P. give $+93{\cdot}23°$.)

The potassium ethyl salt was obtained quite readily from the di-ester.

d-Diethoxysuccinamide.—Dissolves fairly readily in water, slightly in absolute alcohol, but is insoluble in ether. The crystals from rectified spirits are small, light, and feathery, melting when quickly heated at 241–242°, with decomposition. Found, $N=13{\cdot}59$; $C_8H_{16}O_4N_2$ requires $N=13{\cdot}72$ per cent.

Potassium Diethoxysuccinate.—The ester was saponified with alcoholic potash, and the salt crystallised out on cooling. It is very soluble in water, slightly in absolute alcohol, and insoluble in acetone, which precipitates it from alcoholic solutions. Found, $K=27{\cdot}62$; $C_8H_{12}O_6K_2$ requires $K=27{\cdot}66$ per cent.

Dissociation Constant of d-Diethoxysuccinic Acid.—The acid used had a m.p. of 126–128°, and a rotation of $[\alpha]_D^{20}=+66{\cdot}53°$. (P. & P. give 126–128°, and $+66{\cdot}51°$) K was found to have a value of 0·097 at 25° ($\mu\infty=368$).

It will thus be seen that d-diethoxysuccinic acid has a constant which is slightly less than that of tartaric acid (0·100 in the same units.)

The salt obtained by half-saponification of 20 grms. of di-ester was dissolved in 40 c.c. of water. It was found that a more concentrated solution could not be dealt with owing to frothing taking place. A total of 312 grms. of the di-ester was treated, yielding 80 grms. of an oily product.

Aqueous Residue.—Besides potassium carbonate and bicarbonate, this contained ethyl alcohol, formaldehyde, and formic and oxalic acids

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* *Jour. Chem. Soc.*, **75**, 159.

Oily Product.—The oil was nearly optically inactive, indicating that in all probability little or no synthesis had taken place. It was distilled under reduced pressure, and yielded two fractions boiling below 100° :—

1. Below 25° at 15 mm. (4 grms.).

2. 95–103° at 12 mm. (11 grms.). The remainder was distilled at a pressure of ½ mm. using a Gaede mercury pump.

3. 80–100° (13 grms.).

4. 125–155° (33 grms.).

The distillation was stopped at this point as some decomposition was taking place. A residue of weight 18 grms. remained in the flask.

1. This was found to contain ethyl alcohol, formic acid, ethyl formate, and a trace of methyl alcohol.

2. Gave a blue-violet coloration with ferric chloride. It was redistilled under ordinary pressure giving (*a*) a small quantity under 100°, identical with fraction 1; (*b*) about 1 grm. between 100° and 200°, which, after standing for a few months, deposited some crystals of m.p. 101°. They were found to be crystals of hydrated oxalic acid; the remaining liquid was ethyl oxalate; (*c*) a liquid boiling at 200–225.° It could not be further purified by distillation. A molecular weight determination in acetone gave a value of 179. Besides giving the ferric chloride reaction, it coloured Schiff's reagent, reducing Fehling's solution, gave a silver mirror with ammoniacal silver hydroxide, and reacted with phenylhydrazine. The substance was suspected to be ethyl formylethoxyacetate, EtOOC . CH(OEt) . CHO. Found, $C = 51{\cdot}79, 52{\cdot}09, H = 8{\cdot}26, 8{\cdot}32$. Calculated for $C_7H_{12}O_4$, $C = 52{\cdot}47$, $H = 7{\cdot}56$ per cent.

3. This fraction did not colour ferric chloride. On saponification it gave a solid acid, which proved to be oxalic acid. When treated with ammonia, oxamide was obtained. The liquid was ethyl oxalate.

4. A molecular weight determination gave a value of 259. The oil was somewhat viscous, and optically inactive. It gave all the reactions shown by fraction 2 (*c*) above, with the exception of the ferric chloride reaction. It was redistilled under reduced pressure, whereby it was divided into three equal portions:—(*a*) below 120°, (*b*) 120–170°, and (*c*) 170–175° at 12 mm. Each of these now coloured ferric chloride; (*a*) contained some ethyl oxalate; (*c*) was very viscous, and had a molecular weight of 262 in benzene.

In view of the fact that ethyl formylacetate and its derivatives are somewhat unstable compounds, and may exist in two or three isomeric modifications, it was decided at this stage to prepare a specimen of ethyl formylethoxyacetate, which has not hitherto been described, and examine its properties in order to facilitate its identification in the electrolytic oil. This was accordingly done by condensing ethyl formate with ethyl ethoxyacetate in presence of sodium. For description of the preparation and properties of this substance see below. It was found that the behaviour of fraction 4 (wide range of boiling point and abnormal molecular weight) was similar to that of ethyl formylethoxyacetate.

With phenylhydrazine (two mols. for one of ester), a solid derivative was obtained which could easily be recrystallised from chloroform, and then appeared as a canary yellow powder. Its m.p. was 229°. Found, $N = 18·15$; $C_{17}H_{20}O_2N_4$ requires $N = 17·93$ per cent.

There was still a large quantity of aldehyde in the portion of oil which could not be distilled. It was got rid of by shaking with an aqueous solution of sodium hydroxide in the cold. The insoluble oil was extracted with ether and the solution dried with calcium chloride. When heated with hydrobromic acid (constant boiling) to 100° in a sealed tube, it was completely decomposed.

Ethyl formylethoxyacetate.—The method of preparation of the analogous substance ethyl formylphenylacetate [*] was followed. Into a flask containing 15·5 grms. of sodium wire in 600 c.c. of dry ether, a mixture of 90 grms. of ethyl ethoxyacetate and 60 grms. of ethyl formate (a slight excess) was poured. In the course of an hour or two the sodium dissolved leaving a yellow, turbid liquid which deposited a slimy substance on standing. This was the sodium compound of ethyl formylethoxyacetate. The mixture was shaken out twice with water which dissolved the sodium compound, and the oil was liberated on acidification. It boiled at 95–110° at 15 mm.

The compound has an irritating odour when freshly distilled. It is rather viscous, and its solution in alcohol gives an intense blue-violet coloration with ferric chloride. The molecular weight, after standing for a week was 190 in alcohol (Beckmann's boiling-point method), the normal value being 160. After a month the figure had risen to 231 in alcohol and 270 in benzene, but there was no further rise on prolonged standing. On combustion, figures close to the calculated ones were obtained. Found, $C = 52·33$, $H = 7·66$; $C_7H_{12}O_4$ requires $C = 52·47$, $H = 7·56$ per cent.

It was not found possible to narrow the boiling-point range by fractional distillation.

Michael [†] states that ethyl formylpropionate, $EtOOC.CH(CH_3).CHO$, is unstable at ordinary temperatures.

The ester is very readily oxidised, dilute neutral permanganate attacking it at once with formation of oxalic acid. No sodium bisulphite or copper compounds could be obtained from the ester. When treated with two molecular proportions of phenylhydrazine and warmed, the ester yields a solid substance which can readily be recrystallised from chloroform, and is a canary-yellow powder, crystalline when seen under the microscope. It is somewhat soluble in alcohol. It is the phenylhydrazide of the phenylhydrazone of formylethoxyacetic acid, Ph.NH.NH.CO.CH(OEt).CH:N.NH.Ph. When heated suddenly, it melts at 228–229°. Found, $C=65{\cdot}16$, $H=6{\cdot}43$, $N=18{\cdot}09$; $C_{17}H_{20}O_2N_4$ requires $C=65{\cdot}33$, $H=6{\cdot}47$, $N=17{\cdot}93$ per cent.

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[*] Cf. *Ann.*, **291**, 164; and **391**, 248.

[†] *Ber.*, **38**, 44.

ELECTROLYSIS OF POTASSIUM ETHOXYACETATE, $CH_2(OEt).COOK$.

A saturated solution containing 30 grms. of the salt was electrolysed, a current of 3 amps. being passed for two hours. An oily layer was seen to separate soon after the commencement of the electrolysis. This was taken up in ether, the solution dried, and the ether removed. 2·5 grms. of an oily product remained which distilled for the most part at 90° and gave formaldehyde on treatment with dilute acids. It was formaldehyde diethylacetal, $CH_2(OEt)_2$. The aqueous residue contained much formaldehyde and ethyl alcohol.

ELECTROLYSIS OF SODIUM iso-AMYLOXYACETATE, $CH_2(OC_5H_{11}).COONa$.

Owing to the tendency of this substance to froth when the ordinary platinum crucible apparatus is used, the operation was carried out in a small beaker. A layer of mercury functioned as cathode, and a spiral of platinum wire wound round suitably shaped glass tubing was used as anode. A current of water was led through the tubing, and this, together with ice-water in which the beaker was immersed, provided very efficient cooling.

This apparatus was not merely convenient in that it allowed room for frothing; the tendency to froth was almost entirely eliminated. The liquid became acid to litmus shortly after commencing electrolysis, the sodium being retained by the mercury. It was found possible to pass a current of 3 to 4 amps., whereas with the platinum crucible even ¼ amp. produced a very persistent froth of large volume.

40 c.c. of saturated solution (containing about 16 grms. of solid) were electrolysed at a time, yielding 8 to 10 grms. of an oil. This was taken up in ether, and the solution dried with sodium sulphate. A total of 108 grms. of oil was collected in this way.

The aqueous residue had a strong odour of formaldehyde.

The oil was distilled and proved to be a very complex mixture. The main fractions were, however, identified. These were as follows:—

1. 87–100°. On distillation, condensation took place with liberation of water. The substance gave the reactions of an aldehyde, and iso-valeric aldehyde was suspected (b.p. 92°). The sodium bisulphite compound was readily prepared, and an estimation of the sodium content showed the substance to be that aldehyde. Found, Na = 12·3; $C_5H_{11}O_4NaS$ requires Na = 12·1 per cent.

2. 190–215°. It was expected that this fraction would contain the synthetic product if any had been formed. It was boiled with concentrated alcoholic potash to remove any esters present. The mixture was poured into water and extracted with ether. This treatment reduced the quantity only by a little, about 2 c.c. of a clear liquid being left. By repeated fractionation, a substance of b.p. 207–209° was isolated. It was not the diamyl ether of ethylene glycol. When hydrolysed with hydrochloric acid amyl alcohol was obtained, suggesting that it might be formaldehyde diamylacetal, $CH_2(OC_5H_{11})_2$, whose b.p. is 207°. A density determination was made, and a value of 0·837 found. The density of formaldehyde diamylacetal is 0·835. An analysis established the identity of the oil with this substance. Found, C = 69·55, 69·61, H = 12·91, 12·80; $C_{11}H_{24}O_2$ requires C = 70·13, H = 12·86 per cent.

3. 240–250°. This fraction constituted about two-thirds of the mixture. Alcoholic potash saponified it, giving amyl alcohol and an oily acid sparingly soluble in water. From its insoluble silver and copper salts, and its molecular weight by titration, this appeared to be amyloxyacetic acid. The amide was prepared from the original ester, and showed, after recrystallisation a m.p. of 58°. The substance was therefore the amyl ester of amyloxyacetic acid.

Iso-Amyl iso-Amyloxyacetate.—Amyl chloracetate was added in small portions at a time to a solution of sodium amyloxide in amyl alcohol. After standing overnight, the mixture was poured into water, neutralised with dilute acid, and the water separated off. The ester was freed from amyl alcohol and water by distillation. The yield was almost quantitative. B.P. 248°. Sp. gr. at 15/15° = 0·911. Found, C = 66·51, H = 11·06; $C_{12}H_{24}O_3$ requires C = 66·60, H = 11·19 per cent.

Iso-Amyloxyacetamide.—Amyl amyloxyacetate was treated with concentrated ammonia and alcohol, and allowed to stand for several days. The liquid was then concentrated and extracted with ether, the amide being left as a white solid on removal of the ether. It is readily soluble in most of the organic solvents and in water. The m.p. was 58° after three recrystallisations from petroleum ether. Found, $N=9{\cdot}65$; $C_7H_{15}O_2N$ requires $N=9{\cdot}65$ per cent.

SUMMARY.

Four compounds have been electrolysed and the nature of the resulting compounds investigated. These are :—

(*a*) Potassium ethyl ethoxymalonate, which gave ethyl diethoxyacetate, the alcoholate of ethyl glyoxylate, a little ethyl diethoxysuccinate, ethyl alcohol, oxalic acid, formaldehyde, and acetaldehyde.

(*b*) Potassium ethyl diethoxysuccinate, which gave ethyl formylethoxyacetate, oxalic acid, ethyl oxalate, ethyl alcohol, formaldehyde, and formic acid.

(*c*) Potassium ethoxyacetate, which gave formaldehyde, alcohol, and formaldehyde diethylacetal.

(*d*) Sodium amyloxyacetate, which gave amyl amyloxyacetate, formaldehyde diamylacetal, formaldehyde, and iso-valeric aldehyde.

The author wishes to acknowledge his indebtedness to Professor Sir James Walker, F.R.S., for his helpful suggestions and advice in the course of this research.

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(*Issued separately December* 24, 1924.)

V.—The Ionisation of Iodine Vapour by Ultra-Violet Light. By

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(MS. received October 7, 1924. Read December 8, 1924.)

THE ionisation of iodine vapour by light has been sought by several investigators with negative results.* Nevertheless, the application of the quantum theory, which has proved so fruitful with respect to the relation between electron impact on gaseous molecules and the emission of light, leads to the expectation that the ionisation of a gas should be affected by light of sufficiently short wave-length. This ionisation was first observed by Lenard, who investigated the effect of light in the extreme ultra-violet on air and other gases.† More recently ionisation of cæsium vapour by wave-lengths shorter than the limit of the principal series, but not by longer, has been observed.‡ The earlier workers on iodine vapour used quartz windows and the light passed through a considerable layer of air, so that the iodine could not have been illuminated by light of shorter wave-length than $185\mu\mu$; in view of the interest of the result from both the physical and chemical points of view, it was determined to look for the effect in the Schumann region, recent observations on the emission spectrum of iodine having suggested the probability that the search would be successful.§

EXPERIMENTAL.

As the source of light a powerful spark between aluminium terminals was used. Lenard has shown that the radiation from such a source is capable of ionising air and other gases, and Lyman's photograph of the aluminium spark in air shows groups of strong lines at about 176 and $130\mu\mu$ in the fluorite ultra-violet. The spark, contained in an earthed tin box to prevent possible electrical disturbance of the electrometer, was driven by a 10,000 volt transformer, taking a primary current of 10 amperes. In parallel with the spark gap was a condenser of about 0·003 microfarad capacity. The spark gap was placed almost in contact with the fluorite window, so that during the discharge, the spark played against the window which had frequently to be cleaned and eventually became so pitted that it lost much of its transparency. The first experiments were made in glass apparatus to which the window was cemented, but the magnitude of the photo-electric effect on the electrodes caused the method to be abandoned in favour of a streaming method.

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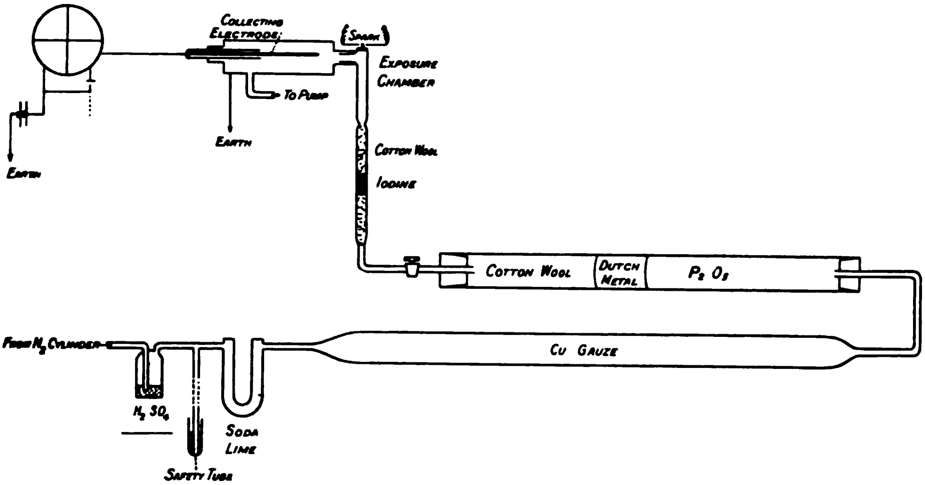
* Henry, *Proc. Camb. Phil. Soc.*, ix, 319 (1898); Whiddington, *ibid.*, xv, 189 (1910).

† Lenard and Ramsauer, *Sitz. d. Heidelberg, Akad. d. Wiss.* (1910–11); Hughes, *Proc. Camb. Phil. Soc.*, xv, 483 (1910).

‡ Williams and King, *Phys. Rev.*, 15, 550 (1920).

§ Ludlam and West, *Proc. Roy. Soc. Edin.*, xliv, 185 (May, 1924).

The exposure chamber consisted of a glass tube on the end of which was cemented a window of clear colourless fluorite, 1·3 mm. thick and 5 mm. in diameter. A short tube lead to the copper ionisation chamber, the outer cylinder of which was earthed. The collecting electrode of copper, insulated from the earthed cylinder by quartz, was connected to one pair of the quadrants of a Dolezalek electrometer, the connecting wire being enclosed in an earthed case. A side tube from the ionisation chamber lead to a rotatory oil pump. The electrometer, enclosed in an earthed iron box, was arranged so that 1 volt gave a deflection of 200 mm. The two sets of quadrants could be joined by means of a key, worked by a string of unspun silk.



Solid iodine was placed in the exposure tube, and a stream of pure nitrogen at a pressure of a few millimetres passed over in order to carry a current of iodine vapour past the window. The solid iodine was protected from the action of the light by a plug of cotton wool which also served to retain any solid particles.

The nitrogen was obtained from a cylinder of compressed gas: it was purified by passing through sulphuric acid, over soda lime and over a long length of copper gauze which was heated in a stream of hydrogen before each series of experiments. It was finally dried by passing over phosphorus pentoxide and passed through a layer of tightly packed cotton wool to remove dust. A safety tube was attached, and between this and the chamber Dutch metal was interposed to catch any traces of mercury vapour. Before making any observations the nitrogen was passed through the apparatus for a considerable time to remove traces of volatile impurities.

Nitrogen was chosen because it was unlikely to be ionised by the light used. The ionisation of air is due to the oxygen which possesses an absorption band between $180\mu\mu$ and $120\mu\mu$. Nitrogen scarcely absorbs at all in this region.

Experiments with Pure Nitrogen.

Experiments were first carried out to see whether ionisation were produced in pure nitrogen. A potential was applied to the quadrants and the collecting electrode and the quadrants were then separated. Perfect insulation was not attained and the needle slowly crept across, the time being noted.

A large number of experiments were made on the rate of leak with and without exposure to the light, using pressures of nitrogen varying from 1 mm. to about 1 cm.; the following are examples of such blank experiments:—

Table I.—Experiments with Nitrogen.

Potential on Electrode − 3 Volts.			Potential on Electrode + 3 Volts.		
Time.	+ Leak without Exposure.	+ Leak with Exposure.	Time.	− Leak without Exposure.	− Leak with Exposure.
	mm.	mm.		mm.	mm.
1			1	14	
2	8		2	12	
3		7	3	12	
4	6		4		10
5		6	5	10	
6	6		6		11
7		7	7	10	
8	6		8		8
9		6	9	10	
10	6		10		7
11		6	11	8	
12	6		12		10
13		6	13	6	
14	5		14		10
15		5	15	6	
16	5		16		5
			17	8	

There is thus seen to be no greater positive leak for illuminated than for non-illuminated nitrogen; there is also no marked surface effect due to photo-electric effects on the electrodes, since the negative leak is scarcely greater with exposure than without.

EXPERIMENTS WITH IODINE.

Iodine was now placed in the tube and the experiments repeated, all other arrangements being exactly as before. Table II. shows the kind of result obtained for exposures of 5 seconds.

TABLE II.—NITROGEN AND IODINE.

Potential on Collecting Electrode −4 Volts.			Potential on Collecting Electrode +4 Volts.		
Time.	+ Leak without Exposure.	+ Leak with Exposure.	Time.	− Leak without Exposure.	− Leak with Exposure.
	mm.	mm.		mm.	mm.
2	14		2	12	
3		32	3		33
4	16		4	13	
5		26	5		24
6	11		6	13	
7		27	7		26
8	12		8	10	
9		25	9		22
10	8		10	10	
11		25	11		26
12	8		12	9	

These experiments, which are only representative of a large number with similar results, show that there is a decided increase in the positive and negative leaks on exposure. For any set of observations the difference between the leak in a minute containing a five-seconds exposure and the leak per minute without illumination is approximately constant; the increased current on exposure is of the order 10^{-12} ampere. In most of the experiments the negative leak is slightly greater than the positive, but evidently there is little surface ionisation.

Some experiments were carried out with the pump out of action. For positive ions there was no greater leak in this case when illuminated, as would occur if recombination of the ions took place before they reached the electric field. For negative ions there appeared to be a slight but real increase in the leak on exposure, even with the pump not working, though the increase was much smaller than when the pump was in action. The formation of negative ions might here be due to slight ionisation caused by the action of the light on the surface; but there is also the possibility of the emission of radiation in the process of recombination between an electron and a positive iodine ion, which would cause a photo-electric effect at the electrodes.

After a number of experiments had been performed a group of results were obtained which apparently contradicted the earlier ones. No increased conductivity was found on exposure even with the pump running and the spark as near as possible to the window. The fluorite was perfectly clean and transparent, but exposures of ten or even twenty seconds failed to increase the conductivity.

It was, however, observed that the ends of the aluminium electrodes were covered with oxide; under these conditions the discharge changes from a spark into an arc, and the aluminium lines in the spectrum are weakened. The light appears to be chiefly temperature radiation from the oxide and is feeble in light of the extreme ultra-violet. This is additional evidence that the effect is due to volume ionisation by the short wave-length ultra-violet radiation and not to any surface effect or electrical disturbance, for copper would easily give photo-electric effects with the arc discharge. On cleaning the terminals the original effect was again strongly produced.

In order to determine whether the effective radiation was strongly absorbed by oxygen, an experiment was performed in which the spark was moved back about a centimetre from the window. The leak was no greater with illumination than without, as is shown by Table III.

TABLE III.—SPARK-GAP 1 CM. FROM WINDOW.

Potential on Collecting Electrode – 4 Volts.					
Time.	+ Leak without Exposure.	+ Leak with Exposure.	Time.	+ Leak without Exposure.	+ Leak with Exposure.
	mm.	mm.		mm.	mm.
1	14		9	14	
2		14	10		12
3	16		11	13	
4		8	12		12
5	20		13	12	
6		16	14		11
7	14		15	11	
8		11			

Returning the spark-gap to its original position, the effect was strongly reproduced.

Now oxygen at atmospheric pressure begins to absorb strongly at about

185$\mu\mu$, hence the effective radiation must be of shorter wave-length than this.

Finally, when the series of experiments with iodine was concluded, the apparatus was cleaned and the blank experiments with nitrogen repeated, with the same results as before.

SUMMARY OF RESULTS.

1. Nitrogen streaming through the apparatus at pressures from 1 mm. to 10 mm. shows no conductivity.

2. When the nitrogen contains iodine vapour at a pressure 0·13 mm. (temperature 15° C.), exposure to the light caused a current of the order 10^{-12} ampere.

3. Both positive and negative charges are given to the electrometer in approximately equal amounts.

4. The effect is not obtained if the vapour is not present.

5. The effect ceases when a film is formed on the fluorite and when the light source is removed 1 cm. away from the clean fluorite.

6. The aluminium sparking terminals need to be kept clean.

DISCUSSION OF RESULTS.

The experimental results indicate that iodine vapour is ionised by light in the extreme ultra-violet, radiation of longer wave-length than 185$\mu\mu$ being inactive. As fluorite itself begins to absorb in the neighbourhood of 120$\mu\mu$, it is probable that the intensity of any radiation of shorter wave-length than this would be too feeble to produce any appreciable effect in these experiments.

Some work has been done on the critical potentials of iodine vapour from which it can be calculated what the frequency of the radiation must be if it is to possess the necessary form of energy to produce ionisation. Smyth and Compton * have found a critical potential at 9·4 volts, which they ascribe to ionisation of the molecule. Mohler and Foote † give the value 10·1 volts; the corresponding wave-lengths are 123 and 131$\mu\mu$. A weak ionisation at 8 volts, corresponding to 154$\mu\mu$, observed by Smyth and Compton, is ascribed to the atom. In his "Spectroscopy of the extreme Ultra-violet," Lyman reproduces photographs of the aluminium spark in air which show strong lines between 170 and 180$\mu\mu$, weaker lines at 160 and 150, and a group of strong lines at 130$\mu\mu$.

As our experiments were carried out at ordinary temperatures there would be very few atoms primarily present, but the possibility of the

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* *Phys. Rev.*, 16, 501 (1920).

† *Ibid.*, 15, 321 (1920).

dissociation of molecules by the radiation must not be overlooked. Keeping, however, to known facts, it is seen that normal iodine molecules would just be ionised by the radiation at 130$\mu\mu$.

It is, however, probable that longer wave-lengths than this will produce ionisation of iodine molecules. Iodine at low pressures fluoresces under the influence of visible light: the molecule absorbs the visible radiation and assumes a state of greater energy than the normal: the return to the normal state liberates the excess energy in the form of fluorescent light. Such an activated molecule will require less energy to raise it to the state of ionisation than will the normal atom. This has already been proved to be the case [*] in the case of electron impact, the difference between the ionisation potentials of the normal and the fluorescing molecules corresponding to the wave-length of the light necessary to produce fluorescence. It has also been observed that for the same potential difference between the electrodes of a discharge tube containing iodine vapour, the glow discharge current is greater for the fluorescing than for the normal vapour.[†] In the same way one might expect that ionisation of the iodine molecule could take place in two stages when subjected to the action of light containing a wide range of wave-lengths: first the activation to the state of fluorescence and then the ionisation of the fluorescing molecule, the energy required in the second quantum being less than that which would be necessary to ionise the normal molecule by the quantum necessary to cause fluorescence.

The wave-length corresponding to 6·8 volts, the ionisation potential of fluorescing iodine, is 181 $\mu\mu$, which is well within the range of the radiation available under the conditions of our experiments, but it would have been absorbed before reaching the iodine in the arrangement used by previous investigators.

We conclude, therefore, that the conductivity observed is due to the ionisation of the molecules by light of wave-length shorter than 180 $\mu\mu$, and as in our experiments many of the molecules might be expected to be in the state of fluorescence it is probable that these contributed the greater share.

We wish to express our thanks to Professor Sir James Walker for his constant interest and support, and to Messrs Brunner Mond & Co. for a grant enabling us to obtain the necessary apparatus.

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[*] Smyth and Compton, *loc. cit.*

[†] Franck and Westphal., *Verh. d. Deutsch. phys. Ges.*, **14**, 159 (1912).

(*Issued separately April* 10, 1925.)

ADDENDUM: February 9, 1925.

Note on the Ultra-Violet Emission Spectra of the Halogens.

At the time of contributing our paper describing the ultra-violet emission spectra of the halogens produced by the Tesla discharge in silica tubes using external electrodes, we were unaware of the results obtained by Strutt and Fowler (*Proc. Roy. Soc. London,* A, lxxxvi, 108, 1911) on the spectrum of the radiation emitted when the halogens react with active nitrogen.

In the case of iodine the band which we had selected as that of the affinity spectrum in our photographs is the most prominent one in theirs. They also found a strong line at 207·5 $\mu\mu$, which is almost certainly the same as the one to which we ascribed the wave-length 206·1 $\mu\mu$ by reference to the zinc line 206·2 $\mu\mu$.

In the case of chlorine and bromine the only bands which they found are those which we attributed to the electron affinity, *i.e.* to the receipt of an electron by a chlorine or bromine atom.

The fact that the same radiation can be excited by two methods so different may lead to some criterion as to the nature of the change accompanying the emission, though experimental work with this object in view will be necessary before conclusions can be drawn.

E. B. L., W. W.

VI.—Experiments and Observations on Crustacea. Part VI: The Mechanism of Massive Movement of the Operculum of Balanus nubilis. By **John Tait**, M.D., D.Sc., and **W. F. Emmons**, M.D., M.Sc. (From the Biological Station, Nanaimo, Vancouver Island, and the Department of Physiology, M'Gill University, Montreal.)

(MS. received August 16, 1924. Read November 24, 1924.)

Balanus nubilis, one of the sessile Cirripedes, grows to a considerable size, a large specimen measuring some 10 to 11 cm. across the base and an equal distance in height. During a visit to Nanaimo Biological Station an opportunity was taken to study in this giant form a few features pertaining to the mechanism of movement of the operculum, which is attached within the cup of the shell by the opercular membrane. Peripherally the membrane is attached close to the downwardly directed sharp edge of an overhanging shelf formed by the inner wall of the shell; centrally it is attached to the free edge of the operculum on its basal aspect. The peripheral attachment, shaped like a circular ring, lies practically in one (horizontal) plane. The line of the central attachment, if modelled in wire, would also form a closed ring; the ring, however, would not lie flat, because the edges of the membrane dip down on each side to follow the line of the downwardly projecting spur on each tergum.

By action of the muscles which connect the operculum to the shell the animal can roll its closed operculum within the cavity of the shell somewhat like the ball of a ball-and-socket joint. These rolling movements can readily be excited by mechanical stimulation of the membrane itself, which, unlike the calcareous shell plates, is provided with abundant sensory organs. If a knife-blade or other thin object is inserted between the shell and the operculum, and lightly drawn over the surface of the membrane on one or other side, the operculum executes a rotatory movement towards the irritated side. Suppose the membrane be stimulated on the right side, the operculum is swung slowly but powerfully backwards and towards that side, meantime rasping the shell face on the same side; it usually returns to its position of rest by a completion of the rotatory movement, but without rasping the opposite side of the shell, the time of the whole swing lasting, say, five or six seconds. Irritation of the left side causes a corresponding rotation in an opposite direction. The effect of these movements is to grind or crush the intruding object between the shell and the tergal plates, which are seen to be polished by frequent repetition of this rubbing action. Some idea of the crushing force exerted in the movement may be gathered from the grinding, grating noise when a small pebble is introduced so as to rest on the membrane, while a finger incautiously inserted into the crevice may be severely crushed. The reaction is undoubtedly a defensive one, serving to protect the membrane against settlement or attack by unwelcome intruders. Incidentally it can be observed that there is an accurate correlation between the site of irritation and the point of application of the greatest crushing force. If a pebble is dropped upon the membrane, the animal makes repeated rotatory movements, the effect of which is ultimately to displace the pebble to the upper surface of the scuta, which are never rubbed against the shell and on which various sessile animals may find a lodging.

One would naturally expect that a heavily constructed and forcefully manipulated organ like the operculum would articulate with the shell by some form of joint involving the contact of solid bearing surfaces, as, for example, in the coxo-basal joint of an isopod, at which a rotatory movement is also possible.* Nevertheless the muscle-moved operculum rests wholly upon soft yielding material, all its powerful movements being effected without the intervention of any rigid hinge. The animal contains an abundant quantity of body-fluid, and the operculum surrounded by its membrane rests in reality upon a cushion of fluid, which is subjected to internal pressure during voluntary or reflex movement.

In seeking to draw blood from the animal by puncture of the opercular membrane it was noted that the body-fluid would fill with unusual rapidity the capillary-pointed glass tube introduced for this purpose; when the tube was withdrawn the body-fluid was frequently ejected from the puncture to a height of 3 or 4 feet. This finding directed attention to the problem of internal pressure, and suggested the use of a mercury manometer for measurement of the pressure. The projecting beak (or tergal spines), which usually rests in contact with the carina, having been pulled rostrumwards and there restrained by a powerful grip under the thumb, a capillary-pointed glass tube, armed with a cork shoulder to prevent leakage around the point of insertion, was pushed through the opercular membrane in the region between the anterior pair of depressor muscles, the opposite end of the tube (which had previously been filled with sea-water) being connected with a mercury manometer. The liquid rose to varying heights in the case of different animals, a common reading being between 200 and 350 mm. of mercury, while 430 mm. was the highest record obtained. As materials were not at the moment available for the construction of a manometer of narrow bore, there was an inevitable escape of considerable fluid from the body in taking these records; this, however, was partially circumvented by using a clip and raising pressure in the manometer before insertion of the tube. Under more favourable conditions of experiment and with freshly caught specimens (we worked with a single batch, the fruit of one day's collecting, and the animals seemed to deteriorate with confinement in the sea-water near the laboratory) it is probable that still higher records might be obtained.

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* Tait, *Proc. Roy. Soc. Edin.*, vol. xxxvii (1918), pp. 252–258.

It would be a mistake to suppose that the body-fluid of the barnacle is constantly under a pressure of such magnitude. The pressure is absent during the condition of rest, and is developed only when the animal seeks to move its operculum. Nor is the pressure maintained at any one constant level during actual or attempted movement. Each successive effort on the part of the animal to withdraw its operculum from the position of restraint in which it is held by the operator, who has on occasion to exert all his finger-strength in opposition, is accompanied by a marked rise of internal pressure, so that the mercury column shows great swings, at one time climbing high and then again subsiding towards zero. In other words, the manometer pressure is a mere index of the force of muscular contraction temporarily exerted by the animal. By mechanical irritation of the opercular membrane the animal may be made to exert itself again so soon as one bout of activity has ceased, and our maximum records were invariably obtained by applying strong sensory stimulation to the membrane. Even without any manometer and by simple palpation of the membrane, as in estimating intraocular pressure in man, one can readily follow these changes of internal pressure, the membrane bulging and becoming resistant when the animal exerts itself, and becoming flaccid when it relaxes its efforts.

Within the interior of *Balanus* there are no rigid walls or inextensible membranes to confine or to localise the application of this fluid pressure, which must be uniformly exerted on all tissues situated beneath the operculum and its membrane. The movable or appendage-carrying body lies in a special sac, the so-called mantle-cavity, which is in free communication with the surrounding sea-water, being filled and partially emptied during ordinary respiratory movements. In the animal at respiratory rest, *i.e.* with the two opercular plates closed, the mantle-cavity still contains a certain residual quantity of sea-water. When, however, the internal body pressure becomes conspicuously raised, the membranous valves guarding the respiratory opening are forced and the contained sea-water is expelled. Thus, under conditions of strong muscular effort, the appendage-carrying body is pressed hard against the deep surface of the operculum.

Let us now return to consider the mechanism of movement and the hinging of the operculum as a whole.

Attached to the operculum on each side are three powerful muscles, the so-called depressors. Each takes origin from a wide area, chiefly on the base of the shell, and converges towards an insertion along the basal edge of the operculum, one going to the tergum and two to the scutum. They are of the parallel-fibred variety, and each is composed of a large number (say 100) of separate fasciculi, 1 to 2 mm. in diameter, arranged side by side like a bundle of twigs, but not bound one to another by connective tissue. In the living condition (the muscles are endowed with marked viability) each is separately stimulable. Thus a single main muscle is in reality a great aggregate of anatomically separate smaller muscles, and the body-fluid freely penetrates between adjacent fasciculi. As the animal has no heart and no circulatory system proper, the occasional contraction of these muscles is no doubt of service in causing an intermingling of the body-fluid and incidentally in determining currents between the fasciculi themselves. After death and immersion of the animal in preservative, these depressor muscles shorten markedly and assume a conical shape not unlike that of the barnacle itself, the apex of the cone being at the point of insertion into the operculum. It will thus be noted that in a normally shaped animal growing from a broad base of attachment the muscular fasciculi converge like the poles of an Indian wigwam from a wide area on the base of the shell to their narrow line of insertion on each opercular plate.

It is, of course, by the properly co-ordinated contraction of these fasciculi of the depressor muscles that the rotatory movements of the operculum are carried out. During life the operculum is situated at a higher level within the conical cavity of the shell, and fits its rigid surroundings more closely than one would infer from examination of a preserved and partially dehydrated specimen, in which the opercular membrane comes conspicuously into view by downward displacement of the operculum. When in the living animal the operculum was artificially lowered by withdrawal of body-fluid, we found that fluid was again absorbed from the sea-water so as, within the space of some few hours, to bring the operculum to its previous level; thus a careful balance is maintained between the adjustment of the opercular articulation and the amount of body-fluid in the animal.

The result of this arrangement is that when the depressors of one side (say the right) contract, lowering the right and elevating the left side of the operculum, the operculum still fits its conical compartment pretty closely. Under such circumstances the opercular membrane on the right side is withdrawn from view, while that on the left bulges conspicuously upwards. At the same time as the operculum is laterally tilted, it undergoes a certain amount of rotation about a vertical axis, by which means the long (or rostrocarinal) axis of the almost circular ellipse formed by its basal periphery is thrown out of alignment with the long axis of the similarly shaped ellipse of the shell-crater. We failed to convince ourselves that there is actual abutment of the left basal edge of the operculum against the shell when the animal rotates to the right (in any case such abutment cannot be by direct contact, for the opercular membrane intervenes), and we finally concluded that the strong lateral pressure of the (right) tergum against the (right) wall of the shell is due to two factors: (1) a horizontal component in the pull of the more lateral, obliquely directed fibres of the depressor muscles, aided and steadied by (2) hydrostatic pressure within the bulging fold of opercular membrane on the opposite side.

As already noted, the depressor muscles in a well-formed specimen have a conical form, the various fasciculi converging from a broad base below to a narrow insertion above. In one specimen at our disposal the basal part of the shell, long and slender, was surmounted by a cup of greater diameter than the base. Now, in this animal the operculum could be readily rocked from side to side with the fingers, all its movements being feebler than those of animals more normally shaped—which finding is in conformity with the supposition that the more obliquely directed muscular fasciculi, present in a normally shaped specimen, are of service in exerting lateral pressure. Similarly, the oblique fibres bordering the gaps between the depressor muscles are those which cause twisting of the operculum about a vertical axis during rotatory movements.

The bulging of the opercular membrane on that side of the operculum which happens at the moment to be elevated is no doubt associated with relaxation or inhibition of the depressor muscles on that side, for fine tendinous bands like the chordæ tendineæ of the auriculo-ventricular valves of the mammalian heart pass from the more lateral muscle fasciculi to the membrane itself (these are especially conspicuous in the case of the pair of tergal depressor muscles). With contraction of the muscles on one side of the body the opercular membrane on that side is simultaneously pulled down, whereas that on the opposite side yields to the internal hydrostatic pressure. In this way an elastic cushion of fluid is inserted between the base of the operculum on its temporarily elevated side and the wall of the shell, thus tending to drive the operculum against the opposite side.

The mechanism by which *Balanus nubilis* is able, without any direct contact of rigid parts, to roll its operculum forcibly within its shell is thus seen to be relatively simple. It involves, however, a high degree of nervous co-ordination among the various muscular elements involved, and the production in certain circumstances of a very high hydrostatic pressure.

(*Issued separately April* 10, 1925.)

VII.—An Investigation of the Absorption of Superposed X-radiations. By Wm. H. Watson, M.A. *Communicated* by Professor C. G. Barkla, F.R.S.

(MS. received February 4, 1925. Read March 9, 1925.)

The first experiments on this subject were suggested by the absorption effects associated with the "J" phenomenon. In a comparison of the absorbability in aluminium and in copper of the radiation emitted in one direction from an X-ray tube, it is found * that in many cases, as the tube is hardened and a certain value of the mass-absorption coefficient reached, there is a *sudden* increase in the absorption by aluminium. Since the radiation is not strictly homogeneous, and since the above effect does not invariably take place, it is evident that the phenomenon is not to be explained simply in terms of a "J" series characteristic radiation similar to K and L characteristic radiations as regards the manner of its excitation. It is evident that certain critical conditions must obtain before the phenomenon occurs, and on account of the abruptness of the change it appears as though the whole wave-length range of the radiation were affected in respect of absorption by aluminium. This seems to be further substantiated by the fact that the discontinuous character of the change in absorption is preserved with a beam which is much more heterogeneous. It is possible, therefore, that the attainment of the critical conditions results in a change in the absorbing properties of the aluminium. This change seemed, when the present experiments were initiated, to be adequately described by the statement that the aluminium, when exposed to radiation beyond a certain frequency, sometimes behaved as the element of next higher atomic number.

If the increase in absorption is due to some temporary transformation of the aluminium, we should expect an increase in the absorption of *another* beam of X-rays of different quality when passing through the transformed aluminium at the same time. We shall refer to a beam in which the critical conditions for such transformation are attained: *i.e.* one in which the additional absorption occurs, as an "effective beam," and one in which they are not attained as "ineffective." The assumption of transformation then leads us to expect that the incidence of an "effective" beam on aluminium would increase the absorption, by that aluminium, of a second beam which by itself is "ineffective." The object of the first series of experiments was to find if this increase in absorption does take place.

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* Barkla, *Bakerian Lecture*, 1916; Barkla and White, *Phil. Mag.*, **34**, 1917.

[Provided no similar J absorption effect is produced in copper, and that the assumption of transformation is valid, the "ineffective" type of beam is such that its representative point in a graph of $\left(\frac{\mu}{\rho}\right)$ Al against $\left(\frac{\mu}{\rho}\right)$ Cu lies on the lower line of Barkla and White *; when both aluminium and copper show the change, the corresponding point lies approximately on the same line; the beam "effective" in Al only is represented by a point on the upper line.]

If an intermittent source of high potential be employed to generate X-rays in such an experiment as that suggested, it is necessary that the two superposed beams should be produced by the same source of high potential, otherwise actual superposition of the beams will not take place. Consideration of the complexity of apparatus which would be introduced by the use of two X-ray tubes led to the adoption of one tube to produce two beams for investigation, differences in character being brought about by the use of radiators and filters.

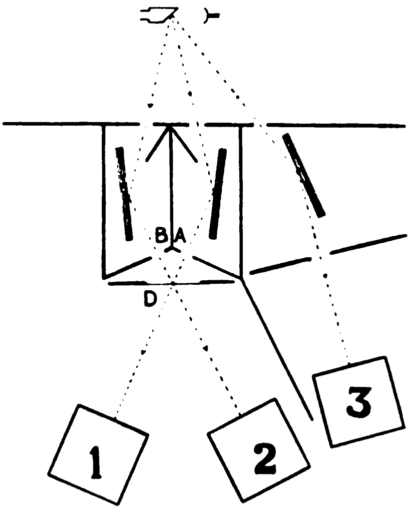

Fig. 1.

As the effect to be looked for was expected to be of the order of a 13 per cent. change in absorption coefficient, it was decided that a definite answer could be obtained if X-ray intensities were measured by ordinary gold-leaf electroscopes with air ionisation. The apparatus was arranged as in fig. 1, in which the heavy lines represent lead screens.

Through each of the circular apertures A, B (of 4 cm. diameter) there passed a beam of X-rays produced by the scattering of a primary beam from an ordinary gas tube (Pt anticathode) operated by induction coil with mercury-gas break. The scattering in each case took place from a slab of aluminium 15 cm. square and 1·8 cm. thick. After having traversed filters specified below, these secondary beams proceeded to the electroscopes 1 and 2.

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* Barkla and White, *Phil. Mag.*, **34**, 1917.

A third scattered beam, produced in a similar fashion, entered electroscope 3. The ionisation current in this electroscope was used as a standard to which the ionisation currents in 1 and 2 were related. Lead screening was introduced where necessary to eliminate stray effects. The position (D) occupied by the aluminium sheets absorbing the rays is indicated in the diagram; the plane of the aluminium was equally inclined to the two beams.

The procedure was as follows:—The deflection (α) of the gold-leaf in electroscope 1 relative to 3 was determined with aperture B closed and A open; then with B and A open (β); finally, A was closed and the effect produced in 1 by the radiation of beam B scattered by the absorbing aluminium into 1 was measured (γ). A series of similar measurements was carried out for the beam B. By increasing the distances of the electroscopes from the absorbing aluminium, the effect γ was reduced to about 3 per cent. of α.

If the ordinary laws of X-ray absorption hold, the total absorption of a beam is equal to the sum of the absorptions which the components of the beam experience separately, and the absorption of such a beam is unaffected by other beams transmitted at the same time, *i.e.* $\alpha = \beta - \gamma$. This was found, within the limits of experimental error, to be the case for two beams from the same tube filtered with different amounts of aluminium before the apertures A and B, when the respective mass-absorption coefficients in aluminium and copper, determined by 50 per cent. absorption, gave values lying on the lower line of Barkla and White (*i.e.* critical conditions were not attained for either beam).

The results in Table I are those in three typical experiments.

TABLE I.

Beam.		(1)		(2) $\left(\frac{\mu}{\rho}\right)$ Al.	
				Cross Beam	
		$\left(\frac{\mu}{\rho}\right)$ Cu.	$\left(\frac{\mu}{\rho}\right)$ Al.	Off.	On.
1	A	7·1	·77	·78	·81
	B	9·0	1·04	1·02	1·05
2	A	6·3	·71	·75	·74
	B	8·5	·85	·82	·80
3	A	6·3	·64	·61	·63
	B	7·3	73	·74	·72

First $\left(\frac{\mu}{\rho}\right)$ Al and $\left(\frac{\mu}{\rho}\right)$ Cu were determined for the beams A and B, with the absorbing sheets placed normal to the path of the rays (col. 1). An amount of aluminium sufficient to cut down the intensity of A by half was then placed at D in position to transmit both beams (fig. 1). From measurements of α and $\beta-\gamma$, and a knowledge of the angles between the aluminium at D and beams A and B, $\left(\frac{\mu}{\rho}\right)$ Al in the cross-beam experiments was calculated. A few typical results are quoted in (2), Table I. The problem and ultimate difficulty of the experiments lay in the controlling of the beams A and B, so that while in one the critical "J" conditions had been obtained, they were not attained in the other.

It was known that the "J" conditions may be produced by transmission (see Barkla, *Nature*, November 22, 1924), and an attempt to achieve the required control of the beams was made by filtering B first with tin and in later experiments with barium carbonate, the elements Sn and Ba having absorption edges in the regions where the "J" discontinuities most frequently appear. These attempts proved unsuccessful, and on the one occasion when success appeared to have been attained the beam known at first to be "ineffective" was subsequently found to be "effective," having been rendered so, not by transmission, but in its genesis: transition to the effective type must have taken place just after the determination of the character of this beam. Thus were observed the two values of the absorption coefficient in Al corresponding to one value of the absorption coefficient in copper. In one the J absorption did not occur; in the other it did. The relative values of $\left(\frac{\mu}{\rho}\right)$ Al and $\left(\frac{\mu}{\rho}\right)$ Cu (Tables I and II) obtained here are not in complete agreement with those of Barkla and White, as it was not the intention in either set of experiments to obtain absolute values of a high degree of accuracy, but to examine the abrupt change; the fractional separation of the two groups of values of $\left(\frac{\mu}{\rho}\right)$ Al is, however, almost exactly the same as in the results of Barkla and White. Measurements were made in the case where both A and B were beams belonging to the upper line of Barkla and White, and no effect of superposition was observed; typical results are incorporated in Table II.

It will be seen that these experiments failed to answer the question which led to their being undertaken, because of the difficulty (as yet unsurmounted) of controlling critical conditions which are at present unknown. This lack of success led to the abandonment of further

TABLE II.

Beam.	(1)		(2) $\left(\frac{\mu}{\rho}\right)$ Al.	
	$\left(\frac{\mu}{\rho}\right)$ Cu.	$\left(\frac{\mu}{\rho}\right)$ Al.	Cross Beam	
			Off.	On.
1 A	5·71	·73	·75	·75
1 B	8·94	1·09	1·03	1·03
2 A	6·5	·78	·79	·81
2 B	6·5	·78	·78	·81
3 A	4·43	·53	·51	·52
3 B	* 6·31	·64	·74	·74

investigation until the publication of C. T. R. Wilson's "Cloud Photographs." †

These photographs show that when X-rays of wave-length less than 0·5 A.U. pass through air,‡ there are produced paired β-ray tracks having their origins very close together and in some cases at the same point. The pairs may be composed of members of the same class or one of each of the two classes of β-rays which such X-rays are capable of ejecting and which differ greatly in range (the difference in kinetic energy generally corresponds to more than 20,000 volts). Further, such paired tracks have *their origin practically without exception within the primary beam of X-rays.*

Wilson suggests that these phenomena may be explained on the assumption that an atom of nitrogen from which a K-electron, in the absence of other influences, cannot be ejected by a quantum of nitrogen K-radiation, may, when exposed to nitrogen K-radiation together with radiation of higher frequency, be able to absorb both radiations (with the ejection of a K-electron) much more readily than either separately. Wilson also suggests "that many features of the β-ray tracks, including the curvature which sometimes appears, may be due to radiations which are excited in atoms by the passage of the β-particle, the radiations continually overtaking the β-particle and affecting the nature of its collisions with electrons in atoms subsequently traversed" (p. 208, *loc. cit.*).

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* Measurement of $\left(\frac{\mu}{\rho}\right)$ Al and $\left(\frac{\mu}{\rho}\right)$ Cu after the performance of the cross-beam experiment gave ·72 and 6·2 respectively ; this is the case referred to above.

† C. T. R. Wilson, *Proc. Roy. Soc.* (A), vol. civ, pp. 1–23 and 190–212.

‡ It is in just this part of the spectrum that the additional J ionisation in air has been observed (Barkla, *loc. cit.*). Evidently Barkla and Wilson observed by different methods the same phenomenon.

In the light of these observations it appeared to us to be probable that a solid element on which a sufficiently hard primary X-ray beam could be directed would absorb a second beam of X-rays characteristic of the element in question to a greater extent when the primary beam was incident than when it was not. It seemed possible, for example, that silver K-radiation falling on silver might be absorbed, when the primary cross-beam was in action, to about the same extent as Sb K-radiation. It was realised, of course, that in two important particulars the conditions holding in Wilson's experiments would not be reproduced in our experiments, where (i) the absorption took place in a solid instead of in a gas, and (ii) the primary radiation interacting with the characteristic radiation did not belong to that identical beam which had excited the characteristic radiation, as the two primary beams were emitted in different directions from the tube.

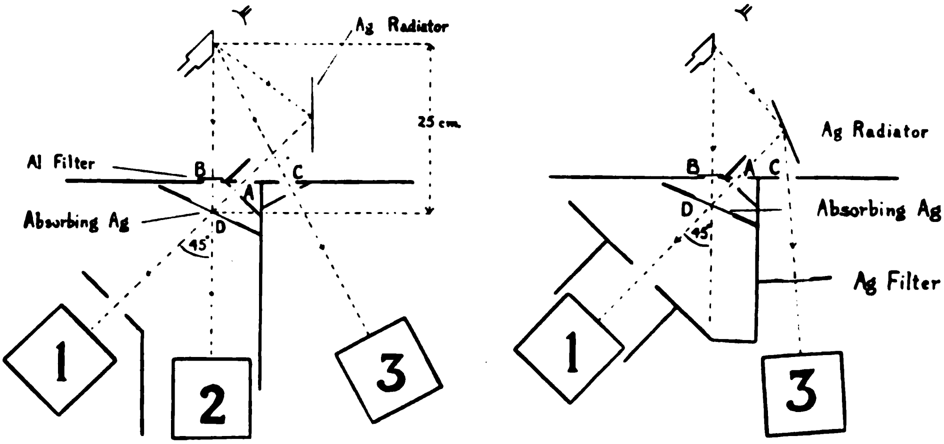
The procedure in this second series of experiments was the same as in the earlier, except that in the later experiments, since it was preferable to have the primary beam relatively intense, it was not possible to cut down the undesirable secondary (scattered and characteristic) radiation produced by the incidence of the primary beam on the absorbing matter. This decreased considerably the accuracy attainable with the apparatus employed, but, in view of the object of the experiments and the magnitude of the effect looked for, the loss of accuracy was not regarded, *a priori*, as serious.

In these experiments it is necessary to consider the selection of a proper standard relative to which measurement of X-ray intensities may be made. Since the electroscope (1 in fig. 2) which receives the approximately homogeneous Ag K-radiation receives also, when the primary beam is superposed on the absorbing silver, a heterogeneous secondary radiation, excited by this primary radiation, variation in the primary radiation from the tube may produce illusory effects. At first a heterogeneous primary beam filtered with aluminium was used to produce the standard ionisation: in some of the subsequent experiments a homogeneous Ag K-beam, filtered with aluminium, and in the remaining experiments filtered with silver, was used as standard (fig. 2, *b*).

In order to reduce to a minimum the amount of scattered radiation present with the Ag K-beam under examination, advantage was taken of the partial polarisation of the primary radiation incident on the radiator, by placing the axis of the X-ray tube parallel to AD (fig. 2, *a* and *b*).

A silver filter (·002 cm.) was placed in the aperture A. The intensity of the primary cross-beam was controlled by an aluminium filter placed before the aperture B. The beams A and B crossed in silver placed at D (25 cm. from the anticathode of the tube), and equally inclined to the two beams which made 45° with each other. The beams A and B entered electroscopes 1 and 2 respectively. Stray radiations were carefully tested for and eliminated.

In making the examination of the absorption of the Ag K-beam in silver, electroscope 1 was employed to measure relative X-ray intensities.



(*a*) Heterogeneous Primary Beam as Standard. (*b*) Ag K-Beam as Standard.

Fig. 2.

Let α = deflection in 1 relative to that in 3 with A open, B closed.
β = „ „ „ A open, B open.
γ = „ „ „ A closed B open.

$\beta-\gamma$ was compared with α, the result being expressed as the ratio $(\beta-\gamma)/\alpha$, which, according to the ordinary theory of X-ray absorption, should be equal to unity.

In each series of observations readings were made in the order $\alpha\beta\gamma$, $\gamma\beta\alpha$, $\alpha\beta$. . . In this way it was hoped to eliminate systematic errors, and to minimise the effect of any changes in quality of the primary radiation. The accuracy of any determination of $\frac{\beta-\gamma}{\alpha}$ depends on the relative values of γ and α, being greatest when γ is small compared with α. For example, taking the comparatively large value which obtained in some of the experiments, and assuming the maximum possible error in α, β, γ to be 2–3 per cent., the maximum error expected in a single determination of $\frac{\beta-\gamma}{a}$ is about 10 per cent. The error in the majority of experiments was, however, much less than this. Mean values were taken to reduce this error for a series.

A preliminary investigation was made to determine if any large effect takes place (*i.e.* $(\beta-\gamma)/a$ differs greatly from 1) over a wide range of primary intensity. This was first carried out with small absorption of the silver characteristic radiation, and later with a 70 per cent. absorption. The results of the preliminary investigations are given in Table III.

TABLE III.

Al Filter in Primary Beam (B).	Thickness of Absorbing Silver.	$\frac{\gamma}{a}$.	$\frac{\beta-\gamma}{a}$.
1·8 cm.	·00135 cm.	·02	·994
1·8 "	"	·01	1·03
1·8 "	"	·02	1·003
·33 "	"	·25	1·039
·165 "	"	·40	1·029
·092 "	"	·45	1·028
·046 "	"	·58	1·027
·046 "	·00064 cm.	·27	1·050
·092 "	"	·24	1·052
·165 "	"	·20	1·020
·33 "	"	·15	1·004
1·8 "	"	·01	1·017

A pin-hole aperture at B was used in a test of the absorption of the primary beam when Ag K-radiation fell on the absorbing silver. No change was observed. The later results on the Ag K-beam are given in Table IV. In addition to Ag K-radiation, Sn, Sb, and Sr K-radiations were employed: of these, Sb K excites K-fluorescence in silver, Sn K does so only to a small extent, and the mass-absorption coefficient of Sr K-radiation in Ag is approximately equal to that of Sb K. The absorption of Ag K-radiation and finally of Ce K-radiation by aluminium, on which a primary beam was directed, was examined in the same way.

From Tables III and IV it is seen that in all cases the absorption of the secondary beam by aluminium was uninfluenced by the transmission through the absorbing substance of the primary cross-beam, although this was varied considerably both in its intensity and in its quality. It is worth noting that though the effect γ to be eliminated varied greatly from experiment to experiment, the corrected absorption

Table IV.

	Radiator.	Filter interposed in Primary (cm. Al).	Thickness of Absorber (cm.).	Absorbing Element.	$\frac{(\gamma)}{(\alpha)}$	No. of Determinations in Series.	$\frac{(\beta)-(\gamma)}{(\alpha)}$
A	Ag	·33 cm.	·00135	Ag	·23	4	1·043
	”	”	·006	”	·89	7	1·052
	”	”	”	”	·88	7	1·038
	”	1·12	”	”	·25	7	1·050
	”	1·43	”	”	·16	7	1·018
	”	·33	”	”	·93	5	1·004
	”	”	”	”	1·06	8	1·008
	”	”	”	”	1·25	9	1·052
	”	”	”	”	·83	6	1·038
	Sn	”	”	”	·78	9	0·997
	Sb	”	”	”	·44	5	1·018
	Ag	”	”	”	1·04	5	1·093*
	”	”	”	Al	·12	3	0·997
B	Ag	·33	·006	Ag	1·41 1·33	3 1	1·098 1·002 *
	”	”	”	”	1·44	9	1·016
	”	·68	”	”	·82	7	1·022
	”	·95	”	”	·57	6	1·013
C	Ag	·95	·006	Ag	·38	6	1·002
	”	”	”	”	·38	10	1·006
	”	·33	”	”	·99	4	1·029
	”	”	”	”	1·00	12	1·039
	”	”	”	”	1·04	5	1·035
	”	”	·092	Al	·10	3	1·009
	Sb (K)	·0013(Ag)	·0011	Ag	·35	11	0·988
	Sr (K)	0	·0011	Ag	·24	10	0·996
	Ce (K)	·33	·360	Al	·58	5	1·017

A, primary beam used as standard.
B, Ag K-beam as standard (filtered with Al).
C, Ag (Sb, Sr, Ce) K-beam as standard (filtered with ·006 cm. Ag).
The quantities (α), (β), (γ) are mean values for the number of determinations in the series given in column 6.

showed no dependence on this factor—a fact which indicates the reliability of the experimental results. (The type of standard employed did not influence appreciably the results obtained.) The secondary radiations employed were the characteristic radiations of the K series of Sr, Ag, Sn, Sb, and Ce; the last named was used because its penetrating power is in the neighbourhood of the J absorption in aluminium.

In the case of absorption of Ag K-radiation by silver, however, the only one on which a very large number of observations was made, there was found a general preponderance of values of $\frac{(\beta-\gamma)}{a}$ in the neighbourhood of 1 and also of 1·04.

A statistical examination was made to test whether or not this grouping of the values is real, and also to find if a change of about 10 per cent. (increase in transmitted energy) indicated by the asterisked values in Table IV does occur more frequently than as a result of error.

Under conditions similar to those obtaining in the concluding experiments with silver reported in Table IV, a series of 120 determinations of the ratio of $\frac{\beta-\gamma}{a}$ was carried through. The curve (fig. 3) shows the distribution of the values of the ratio obtained. It reveals two maxima—at ·99 and at 1·03—a grouping which was evident even in the course of tabulation of the values for the purpose of obtaining the distribution curve. There is no evidence of a peak at 1·10.

The mean value of the 120 observations is 1·011 midway between the peaks in fig. 3. This suggests the possibility of two effects nearly equally frequent in their occurrence: one in which there is no change in the amount of energy transmitted by the silver when a primary cross-beam is thrown on, and the other in which there is a 4 per cent. increase in the ionisation in electroscope 1 when a primary beam is thrown on at 45° to the Ag K-beam under observation. (The difference between ·99 and 1·0 has been neglected.)

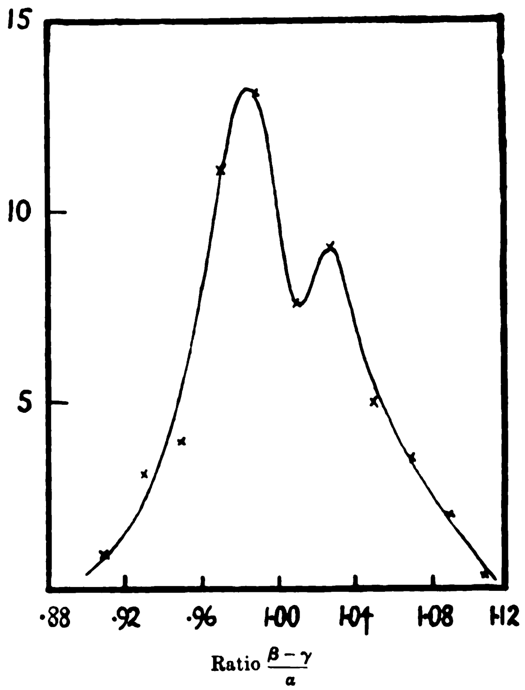


Fig. 3.—Ordinate: number of observations within the range 0·01 containing the value of the abscissa.

The evidence cannot be regarded as conclusive, and the effect is a small one, requiring fuller investigation before it will be possible to give an explanation in terms either of (1) an absorption fatigue effect in the silver, or of (2) a softening of the transmitted radiation so that the net effect is a 4 per cent. increase in the ionisation in the electroscope (since energy is measured in terms of the ionisation in a thin layer of air).

The experiments suggested by Wilson's photographs failed to yield the expected results, and if, as we have reason to believe, the primary beam was sufficiently penetrating, we are led to one or other of the following conclusions: (1) that the effect of superposed radiations on the electronic emission (observed by Wilson) does not appreciably affect the absorption. This might be because the additional electronic emission confined to the gas exposed to primary rays is due to accumulations of energy already absorbed and passed from atom to atom in the manner described by Wilson; (2) that Wilson's phenomena did not take place in these experiments. This might be due to the solid state of the absorber used here—though this would seem to be more favourable to the phenomenon than the gaseous state. On the other hand, the phenomena may not have taken place because the primary cross-beam was not that which had excited the characteristic radiation on which measurements were made. It is also possible, but by no means probable, that Wilson's phenomena take place only in light gases.

All the experiments described above, therefore, have failed to reveal any evidence of effects of the magnitude to be expected on the grounds which suggested the investigation.

It gives me great pleasure to record here my indebtedness to Professor Barkla for his having suggested this research, and also for his constant helpful criticism throughout the progress of the work.

(*Issued separately April* 23, 1925.)

VIII.—The Stability of Suspensions. I. The Rate of Sedimentation of Kaolin Suspensions by Salts at Varying Hydrogen Ion Concentrations.

By **William Ogilvy Kermack** and **William Turner Horace Williamson**, from The Laboratory of the Royal College of Physicians, Edinburgh, and the Edinburgh and East of Scotland College of Agriculture. *Communicated by* Dr ALEXANDER LAUDER.

(MS. received November 26, 1924. Read December 8, 1924.)

THE experiments described in this paper have been undertaken partly with relation to researches on the general question of the stability of colloidal suspensions, partly and more particularly with relation to experiments on the stability of soil suspensions which it is proposed to carry out. Much work along these lines has been described in which one factor, *e.g.* the concentration of the electrolyte present, has been varied and the resulting rates of sedimentation compared, but experiments in which two factors have been varied simultaneously do not appear to be so numerous. It appeared to us to be of interest to carry out experiments on the rate of sedimentation of a suitable suspension where both the concentration of the salt present and the pH of the medium were simultaneously varied. For initial work it is advantageous to use a suspension of a comparatively simple nature and of known chemical composition. Kaolin was consequently chosen as approximately fulfilling these two conditions.

EXPERIMENTAL.

The kaolin used was the ordinary commercially pure variety. Ten grams of this were triturated with a rubber pestle in a beaker containing 0·03 N ammonium hydroxide solution. The beaker was then filled to a depth of 8·5 cm. with the ammonium hydroxide solution and allowed to stand for 24 hours, when the supernatant liquid was decanted off. This process was repeated usually on five successive days, thus removing particles below 0·002 mm. diameter. It may be noted that the suspension obtained with the first washing is less dense than the succeeding ones, presumably because of the neutralisation of part of the ammonium hydroxide by acids in the kaolin. This neutralisation was shown by titration. The solid remaining in the beaker after these washings was then vigorously stirred up with 0·03 N ammonium hydroxide solution, the beaker was filled up to the 8·5 cm. mark with the same solution, and the suspension was allowed to sediment for 30 minutes. The supernatant suspension decanted off at the end of this period was used for the experiments described in this paper. Suspensions prepared in this way contained from 0·35 gm. to 0·52 gm. kaolin in 100 c.c. and were shown by titration to contain 0·025 N ammonium hydroxide, the *pH* being between 9 and 10. One cubic centimetre of suspension diluted with its own volume of distilled water had cleared completely only after about 12 hours. After the addition of the various salt solutions used, readings were taken at the end of periods ranging from 5 to 15 minutes. At the end of such times no sediment had appeared in the control tube containing kaolin suspension and water.

Solutions of acid of various concentrations were prepared and 0·5 c.c. of these were added to each of a row of tubes. To each tube in the various rows was then added 0·5 c.c. of solutions of increasing concentration of the salt investigated. One cubic centimetre of the suspension was then rapidly added to each tube, and the tubes were shaken as nearly as possible all at the same time, allowed to stand, and readings of the degree of sedimentation taken. The shaking was repeated and a second reading taken after the tubes had stood for twelve or more hours. The second readings did not differ significantly from the first except in the cases of calcium sulphate and lanthanum chloride. These exceptions will be remarked upon later. The *pH* of the supernatant liquid was then determined colorimetrically.

The results are best represented in the series of figs. 1–16. All concentrations given are those of the 1 c.c. of solution to which 1 c.c. of kaolin suspension was added. The whole area of each figure is divided by heavy lines into a number of regions. Each region corresponds to a particular degree of sedimentation, which is denoted by a Roman numeral according to the following convention:—The amount of precipitation after 15 minutes in the tube containing 0·06 N hydrochloric acid and no salt is denoted by IV. It was found that the opacity of the supernatant liquid in this tube corresponded to that of No. 6 of a series of opacity tubes prepared as described by Brown (1919).* The reading of the tube to which neither acid nor salt has been added is denoted by O, and intermediate degrees of sedimentation are indicated accordingly. Comparisons between the results of different experiments could be made by means of opacity tube readings. In cases where a combination of *pH* and salt gave a degree of sedimentation greater than that denoted by IV, numbers higher than four were employed to denote such degrees of sedimentation, for instance, where the opacity reading of the supernatant fluid was 1, as determined by Brown's tubes the sedimentation is denoted by VIII, and an opacity reading of 3 is denoted by VI. The *pH* values are shown in the figures by the ordinary numerals within brackets. Where a + or − sign occurs in the figures it refers to the nature of the electric charge on the particles as determined by cataphoresis.

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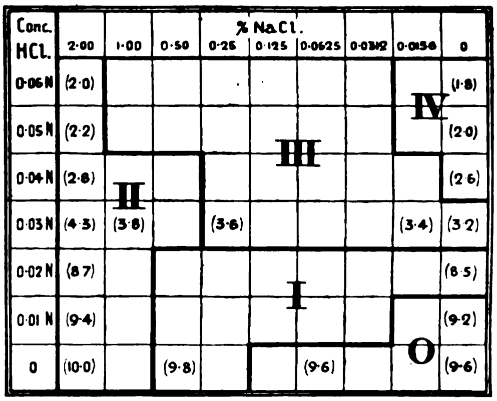
* These tubes are prepared from a 1 per cent. suspension of barium sulphate in a 1 per cent. sodium citrate solution. This 1 per cent. suspension is diluted with seven times its volume of 1 per cent. sodium citrate solution. The tubes 1, 2, 3 . . . 10 contain respectively 1 c.c., 2 c.c., 3 c.c. . . . 10 c.c. of this 1 in 800 suspension made up to 10 c.c. with 1 per cent. sodium citrate.

Discussion.

For purposes of discussion the salts investigated may be conveniently classified as containing uni-, bi-, or tervalent cations. These groups will, therefore, be considered separately.

Univalent Cations.

From consideration of figs. 1–5 it appears that the action of salts with univalent cations varies according as to whether the medium is alkaline or acid. At *pH* 2, for example, sodium chloride does not, in any concentration, increase the rate of sedimentation, and, in fact, in concentrations exceeding 0·5 per cent., causes considerable decrease. On the other hand, at *pH* 9 to 10 the same salt always causes an increase in the rate of sedimentation when present in concentrations greater than 0·5 per cent. Lower concentrations have little effect. The result of this is that although, where there is no salt, there is a very marked difference in the rate of sedimentation at a *pH* of 2 compared with that at a *pH* of 10, yet in presence of 1–2 per cent. sodium chloride the rate of sedimentation is practically independent

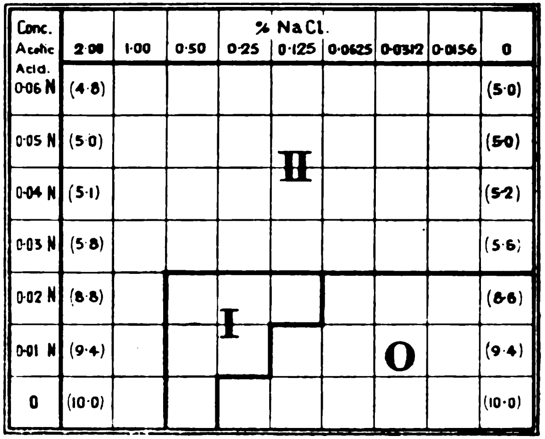


Conc. HCl.	% NaCl. 2·00	1·00	0·50	0·25	0·125	0·0625	0·0312	0·0156	0
0·06 N	(2·0)								(1·8)
0·05 N	(2·2)								(2·0)
0·04 N	(2·8)								(2·6)
0·03 N	(4·3)	(3·8)		(3·6)				(3·4)	(3·2)
0·02 N	(8·7)								(8·5)
0·01 N	(9·4)								(9·2)
0	(10·0)		(9·8)			(9·6)			(9·6)

Fig. 1.—Sodium Chloride and Hydrochloric Acid. (Sedimentation after 15 minutes.)

Kaolin per 100 c.c. suspension = 0·35 gm.

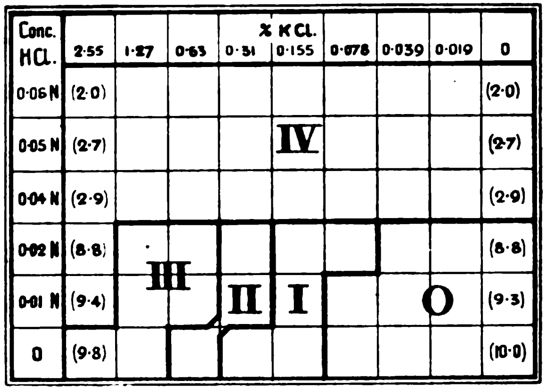
(The Roman numerals indicate the degree of precipitation and the ordinary numerals within brackets the *pH* values.)



Conc. Acetic Acid.	% NaCl. 2·00	1·00	0·50	0·25	0·125	0·0625	0·0312	0·0156	0
0·06 N	(4·8)								(5·0)
0·05 N	(5·0)				II				(5·0)
0·04 N	(5·1)								(5·2)
0·03 N	(5·8)								(5·6)
0·02 N	(8·8)								(8·6)
0·01 N	(9·4)			I			O		(9·4)
0	(10·0)								(10·0)

FIG. 2.— Sodium Chloride and Acetic Acid. (Sedimentation after 15 minutes.)

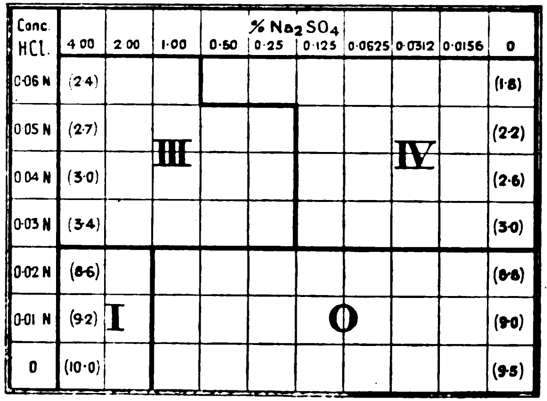
Kaolin per 100 c.c. suspension = 0·35 gm.



Conc. HCl.	% KCl. 2·55	1·27	0·63	0·31	0·155	0·078	0·039	0·019	0
0·06 N	(2·0)								(2·0)
0·05 N	(2·7)				IV				(2·7)
0·04 N	(2·9)								(2·9)
0·02 N	(8·8)	III							(8·8)
0·01 N	(9·4)			II	I		O		(9·3)
0	(9·8)								(10·0)

FIG. 3.—Potassium Chloride and Hydrochloric Acid. (Sedimentation after 15 minutes.)

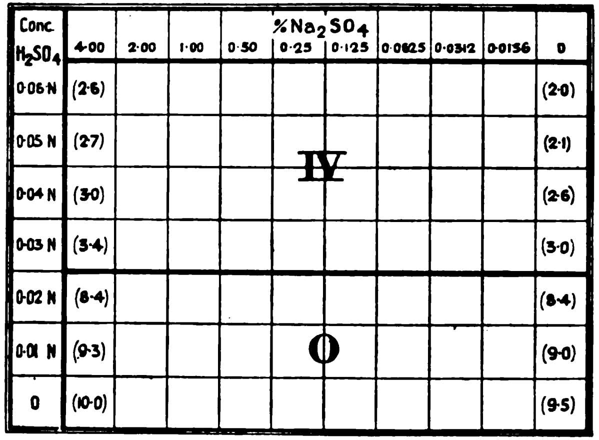
Kaolin per 100 c.c. suspension = 0·37 gm.



Conc. HCl.	% Na_2SO_4 4·00	2·00	1·00	0·50	0·25	0·125	0·0625	0·0312	0·0156	0
0·06 N	(2·4)									(1·8)
0·05 N	(2·7)		III					IV		(2·2)
0·04 N	(3·0)									(2·6)
0·03 N	(3·4)									(3·0)
0·02 N	(8·6)									(8·8)
0·01 N	(9·2)	I					O			(9·0)
0	(10·0)									(9·5)

FIG. 4.—Sodium Sulphate and Hydrochloric Acid. (Sedimentation after 15 minutes.)

Kaolin per 100 c.c. suspension = 0·35 gm.



Conc. H_2SO_4	% Na_2SO_4 4·00	2·00	1·00	0·50	0·25	0·125	0·0625	0·0312	0·0156	0
0·06 N	(2·6)									(2·0)
0·05 N	(2·7)				IV					(2·1)
0·04 N	(3·0)									(2·6)
0·03 N	(3·4)									(3·0)
0·02 N	(8·4)									(8·4)
0·01 N	(9·3)				O					(9·0)
0	(10·0)									(9·5)

FIG. 5.—Sodium Sulphate and Sulphuric Acid. (Sedimentation after 15 minutes.)

Kaolin per 100 c.c. suspension = 0·35 gm.

of the *pH*. If the sodium ion is coupled with a bivalent anion instead of a univalent one, that is, if sodium sulphate be used instead of sodium chloride, the same tendencies may be observed even though in a very much less marked degree. Potassium chloride causes at *pH* 9 to 10 a more marked increase in the rate of sedimentation than does sodium chloride at the same equivalent concentration. This is evident on comparing the precipitation occurring when potassium chloride of 2·55 per cent. or 0·342 N is used with that occurring when sodium chloride of 2 per cent., that is, of the same normality, is used. On the other hand it has apparently no action in decreasing the rate of sedimentation at *pH* 2 to 3. From this it is evident that sodium chloride has a peculiar specific action.

Conc. HCL.	% Na_2HPO_4								
	2·00	1·00	0·50	0·25	0·125	0·0625	0·0312	0·0156	0
0·06 N	(6·0)	III (3·0)							(1·8)
0·05 N	(7·0)	(5·8)	(3·0)			IV			(2·0)
0·04 N	(7·2)	(6·8)	(4·0)	(3·0)					(2·2)
0·03 N	(7·7)		(7·0)	I (8·0)	(5·1)		(3·6)	(2·8)	(2·8)
0·02 N	(8·4)				(8·4)				(8·2)
0·01 N	(9·1)				O				(9·0)
0	(9·6)								(9·4)

Fig. 6.—Di-Sodium Hydrogen Phosphate and Hydrochloric Acid. (Sedimentation after 15 minutes.)

Kaolin per 100 c.c. suspension = 0·52 gm.

When the sodium ion is coupled with a tervalent anion, as in sodium phosphate or sodium citrate (figs. 6–8), a certain complication is introduced in as far as the *pH* varies from tube to tube in the row. However, making allowance for this change, two facts emerge:—Firstly, the presence of the tervalent anions obscures the effect of the sodium ion even more than in the case of the bivalent anions, which also have a less pronounced effect in the same direction, *e.g.* in presence of 0·5 per cent. di-sodium hydrogen phosphate the rate of sedimentation varies from O at *pH* 9·4 to IV at *pH* 3 to 4. These figures are the same as those obtained when no salt is present. Secondly, the point at which transition takes place from no sedimentation to a marked sedimentation in presence of 0·125 per

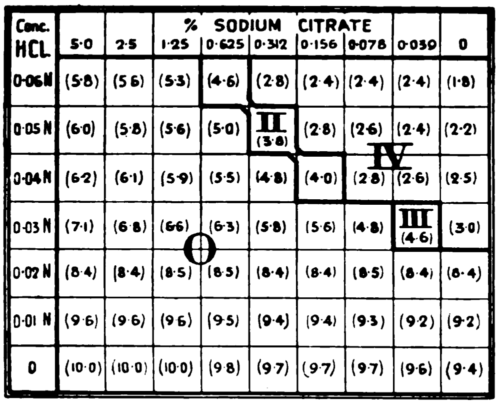
Conc. H_3PO_4	% Na_2HPO_4								
	2·00	1·00	0·50	0·25	0·125	0·0625	0·0312	0·0156	0
0·05 N*	(6·3)	(5·4)	(3·0)			IV			(2·8)
0·04 N	(6·8)		(5·2)	(3·0)	(3·0)	(3·0)			(2·8)
0·03 N	(7·1)			(5·6)	II (4·2)	III (3·8)	(3·8)	(3·0)	(3·0)
0·02 N	(7·4)	(7·4)	(7·2)	(7·0)	(6·8)	(6·8)	(6·6)	(6·8)	(6·2)
0·01 N	(8·6)			O					(8·4)
0	(9·6)								(9·2)

Fig. 7.—Di-Sodium Hydrogen Phosphate and Phosphoric Acid. (Sedimentation after 15 minutes.)

Kaolin per 100 c.c. suspension = 0·52 gm.

* Calculated as monobasic acid.

cent. sodium phosphate and phosphoric acid is about pH 4, and in presence of 0·125 per cent. sodium phosphate and hydrochloric acid it is about pH 5. In fig. 2, with acetic acid alone, this transition point is between pH 5·6 and 8·6. As might be expected the presence of increasing amounts of tervalent anions makes it necessary to reach a higher hydrogen ion concentration in order to effect the same rate of sedimentation. From this point of view the tervalent anions may be looked upon as having a definite protective action. On the other hand, this protective action is barely perceptible if we merely consider the effect of the salts at a definite *pH*. Gengou (1908–9), for example, found that sodium citrate is very potent in retarding the sedimentation of suspensions of barium sulphate. However, in the case of suspensions of kaolin, under the conditions of the experiment, this effect was not nearly so marked. It will be seen from fig. 8 that in acid solution retardation of sedimentation does occur. It will be observed, however, that this effect is almost completely accounted for by the buffering action of the sodium citrate, and that practically no retarding action was evident apart from that due to change of *pH*. It may be noted that there were present in these acid solutions small amounts of ammonium chloride which may have been sufficient to neutralise any retarding effect. On the alkaline side, so far from there being any protection, readings after half-an-hour showed greater sedimentation in the tubes containing sodium citrate than in those containing none.



Conc. HCL	% SODIUM CITRATE 5·0	2·5	1·25	0·625	0·312	0·156	0·078	0·039	0
0·06 N	(5·8)	(5·6)	(5·3)	(4·6)	(2·8)	(2·4)	(2·4)	(2·4)	(1·8)
0·05 N	(6·0)	(5·8)	(5·6)	(5·0)	II (3·8)	(2·8)	(2·6)	(2·4)	(2·2)
0·04 N	(6·2)	(6·1)	(5·9)	(5·5)	(4·8)	(4·0)	(2·8)	(2·6)	(2·5)
0·03 N	(7·1)	(6·8)	(6·6)	(6·3)	(5·8)	(5·6)	(4·8)	III (4·6)	(3·0)
0·02 N	(8·4)	(8·4)	(8·5)	(8·5)	(8·4)	(8·4)	(8·5)	(8·4)	(8·4)
0·01 N	(9·6)	(9·6)	(9·6)	(9·5)	(9·4)	(9·4)	(9·3)	(9·2)	(9·2)
0	(10·0)	(10·0)	(10·0)	(9·8)	(9·7)	(9·7)	(9·7)	(9·6)	(9·4)

Fig. 8.—Sodium Citrate and Hydrochloric Acid. (Sedimentation after 15 minutes.)
Kaolin per 100 c.c. suspension = 0·37 gm.

It is difficult to suggest any explanation to account for the action of, for example, sodium chloride in preventing flocculation by hydrochloric acid. This phenomenon is unlike what is usually found with colloidal suspensions, and presumably is related to chemical actions taking place on the surface of the particles.

Bivalent Cations.

Calcium was chosen as an example of a bivalent cation because of the important use of certain calcium compounds in agriculture. In the simple case of calcium chloride and hydrochloric acid (fig. 9) no unexpected features appear. The calcium ion as compared with the sodium ion or the potassium ion is, as one would expect, a much more efficient flocculator. In the case of calcium sulphate and sulphuric acid two figures are given (figs. 10 and 10A), the second containing the readings observed for a sedimentation time of 10 minutes, the tubes having been reshaken after standing for a period of 24 hours from the time of the first reading. The differences noted in the alkaline region have been repeatedly verified, and occur also when hydrochloric acid is used instead of sulphuric acid. There is, as would be expected, a certain hindering of the sedimentation by the sulphate ion noticeable in the alkaline region in both figs. 10 and 10A, but otherwise the results are substantially the same as those obtained for similar concentrations with calcium chloride and hydrochloric acid. When,

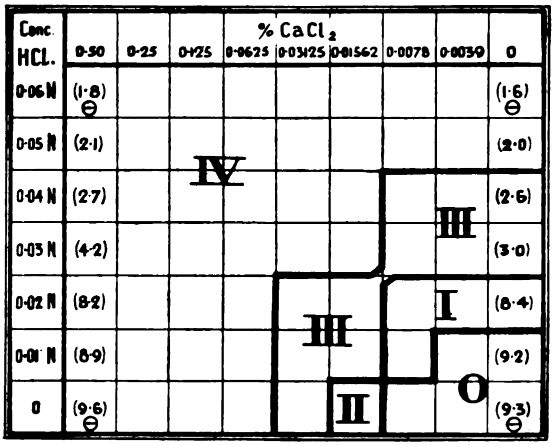


FIG. 9.—Calcium Chloride and Hydrochloric Acid. (Sedimentation after 10 minutes.)

Kaolin per 100 c.c. suspension = 0·35 gm.

Conc. HCl.	% CaCl₂ 0·50	0·25	0·125	0·0625	0·03125	0·01562	0·0078	0·0039	0
0·06 N	(1·8) ⊖								(1·6) ⊖
0·05 N	(2·1)		IV						(2·0)
0·04 N	(2·7)							III	(2·6)
0·03 N	(4·2)								(3·0)
0·02 N	(8·2)							I	(8·4)
0·01 N	(8·9)				III				(9·2)
0	(9·6) ⊖					II		O	(9·3) ⊖

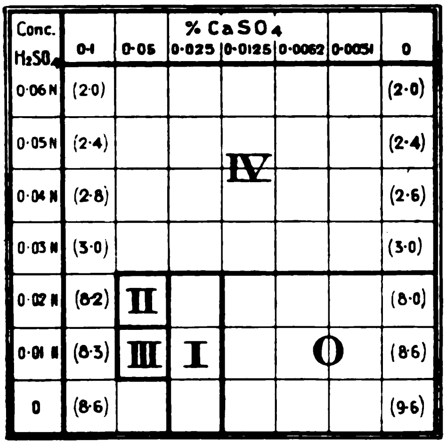


FIG. 10.—Calcium Sulphate and Sulphuric Acid. (Sedimentation after 10 minutes.)

Kaolin per 100 c.c. suspension = 0·52 gm.

Conc. H₂SO₄	% CaSO₄ 0·1	0·05	0·025	0·0125	0·0062	0·0031	0
0·06 N	(2·0)						(2·0)
0·05 N	(2·4)			IV			(2·4)
0·04 N	(2·8)						(2·6)
0·03 N	(3·0)						(3·0)
0·02 N	(8·2)	II					(8·0)
0·01 N	(8·3)	III	I		O		(8·6)
0	(8·6)						(9·6)

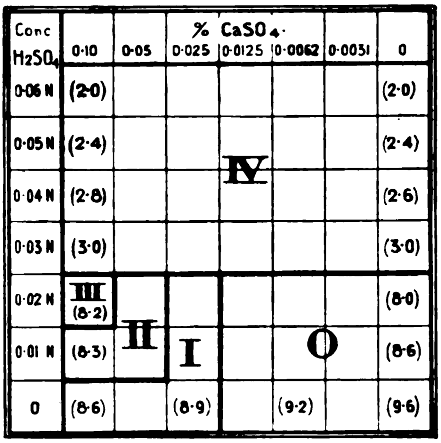
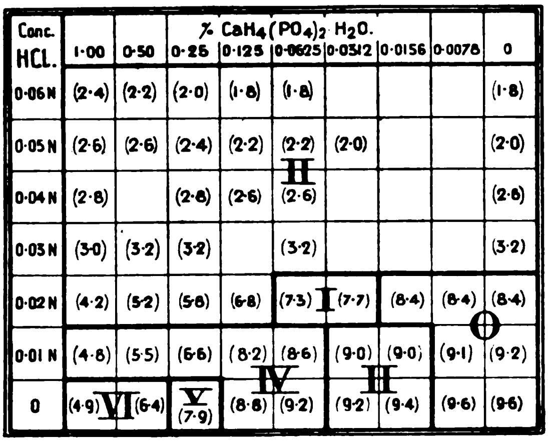


FIG. 10A.—Calcium Sulphate and Sulphuric Acid. (Sedimentation after 10 minutes.) 24 hours later.

Kaolin per 100 c.c. suspension = 0·52 gm.

Conc. H₂SO₄	% CaSO₄ 0·10	0·05	0·025	0·0125	0·0062	0·0031	0
0·06 N	(2·0)						(2·0)
0·05 N	(2·4)			IV			(2·4)
0·04 N	(2·8)						(2·6)
0·03 N	(3·0)						(3·0)
0·02 N	III (8·2)						(8·0)
0·01 N	(8·3)	II	I		O		(8·6)
0	(8·6)		(8·9)		(9·2)		(9·6)

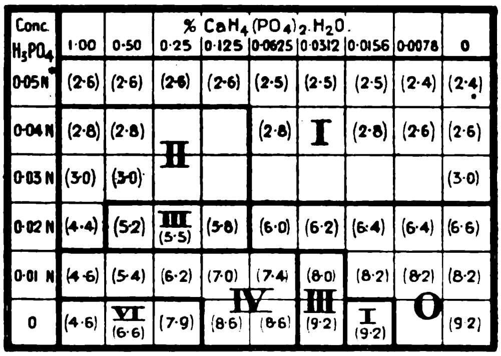
however, figs. 11 and 12, dealing with the effect of monocalcium phosphate, are examined, certain peculiarities are at once observed. In the case of monocalcium phosphate and hydrochloric acid the outstanding fact is, that where a concentration of the phosphate greater than 0·06 per cent. is present, the degree of sedimentation is actually increased by increase of *pH*. At a concentration of 0·25 per cent. there is a change from II at a *pH* of 2 to V at a *pH* of 7·9, and at a concentration of 1 per cent. a change from a sedimentation of II at *pH* between 2·4 and 4·2 to one of VI at *pH* 4·9. If phosphoric acid is used instead of hydrochloric acid, these phenomena are even more marked. It is further to be noted that in the case of these abnormal sedimentations a very bulky and flocculent sediment separated in a very short time, leaving a practically clear supernatant fluid. It seems probable that precipitation of basic calcium phosphate over the surface of the kaolin particles takes place, as a bulky precipitate of the same type separates when a solution of 0·025 N ammonium hydroxide is added to solutions of monocalcium phosphate of the concentrations showing this abnormal effect with kaolin.



Conc. HCL.	% $CaH_4(PO_4)_2 \cdot H_2O$. 1·00	0·50	0·25	0·125	0·0625	0·0312	0·0156	0·0078	0
0·06 N	(2·4)	(2·2)	(2·0)	(1·8)	(1·8)				(1·8)
0·05 N	(2·6)	(2·6)	(2·4)	(2·2)	(2·2)	(2·0)			(2·0)
0·04 N	(2·8)		(2·8)	(2·6)	II (2·6)				(2·8)
0·03 N	(3·0)	(3·2)	(3·2)		(3·2)				(3·2)
0·02 N	(4·2)	(5·2)	(5·8)	(6·8)	(7·3)	I (7·7)	(8·4)	(8·4)	(8·4)
0·01 N	(4·8)	(5·5)	(6·6)	(8·2)	(8·6)	(9·0)	(9·0)	(9·1)	O (9·2)
0	(4·9) VI	(6·4)	V (7·9)	(8·8)	IV (9·2)	(9·2)	III (9·4)	(9·6)	(9·6)

FIG. 11.—Monocalcium Phosphate and Hydrochloric Acid. (Sedimentation after 5 minutes.)

Kaolin per 100 c.c. suspension = 0·52 gm.



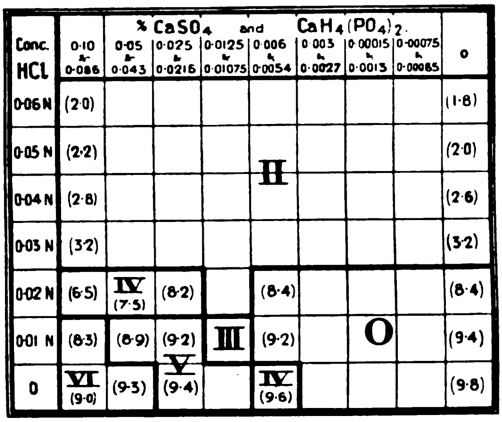
Conc. H_3PO_4	% $CaH_4(PO_4)_2 \cdot H_2O$. 1·00	0·50	0·25	0·125	0·0625	0·0312	0·0156	0·0078	0
0·05 N*	(2·6)	(2·6)	(2·6)	(2·6)	(2·5)	(2·5)	(2·5)	(2·4)	(2·4)
0·04 N	(2·8)	(2·8)	II		(2·8)	I	(2·8)	(2·6)	(2·6)
0·03 N	(3·0)	(3·0)							(3·0)
0·02 N	(4·4)	(5·2)	III (5·5)	(5·8)	(6·0)	(6·2)	(6·4)	(6·4)	(6·6)
0·01 N	(4·6)	(5·4)	(6·2)	(7·0)	(7·4)	(8·0)	(8·2)	(8·2)	(8·2)
0	(4·6)	VI (6·6)	(7·9)	(8·6)	IV (8·6)	III (9·2)	I (9·2)	O	(9·2)

FIG. 12.—Monocalcium Phosphate and Phosphoric Acid. (Sedimentation after 5 minutes.)

Kaolin per 100 c.c. suspension = 0·52 gm.

* Calculated as monobasic acid.

In view of the importance of "superphosphate" as a fertiliser, the action of a mixture containing amounts of monocalcium phosphate and calcium sulphate proportionally equivalent to those present in "superphosphate" was investigated (fig. 13). The rather surprising result was obtained that the abnormal sedimentation was even more marked and more extensive as compared with that caused by monocalcium phosphate alone. In comparing this figure with fig. 11 it is to be observed that whereas the latter commences with a calcium phosphate concentration of 1 per cent. or a calcium ion concentration of 0·16 per cent., the "superphosphate" commences with a total salt concentration of 0·186 per cent. or a calcium ion concentration of approximately 0·04 per cent. At the latter concentration the monocalcium phosphate alone does not show the effect in nearly so marked a degree. This may be related to the fact that a precipitate of basic phosphate is more easily obtained from the mixture of calcium sulphate and phosphate than from the phosphate alone.



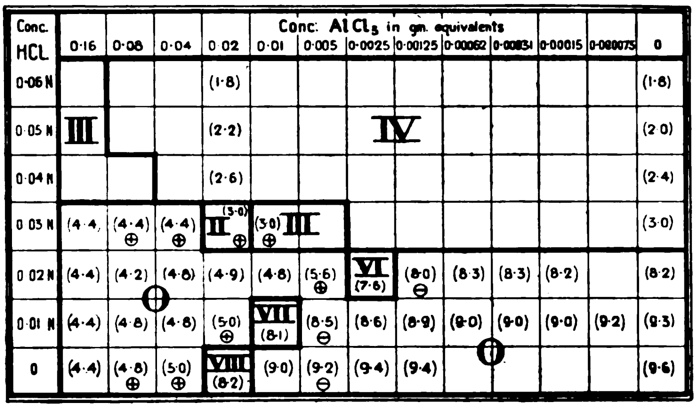
Conc. HCl	% $CaSO_4$ and $CaH_4(PO_4)_2$								
	0·10 & 0·086	0·05 & 0·043	0·025 & 0·0216	0·0125 & 0·01075	0·006 & 0·0054	0·003 & 0·0027	0·00015 & 0·0013	0·00075 & 0·00065	0
0·06 N	(2·0)								(1·8)
0·05 N	(2·2)				II				(2·0)
0·04 N	(2·8)								(2·6)
0·03 N	(3·2)								(3·2)
0·02 N	(6·5)	IV (7·5)	(8·2)		(8·4)				(8·4)
0·01 N	(8·3)	(8·9)	(9·2)	III	(9·2)		O		(9·4)
0	VI (9·0)	(9·3)	V (9·4)		IV (9·6)				(9·8)

Fig. 13.—Calcium Sulphate, Monocalcium Phosphate and Hydrochloric Acid. (Sedimentation after 5 minutes.)

Kaolin per 100 c.c. suspension = 0·37 gm.

Tervalent Cations.

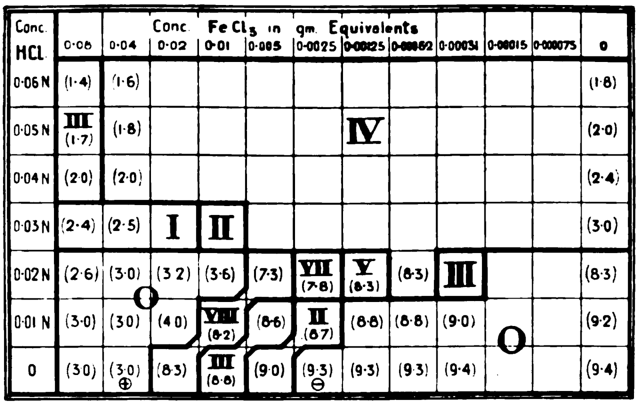
The results of the experiments with aluminium, ferric, and lanthanum chlorides, represented in figs. 14–16, are similar to those already described in as far as the action in all cases is very different in acid and in alkaline solution. In acid solution the action of the tervalent ion is not marked. This is rather surprising, as these tervalent ions have, as is well known, a powerful flocculating action, but a possible explanation may be that the hydrogen ion alone has a sedimenting action on kaolin so marked that the addition of further ions, however active, is without appreciable effect. In alkaline solutions, on the other hand, the sedimenting effect is practically zero, except at a pH of 7–8, where, if the concentration is appropriate, very marked precipitation occurs. The sediment formed at this point is very bulky, and, as it separates, leaves a very clear supernatant fluid.



Conc. HCL	Conc. $AlCl_3$ in gm. equivalents												
	0·16	0·08	0·04	0·02	0·01	0·005	0·0025	0·00125	0·00062	0·00031	0·00015	0·000075	0
0·06 N				(1·8)									(1·8)
0·05 N	III			(2·2)			IV						(2·0)
0·04 N				(2·6)									(2·4)
0·03 N	(4·4,	(4·4) ⊕	(4·4) ⊕	(3·0) II ⊕	(3·0) III ⊕								(3·0)
0·02 N	(4·4)	(4·2)	(4·8)	(4·9)	(4·8)	(5·6) ⊕	VI (7·6)	(8·0) ⊖	(8·3)	(8·3)	(8·2)		(8·2)
0·01 N	(4·4)	(4·8) O	(4·8)	(5·0) ⊕	VII (8·1)	(8·5) ⊖	(8·6)	(8·9)	(9·0)	(9·0)	(9·0)	(9·2)	(9·3)
0	(4·4)	(4·8) ⊕	(5·0) ⊕	VIII (8·2)	(9·0)	(9·2) ⊖	(9·4)	(9·4)	O				(9·6)

Fig. 14.—Aluminium Chloride and Hydrochloric Acid. (Sedimentation after 10 minutes.)

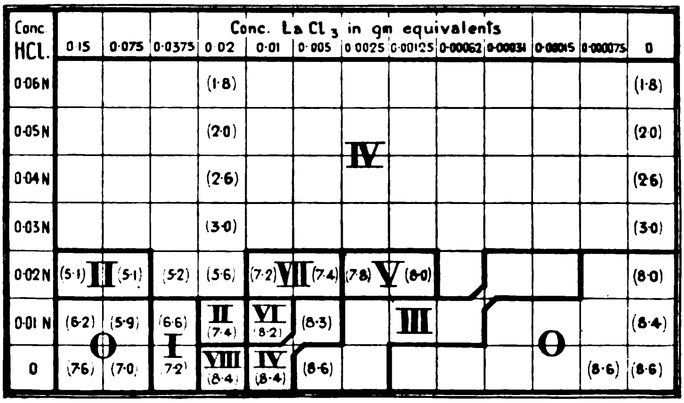
Kaolin per 100 c.c. suspension = 0·37 gm.



Conc. HCL	Conc. $FeCl_3$ in gm. Equivalents											
	0·08	0·04	0·02	0·01	0·005	0·0025	0·00125	0·00062	0·00031	0·00015	0·000075	0
0·06 N	(1·4)	(1·6)										(1·8)
0·05 N	III (1·7)	(1·8)					IV					(2·0)
0·04 N	(2·0)	(2·0)										(2·4)
0·03 N	(2·4)	(2·5)	I	II								(3·0)
0·02 N	(2·6)	(3·0)	(3 2)	(3·6)	(7·3)	VII (7·8)	V (8·3)	(8·3)	III			(8·3)
0·01 N	(3·0)	(3·0) O	(4·0)	VIII (8·2)	(8·6)	II (8·7)	(8·8)	(8·8)	(9·0)	O		(9·2)
0	(3·0)	(3·0) ⊕	(8·3)	III (8·8)	(9·0)	(9·3) ⊖	(9·3)	(9·3)	(9·4)			(9·4)

Fig. 15.—Ferric Chloride and Hydrochloric Acid. (Sedimentation after 10 minutes.)

Kaolin per 100 c.c. suspension = 0·37 gm.



Conc. HCL.	Conc. $LaCl_3$ in gm equivalents												
	0·15	0·075	0·0375	0·02	0·01	0·005	0·0025	0·00125	0·00062	0·00031	0·00015	0·000075	0
0·06 N				(1·8)									(1·8)
0·05 N				(2·0)			IV						(2·0)
0·04 N				(2·6)									(2·6)
0·03 N				(3·0)									(3·0)
0·02 N	(5·1) II	(5·1)	(5·2)	(5·6)	(7·2) VII	(7·4)	(7·8) V	(8·0)					(8·0)
0·01 N	(6·2)	(5·9)	(6·6)	II (7·4)	VI (8·2)	(8·3)		III					(8·4)
0	(7·6) O	(7·0)	I (7·2)	VIII (8·4)	IV (8·4)	(8·6)					O	(8·6)	(8·6)

Fig. 16.—Lanthanum Chloride and Hydrochloric Acid. (Sedimentation after 10 minutes.)

Kaolin per 100 c.c. suspension = 0·37 gm.

From the fact that in the case of ferric chloride the precipitate is yellow in colour, and the supernatant fluid has lost its original yellow colour, it is to be inferred that these precipitates contain the metallic hydroxides probably in an adsorbed condition. Mere precipitation of the metallic hydroxide by the alkali is not sufficient to effect flocculation of the kaolin, as shown by the fact that in the cases of iron and aluminium chlorides a precipitate of the metallic hydroxide could be observed in certain tubes although the kaolin was not sedimented at all. At a *pH* less than 7 a very rapid reduction in precipitation takes place, and, in fact, the particles become protected, and acquire a positive charge at *pH* between 5 and 3. This is particularly marked in the cases of aluminium and ferric chlorides. In the case of lanthanum chloride the results are slightly different. The regions of precipitation and of protection are not nearly so well defined. In addition, the zone of maximum precipitation was observed after a few days to have shifted towards the region of higher concentration of lanthanum. These differences in the case of lanthanum may be related to the fact that lanthanum hydroxide does not, like aluminium and ferric hydroxides, form stable colloidal solutions at *pH* between 3 and 7.

These characteristic zones of precipitation which are observed with salts containing tervalent cations resemble very much the zones of precipitation which occur when these salts in the proper concentration and at the necessary *pH* are added to ordinary colloidal sols such as gold and gum benzoin (Kermack and Voge, 1925), a similarity particularly emphasised by the fact that the zone of precipitation separates a region where the particles are negatively charged from a region within which they are positively charged. These results are similar to those obtained by Comber (1921) with soil clay, and support his view that this characteristic flocculating action of tervalent ions falls into line with the results obtained with ordinary colloids.

On the other hand, it is very difficult to reconcile the anomalous action of a univalent cation such as sodium in hindering sedimentation by acid with the usual precipitating effect of its action. Usually the sodium and hydrogen ions assist each other in their precipitating action. Here, however, exactly the opposite takes place, and at present it is difficult to offer any explanation. It is possible that it is related to the occurrence of chemical reactions which, it is well known, are particularly liable to take place with complex silicates in the presence of simple inorganic salts, and which might modify the surface forces, and so possibly cause these abnormalities.

SUMMARY.

1. The rates of sedimentation of a kaolin suspension in presence of varying concentrations of a salt (sodium chloride, potassium chloride, sodium sulphate, di-sodium hydrogen phosphate, sodium citrate, calcium chloride, calcium sulphate, monocalcium phosphate, "superphosphate," aluminium chloride, ferric chloride or lanthanum chloride) have been compared at various *pH* values. Abnormal results are obtained with sodium chloride, monocalcium phosphate, aluminium, ferric and lanthanum chlorides.

2. Sodium chloride increases the rate of sedimentation in alkaline solution, but actually inhibits it in acid solution.

3. In concentrations of monocalcium phosphate above 0·06 per cent. abnormal sedimentation in alkaline solution takes place, with the result that it is much more complete than at the corresponding concentrations in acid solution.

4. In acid solution the tervalent ions, aluminium, ferric and lanthanum have little effect, but a zone of very marked flocculation occurs at *pH* 7–8. This zone separates a region within which the unsedimented particles are negatively charged from a region within which they are positively charged.

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(*Issued separately April* 29, 1925.)

IX.—The Influence of Gelatin on the Stability of a Colloidal Solution of Cholesterol, and on the Charge on the Particles.

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INTRODUCTION.

IN a previous paper (Wright and Kermack, 1923) it has been shown that colloidal gum benzoin is precipitated by low concentrations of gelatin within a certain range, at a *pH* less than 4·7, that is, by positively charged gelatin in concentrations too low to protect, but sufficiently high to neutralise, the negatively charged colloidal particles of gum benzoin. It appeared of interest to attempt to investigate quantitatively the changes in the charge on the particles of such a colloid, with variation of the concentration of gelatin present and of the hydrogen ion concentration. For such investigations a colloidal solution of gum benzoin is not ideal: it it of a somewhat variable and indefinite composition, and it contains substances of an acidic nature which influence the *pH* and the conductivity. From some investigations on similar lines on the properties of Wassermann "antigens," which contain a certain amount of cholesterol, we were led to the study of the colloidal properties of cholesterol itself.

It is well known that cholesterol plays a very important part in these Wassermann "antigens." It therefore appeared to us of importance to investigate the properties of colloidal solutions of cholesterol, more especially with a view to finding out how far they are influenced by the presence of various other substances. Some results, more particularly on the influence of protein (gelatin) on the stability of colloidal cholesterol, are given in the following paper.

It was found that certain commercial samples of cholesterol formed a colloidal sol with the greatest ease when an alcoholic solution was added to water. It is to be noted that pure recrystallised cholesterol did not show this property, but if to an alcoholic solution of pure cholesterol a sufficient quantity of sodium taurocholate is added, a stable sol results when this solution is added to water. Since there is practically no electrolyte present, such a sol as that obtained by the use of commercial cholesterol is very convenient for measurements of the cataphoretic potential difference (P.D.) between the particles and the medium. There is the further consideration that cholesterol is widely distributed in nature as a constituent of cells.

There are comparatively few estimations of the mobility of the particles of such sols, or of the potential difference between them and the medium, to be found in the literature. The values given by Burton (1921) for the cataphoretic P.D.s of various negatively charged suspensoids and emulsoids vary from 14 millivolts (chloroform) to 58 m.v. (gold chemically prepared). It is of interest that the values obtained for the cholesterol sol indicate a P.D. of about 20 m.v.

As the P.D. between particles and medium is intimately related to their stability, experiments were carried out, similar to those described in the paper on colloidal gum benzoin, on the behaviour of gelatin as a precipitating and as a protecting colloid. The experimental work therefore naturally falls into four sections:—

1. General.
2. Precipitation of cholesterol sol by gelatin solutions.
3. Determination of the P.D. between the cholesterol particles and the medium in the presence of gelatin of varying concentration and *pH*.
4. Protection by gelatin of the cholesterol particles against precipitation.

I. GENERAL.

Preparation of Cholesterol Sol.

Commercial cholesterol (1 gm.) is dissolved in absolute ethyl alcohol (100 gm.) if necessary with gentle warming. The solution is, of course, stable, but occasionally, on prolonged standing, a small precipitate has been noticed. The cholesterol used was obtained from the British Drug Houses, Ltd., and melted at 146°–146·5° C. The melting-point was not altered by recrystallisation, although the recrystallised specimen did not form nearly as stable a sol.

To about 80 c.c. of distilled water heated to 65° C., 5 c.c. of a 1 per cent. alcoholic solution of commercial cholesterol is added. Almost with the first drop of the alcoholic solution the water becomes slightly milky, and this milkiness becomes more pronounced as the addition is continued. The water is shaken vigorously during the addition (which takes about thirty seconds), and the tip of the pipette is kept below the surface of the water.

This tends to ensure the formation of a more homogeneous sol. Distilled water is added to make the total volume about 100 c.c., and the whole is filtered through a large Whatman No. 1 filter paper to remove a very slight amount of flocculated material and allowed to cool.

The sol so prepared (containing 5 c.c. of the alcoholic solution per 100 c.c. of sol) is a milk-like liquid with a slightly blue tinge. When only 0·5 c.c. of alcoholic solution of cholesterol is added, the solution is much less milky and more opalescent and the blue colour is more marked.

An electrometric apparatus not being available, the determination of the hydrogen ion concentration in the following experiments was carried out colorimetrically.

Precipitation by Sodium Chloride.

The amount of neutral sodium chloride solution required to precipitate the sol depends on the concentration of the sol, and, as will be seen from the following table, the more dilute the sol, the greater the concentration of sodium chloride required to precipitate it. (Table I).

Precipitation by Acid.

In the same way the more dilute the sol, the more acid is required to precipitate it, and the lower is the *pH* of the supernatant fluid when precipitation just takes place. This is shown in Tables II and III.

It may be mentioned that this sol containing 2·5–5 per cent. of 1 per cent. alcoholic solution of cholesterol is very stable and keeps many months at least. Further, even centrifugalisation at 3000 revolutions per minute in a centrifuge of 11 cm. radius for five minutes effects no sedimentation. Under the ultra-microscope the particles show a lively Brownian movement, and measurements show them to have a diameter of approximately 200 $\mu\mu$. They are visible under dark ground illumination, but are not distinguishable with ordinary illumination even with a powerful condenser.

It is of interest to notice that the stability of these sols appears to increase to some extent on standing. The results of some experiments illustrating this are given in Table IV.

Precipitation by Sodium Chloride. Relationship to pH.

The amount of sodium chloride required for the precipitation of the sol depends on the *pH* of the mixture. The results represented in fig. 1 have been obtained in a manner similar to that described in a previous

TABLE I.—PRECIPITATION OF CHOLESTEROL SOLS BY SODIUM CHLORIDE.

Conc. of Sodium Chloride:—		5·0%.		2·5%.		1·25%.		0·625%.		0·3125%.		0·156%.		0·078%.		0·039%.		0·0195%.		0·0097%.	
Cholesterol Sol.	Prepared at	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*
1 per cent.	65° C.	4	...	4	6·7	4	...	2	...	1	...	0	...	0	6·6	0	...	0	6·6	0	...
5 ,,	Room temperature	4	...	4	6·8	4	...	4	...	4	...	3½	...	1	6·7	0	...	0	...	0	6·6
5 ,,	65° C.	4	...	4	6·8	4	...	4	...	4	...	3½	...	1	6·6	0	...	0	...	0	6·6

Note.—The figures indicate the degree of precipitation of the cholesterol sol.

4 = Complete precipitation.
3 = Almost complete precipitation.
2 = Marked deposit, supernatant fluid still milky.
1 = Slight deposit.
0 = No precipitation.

TABLE II.—PRECIPITATION OF CHOLESTEROL BY SULPHURIC ACID.

Conc. of Sulphuric Acid:—		N/40.		N/80.		N/160.		N/320.		N/640.		N/1280.		N/2560.		N/5120.		N/10240		N/20480.	
Cholesterol Sol.	Prepared at	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*
1 per cent.	65° C.	4	...	4	...	4	...	4	...	4	...	4	3·0	2	3·4	0	4·2	0	...	0	...
5 ,,	Room temperature	4	...	4	...	4	...	4	...	4	...	4	...	4	3·3	2	3·7	0	4·4	...	...
5 ,,	65° C.	4	...	4	...	4	...	4	...	4	...	4	...	4	3·3	2	3·7	0	4·4	0	...

Note.—Where precipitation has occurred the *pH* values recorded are those of the supernatant fluid.

TABLE III.—PRECIPITATION OF SOLS OF DIFFERENT CONCENTRATIONS BY HYDROCHLORIC ACID.

Conc. of Hydrochloric Acid :—	N/50.		N/100.		N/200.		N/400.		N/800.		N/1600.		N/3200.		N/6400.	
Cholesterol Sol.	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*
0·5 per cent.	4	...	4	...	4	...	4	2·8	1	3·1	0	...	0	...	0	...
1 ,, ...	4	...	4	...	4	...	4	2·8	1	3·1	0	...	0	...	0	...
2·5 ,, ...	4	...	4	...	4	...	4	...	4	3·1	2	3·6	0	...	0	...
10 ,, ...	4	...	4	...	4	...	4	...	4	...	4	3·5	1	4·1	0	...
1 per cent. + 9 per cent. added alcohol	4	...	4	...	4	...	4	2·8	3	3·1	0	...	0	...	0	...
10 per cent. diluted 1 in 10 .	4	...	4	...	4	...	4	2·8	2	3·1	0	...	0	...	0	...

Note.—All sols about 12 days old. Readings after 16 hours.

TABLE IV.

(*a*) Fresh 10 per cent. Cholesterol Sol.

Sodium Chloride Concentration :—	0·05%.	0·075%.	0·1%.	0·125%.	0·15%.	0·175%.	0·2%.	0·25%.
Precipitation	0	4	4	4	4	4	4	4

(*b*) 10 per cent. Cholesterol Sol, 16 days old.

Sodium Chloride Concentration :—	0·03%.	0·04%.	0·05%.	0·06%.	0·07%.	0·08%.	0·09%.	0·1%.
Precipitation	0	0	0	0	1	1	2	4

paper (Wright and Kermack, 1923). To each tube in a particular row of tubes, 0·5 c.c. of hydrochloric acid or of sodium hydroxide of a definite and constant concentration was added, then 0·5 c.c. of a variable concentration of sodium chloride. 1 c.c. of cholesterol sol (5 per cent.) was then added and the tubes shaken. By arranging a number of rows and using different concentrations of hydrochloric acid or of sodium hydroxide, so that the pH was graded from row to row, the concentration of sodium chloride required to produce precipitation could be found for the various pH values. It will be seen that this varied in a manner similar to that found in the case of sols of colloidal gum benzoin.

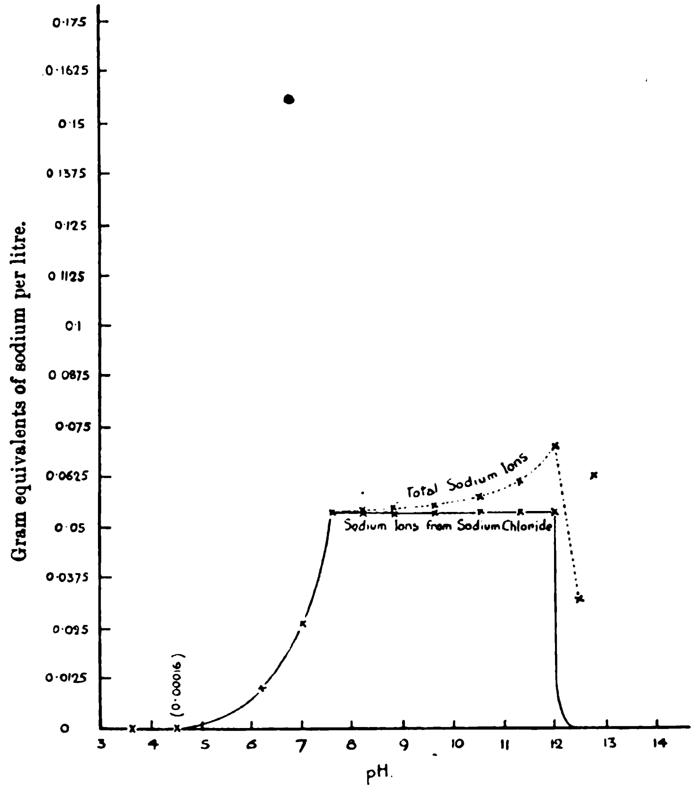


FIG. 1.—Effect of pH on the concentration of Sodium Chloride required to precipitate a 5 per cent. Cholesterol Sol.

Moreover, with the cholesterol sol readings could be made at quite high pH values, whereas gum benzoin is dissolved by the alkali under these conditions. The very interesting fact will be noted that the curve indicating the minimum amount of sodium chloride causing precipitation returns to zero at $pH > 12$, *i.e.* precipitation occurs with alkali alone.

When the sodium ions due to the sodium hydroxide are included so that the total sodium ions are plotted against pH, the result is indicated by the dotted line in fig. 1.

II. Precipitation of Cholesterol Sol by Gelatin Solutions.

In the previous paper by Wright and Kermack (*loc. cit.*), the question was raised as to whether lyophobe sols in general were precipitated by lyophile colloids under suitable conditions in the same way as colloidal "gum benzoin" is precipitated by dilute solutions of gelatin. As mentioned in that paper, ordinary Wassermann antigen shows this property, and it will now be demonstrated that the same phenomenon occurs with a colloidal solution of cholesterol.

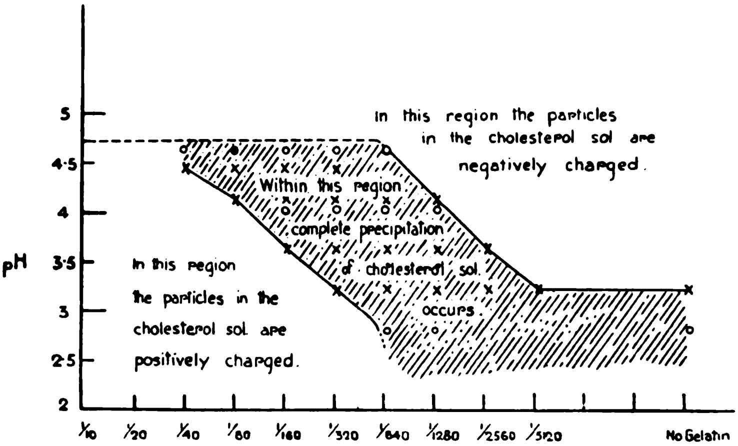


Fig. 2.—Effect of Gelatin on the stability of a 5 per cent. Cholesterol Sol at various pH values.
Note.—0 and × indicate separate experiments.

The details of the experiments were exactly parallel to those described in the paper referred to (p. 645). A series of dilutions of isoelectric gelatin, hydrochloric acid, and of sodium hydroxide were prepared. To each tube of a series of tubes were added 0·5 c.c. of gelatin solution (the concentration varying from tube to tube), 0·5 c.c. of the appropriate acid or alkali solution, and finally 1 c.c. of 5 per cent. cholesterol sol. The tubes were then well shaken and corks inserted. The readings were made after standing over night at room temperature. The results are collected in Table V and fig. 2. The concentrations given are the final concentrations, *i.e.* those of the resultant mixture of acid or alkali, gelatin and sol. This method of expressing concentrations was used as it was found more convenient when dealing with the single sols which were prepared for cataphoresis (see p. 80).

TABLE V.—PRECIPITATION OF CHOLESTEROL BY GELATIN, EFFECT OF *pH*.

Concentrations of Gelatin × 10^3:—		1/10.		1/20.		1/40.		1/80		1/160.		1/320.		1/640		1/1280.		1/2560		1/5120.		H_2O.	
		Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*	Pptn.	*pH.*
	H_2O	**0**	5·4	½	3·6	**0**	5·4	**0**	...	**0**	...	**0**	...	**0**	...	**0**	...	**0**	...	**0**	5·6	...	...
Concentrations of Hydrochloric Acid.	N/40960	**0**	5·4	½	5·2	**1**	5·0	**4**	4·6	**0**	5·2	**0**	...	**0**	...	**0**	...	**0**	...	**0**	...	...	...
	N/20480	**1**	4·4	**1**	4·4	**4**	...	**4**	...	**4**	...	**3**	...	**2**	...	**1**	...	**1**	...	**0**	4·4	...	...
	N/10240	½	4·2	½	..	**1**	4·1	**3**	...	**4**	...	**4**	...	**4**	...	**3**	...	**1**	...	**1**	4·0	...	...
	N/5120	½	3·8	½	...	½	...	½	3·6	**3**	...	**4**	...	**4**	...	**4**	...	**4**	...	**2**	3·6	½	...
	N/2560	½	...	½	...	½	...	½	...	½	3·2	**3**	...	**4**	...	**4**	...	**4**	...	**4**	3·2	**4**	...

By the shaded area in fig. 2 precipitations of the degree 3 or 4 (Table V) are indicated.

The determination of the nature of the charges on the particles was made in the first instance by means of the ultra-microscope as described previously (Wright and Kermack, *loc. cit.*, p. 649). It will be seen that the region of precipitation exactly separates the region of positively charged from that of negatively charged colloid, verifying the results obtained with colloidal "gum benzoin." No results similar to those described by Walpole (1913) for the effect of acid and gelatin on oil emulsion, in which the region of flocculation did not coincide with the change from positively to negatively charged colloid, were obtained. This is mentioned the more confidently as it has been confirmed by the quantitative measurements of potential difference between the particles and the medium now to be described.

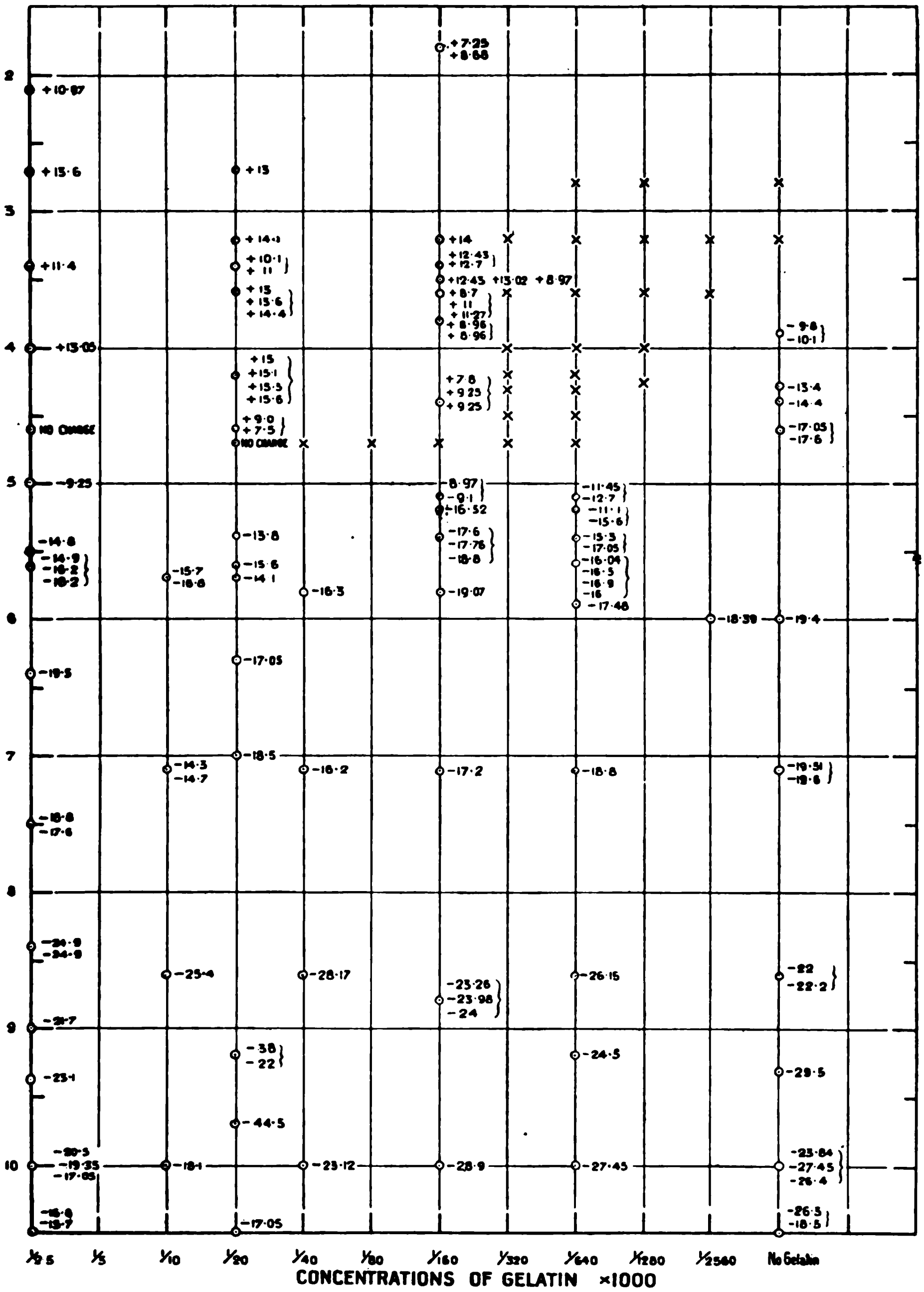
III. DETERMINATION OF THE P.D. BETWEEN THE CHOLESTEROL PARTICLES AND THE MEDIUM IN THE PRESENCE OF GELATIN OF VARYING CONCENTRATION AND *pH*.

It is of great interest to trace the variations in the P.D. of the cholesterol particles in the presence of small quantities of protein. Such measurements may usually be carried out by observing the particles under the microscope and measuring the rate of travel in an electric field of known potential gradient or by the U-tube method, in which the boundary of the sol is observed and its rate of travel measured. It was decided first of all to attempt to measure the P.D. by the U-tube method, and for this purpose the apparatus described by Northrop and Cullen (1922) was used. The ordinary U-tube cannot be used, as owing to the alcohol which it contains the cholesterol sol rises to the surface. Northrop and Cullen's apparatus is, however, particularly suitable in such a case. It is essentially an inverted U-tube. The electrodes, zinc in zinc sulphate, are at the ends of the limbs, and the sol under observation occupies the bend at the top of the apparatus. In each limb between the zinc sulphate below and the sol occupying the bend is a solution of sucrose of suitable concentration. Northrop and Cullen recommend the use of a $\frac{M}{5}$ sucrose solution, but it was found that $\frac{M}{10}$ sucrose solution acted equally well if not better, and this concentration was adopted.

Preliminary experiments showed that with an apparatus of the dimensions recommended by Northrop and Cullen, in which there is a distance between the electrodes of approximately 19 cm., a potential of about 222·5 volts gave reasonable movement. The following series of readings was ultimately found to be convenient. First, the applied electric field was allowed to act in one direction for six minutes, readings being taken every two minutes. Then the field was reversed for two minutes and shut off completely for one minute, during which time a very careful reading was taken. The difference between this reading and the initial one gives the net movement for four minutes. The field was then applied in the same (reversed) direction for ten minutes, and then again reversed (to the initial direction) for two minutes, readings being taken every two minutes as before. The difference between the final and the six-minute reading gives the net movement resulting from the application of the field for eight minutes. These two readings for four and eight minutes respectively, in good experiments gave for the cataphoretic P.D. between the particles and the medium values agreeing to within 2–3 mv. The reason for the two minutes' reversal of field is that the boundary surfaces of the sol at the end of four, or more particularly of eight minutes, sometimes became very indistinct, possibly from inequality of P.D., and so inequality of rate of travel of the individual particles. After reversal for two minutes the edge usually becomes much sharper. Indeed, this effect of clearing up an indistinct boundary surface by reversal of the field is often very astonishing.

In preparing sols for experiments upon the effect of gelatin in presence of acid or alkali on the P.D., the following mode of procedure was adopted. Into each of a number of clean, large boiling tubes was put 50 c.c. of distilled water. By adding 50 c.c. of gelatin solution to the first, transferring 50 c.c. of the resultant solution to the second, 50 c.c. of this new solution to the third, and so on, a series of increasing dilutions was obtained. To each tube was then added 50 c.c. of a solution of acid or alkali of a given concentration, and after thorough mixing 50 c.c. of the resultant solution was transferred from each tube to one of a series of empty tubes. This gave two rows of tubes containing exactly corresponding dilutions of gelatin at a particular *pH*. To one row was then added in each tube 50 c.c. of cholesterol sol (5 c.c. of alcoholic solution in 100 c.c. of distilled water), and to the other 50 c.c. of a $\frac{M}{10}$ sucrose solution. In performing the cataphoretic experiments the sucrose solution was interposed in the limbs of the Northrop apparatus between the zinc-zinc sulphate electrodes and its corresponding cholesterol sol, thus ensuring that both

Table VI.—Effect of *pH* on the Cataphoretic P.D. between the Particles of Cholesterol and the Medium in the presence of Gelatin of varying Concentration. × indicates Precipitation.



were of the same conductivity. In this way it is possible to prepare sets of such cholesterol sols and sucrose solutions containing the same amount of gelatin but differing in *pH*. Concentrated glucose did not precipitate the sol.

The results are given in Table VI, and represented graphically in fig. 3.

Practically all these measurements have been confirmed by repetition. In the few cases where they have not, they have been verified by the agreement of the results obtained in two groups of experiments, one in which sets were made containing constant gelatin and varying acid and alkali, and the other in which they contained constant acid or alkali and varying gelatin.

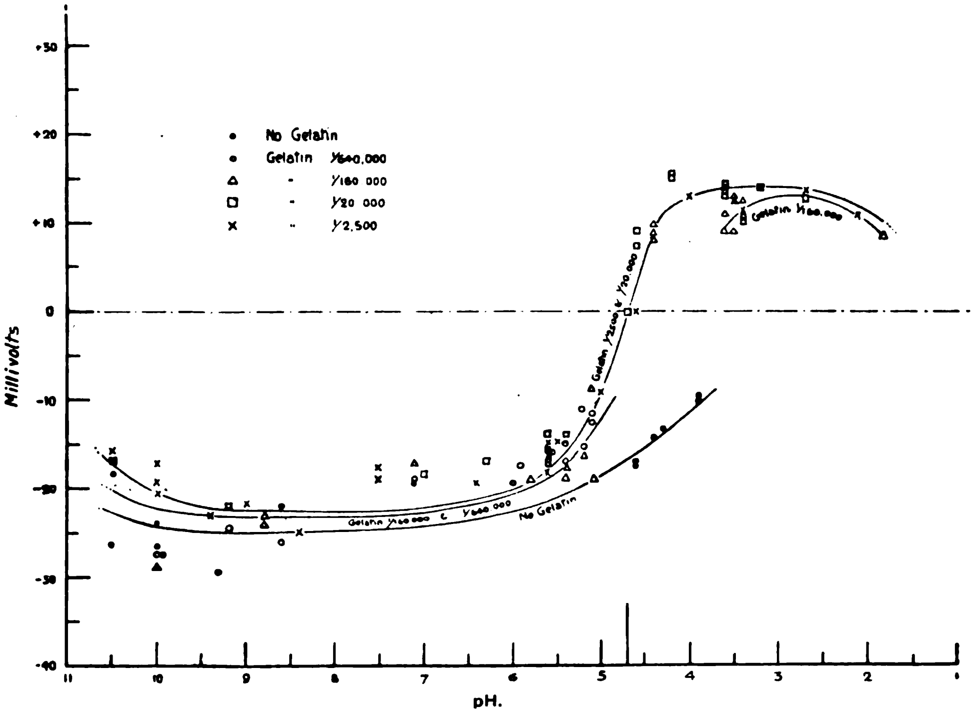


Fig. 3.—Effect of *pH* on the P.D. between the Cholesterol Particles and the Medium in the presence of Gelatin of varying Concentrations.

The calculation of the potential difference between the particles and the medium was made by means of the formula given by Northrop and Cullen (*loc. cit.*).

$$\text{P.D. in millivolts} = \frac{4{\cdot}5 \times \text{velocity in mm. per hour}}{\text{potential gradient in volts per cm.}}$$

The potential gradient actually employed was 11·7 volts per cm.

The general results can be read directly from the above figures. It will be noticed that when only very small amounts of gelatin are present (less than 1/40000), precipitation occurs when the P.D. between the particles and the medium falls below about −9 m.v. These results are discussed more fully on pp. 86–89.

Certain differences in the values obtained are attributable partly to experimental error, but partly also to the difficulty of obtaining identical sols at different times (*cf.* effect of age on the stability of the sol, p. 73).

Effect of Sodium Chloride on the Precipitation of Cholesterol by Gelatin Solutions.

The action of sodium chloride on the precipitation of a cholesterol sol by gelatin is shown in Table VII. A difficulty which occurs in these experiments may be noted. Dilute gelatin solutions, without acid or alkali, when added to the cholesterol sol have a *pH* above the isoelectric point of gelatin, so no precipitation by the gelatin occurs. In order, therefore, to obtain precipitation, it is necessary to render the mixture acid. In the case of gum

TABLE VII.—EFFECT OF SODIUM CHLORIDE ON THE PRECIPITATION OF CHOLESTEROL SOL BY GELATIN. (*pH* throughout 4·2.)

FINAL CONCENTRATION OF GELATIN ×1000

Final percentage Concentration of Sodium Chloride

	1/2·5	1/5	1/10	1/20	1/40	1/80	1/160	1/320	1/640	1/1280	1/2560	1/5120	1/10240	1/20480	No Gelatin
5·0	0	0	0	0	4	4	4	4	4						4
2·5	0	0	0	0	4	4	4	4	4						4
1·25	0	0	0	0	4	4	4	4	4						4
0·625	0	0	0	0	4	4	4	4	4						4
0·15	0	0	0	0	4	4	4	4	4						4
0·1	0	0	0	0	4	4	4	4	4						4
0·075	0	0	0	0	4	4	4	4	4						4
0·05	0	0	0	0	4	4	4								4
0·025	0	0	0	0	4	4	4	4	4	4	4	4	4	4	4
0·0185									4	4	4	?	3	3	3
0·0125									4	4	4	4	2	2	2
0·00675									4	2	½	0	0	0	0
No Sodium Chloride				0	0	4	4	4	3	0	0	0	0	0	0

benzoin this does not require to be done, as the gum benzoin is itself acid. By experiment it was found that a suitable sol was obtained by adding 0·5 c.c. of N/50 H_2SO_4 to 100 c.c. of the filtered cold cholesterol sol. The resulting sol, when added to the sodium chloride-gelatin mixture, gave a *pH* of 4·2.

It will be seen that, as in the case of colloidal gum benzoin, the effect of the sodium chloride is to widen the zone of precipitation, particularly towards the side with gelatin of lower concentration where the cholesterol particles are negatively charged. The effect is very different from that of acid where the zone, instead of being widened, is altered in position. These results may also be compared with those shown in Table VIII, which may be looked on as representing a similar experiment at *pH* about 6·6, a *pH* at which no precipitation takes place with gelatin alone.

IV. PROTECTION BY GELATIN OF THE CHOLESTEROL PARTICLES AGAINST PRECIPITATION.

(a) *Action of Gelatin in Protection of Cholesterol against Precipitation by Sodium Chloride.*

The following experiment was carried out to obtain information on the effect of *pH* on the action of gelatin in protecting cholesterol against precipitation by sodium chloride. A series of dilutions of sodium hydroxide, hydrochloric acid, and of gelatin was prepared, and 0·5 c.c. of alkali (or acid) and 0·5 c.c. of gelatin solution were placed in each of a set of tubes arranged in rows and columns, each row having the same concentration of alkali (or acid) and each column the same concentration of gelatin. One c.c. of 10 per cent. sodium chloride solution was then added to each tube, followed after about thirty minutes by 2 c.c. of 5 per cent. cholesterol sol. The results, read after standing twenty-four hours, are shown in the following table (Table IX).

This may be compared with the corresponding table (Wright and Kermack, 1923, p. 653) for colloidal gum benzoin. In that case gelatin showed a minimum protecting power at the isoelectric point. Here there is no such minimum, but it may be noted that from *pH* 3·2 to *pH* 5 there is very little change in the amount of precipitation, the tube containing 1/40000 of gelatin being slightly protected in every case. As hydrochloric acid is itself a precipitating agent, and as it is increasing constantly as the solution becomes more acid (particularly in the case of cholesterol, which requires more acid to reach a definite *pH* than the corresponding benzoin sol), there is really quite probably a minimum region of protection about the isoelectric point of gelatin.

TABLE VIII.—EFFECT OF GELATIN ON THE PRECIPITATION OF CHOLESTEROL BY SODIUM CHLORIDE.

Conc. of Gelatin × 1000 :—		1/2·5		1/5.		1/10.		1/20.		1/40.		1/80.		1/160.		H_2O.	
		Pptn.	*pH*.	Pptn.	*pH*.	Pptn.	*pH*.	Pptn.	*pH*.	Pptn.	*pH*.	Pptn.	*pH*.	Pptn.	*pH*.	Pptn.	*pH*.
Concentrations of Sodium Chloride.	5 per cent.	0	6·8	0	...	½	...	1	...	3½	6·8	4	...	4	...	4	6·8
	2·5 ,,	0	6·5	0	...	½	...	1	...	3½	6·6	4	...	4	...	4	6·8
	1·25 ,,	0	5·7	0	...	0	...	1	...	3	6·6	4	...	4	...	4	6·8
	·625 ,,	0	5·5	0	...	0	...	1	...	2	6·5	4	...	4	...	4	6·6
	·15 ,,	0	5·3	0	...	0	...	0	...	0	6·5	1	6·5	3	6·6	4	6·6
	·075 ,,	0	5·5	0	...	0	...	0	...	0	...	0	...	0	...	0	6·6

TABLE IX.—EFFECT OF GELATIN ON THE PRECIPITATION OF CHOLESTEROL BY 2·5 PER CENT. SODIUM CHLORIDE AT DIFFERING *pH* VALUES.

Final Conc. of Gelatin × 10^3 :—		1/5.		1/10.		1/20.		1/40.		1/80.		1/160.		1/320.		1/640		H_2O.	
		Pptn.	*pH*.	Pptn.	*pH*.	Pptn.	*pH*.	Pptn.	*pH*.	Pptn.	*pH*.	Pptn.	*pH*.	Pptn.	*pH*.	Pptn.	*pH*.	Pptn.	*pH*.
Final Concentration of Sodium Hydroxide.	N/320	0	10·5	0	...	0	...	½	...	1½	...	3	10·5	3½	...	3½	...	4	10·5
	N/1280	0	9·2	0	...	0	...	1	...	2	...	3½	9·2	4	...	4	...	4	9·6
	N/5120	0	7·5	0	...	½	...	1	...	2	...	4	7·4	4	...	4	...	4	8·2
	H_2O	0	7·0	0	...	½	...	1	...	3	...	4	7·0	4	...	4	...	4	7·0
Final Concentration of Hydrochloric Acid.	N/81920	0	6·8	0	...	½	...	½	...	2½	6·8	4	...	4	...	4	...	4	6·8
	N/40960	0	6·8	0	...	½	...	1	...	3½	6·8	4	...	4	...	4	...	4	6·8
	N/20480	0	6·8	0	...	½	...	1	...	3½	6·8	4	...	4	...	4	...	4	6·6
	N/10240	0	6·0	0	...	0	...	1	...	3½	6·4	4	...	4	...	4	...	4	6·5
	N/5120	0	5·0	0	...	0	...	2	...	4	5·0	4	...	4	...	4	...	4	5·2
	N/2560	0	4·2	0	...	½	...	3	4·2	4	...	4	...	4	...	4	...	4	4·2
	N/1280	0	3·7	0	...	1	...	3½	3·6	4	...	4	...	4	...	4	...	4	3·6
	N/640	0	3·2	0	...	1	...	3½	3·2	4	...	4	...	4	...	4	...	4	3·2
	N/320	0	2·8	0	...	1	...	4	2·8	4	...	4	...	4	...	4	...	4	2·8

Note.—The *pH* recorded, if precipitation has occurred, is that of the supernatant fluid.

(b) *Action of Gelatin in Protection of Cholesterol against Precipitation by Concentrated Alkali and Concentrated Acid.*

In the case of cholesterol these effects can also be investigated when relatively concentrated alkali is used. It is found that alkali itself in high concentrations precipitates the sol. Approximately 0·06 N alkali causes precipitation when no gelatin is present, whilst in the presence of 1/10000 gelatin approximately 0·937 N alkali is required.

In the same way the protective effect of gelatin against precipitation by relatively concentrated acid has been investigated.

In fig. 4 the results of experiments on this protective effect of gelatin against precipitation by alkali and by acid are represented, the *pH* values, where these are beyond the range of ordinary indicators, being calculated from the known concentrations of alkali and acid. It will be noticed that the region of precipitation of the sol by acid is continuous with that of its precipitation by gelatin shown in fig. 2.

Readings in these experiments were made after twenty-four hours. In the case of the experiments with concentrated alkali, it was noted that after forty-eight hours some of the tubes adjacent to those already precipitated (but not precipitated themselves) showed increasing precipitation. This is no doubt to be referred to slow hydrolysis of the gelatin by the concentrated alkali. The concentrated acid showed no such effect. It would not be expected that acid in these concentrations would hydrolyse the gelatin as easily as the alkali.

DISCUSSION.

The results obtained more particularly of interest, on the one hand, are the P.D. between the particles and the medium and its relation to precipitation, and, on the other, the phenomenon of protection. The figures obtained for the cataphoretic P.D.s agree generally with those obtained by Loeb (1924), who determined the charge on suspensions and colloidal particles in the presence of protein at various hydrogen ion concentrations. It is of interest to note that, where sufficient gelatin is present, the cholesterol particles show a zero P.D. at *pH* 4·7, the isoelectric point of gelatin. On decreasing the *pH* the particles acquire a positive P.D. which increases until a P.D. of about +14 m.v. is reached, and at lower *pH* values the charge remains approximately constant or even decreases. Similarly, on increasing the *pH* above 4·7 the P.D. becomes negative up to values approximating to 25 m.v., and thereafter remains constant or at very high *pH* values decreases. Indeed, the action of gelatin on the P.D. of the cholesterol particles is probably to be correlated with its behaviour in regard to those properties such as viscosity, osmotic pressure, etc., which are at a minimum at *pH* 4·7. This suggests that, under these conditions, forces may make themselves manifest similar to those which characterise the Donnan equilibrium, and indeed it is clear that if we postulate an excess in the gelatin concentration at the cholesterol-water interface, a Donnan equilibrium must exist between this denser layer and the rest of the solution. From considerations given below, it seems probable that very small quantities of gelatin would tend to spread themselves in a thin

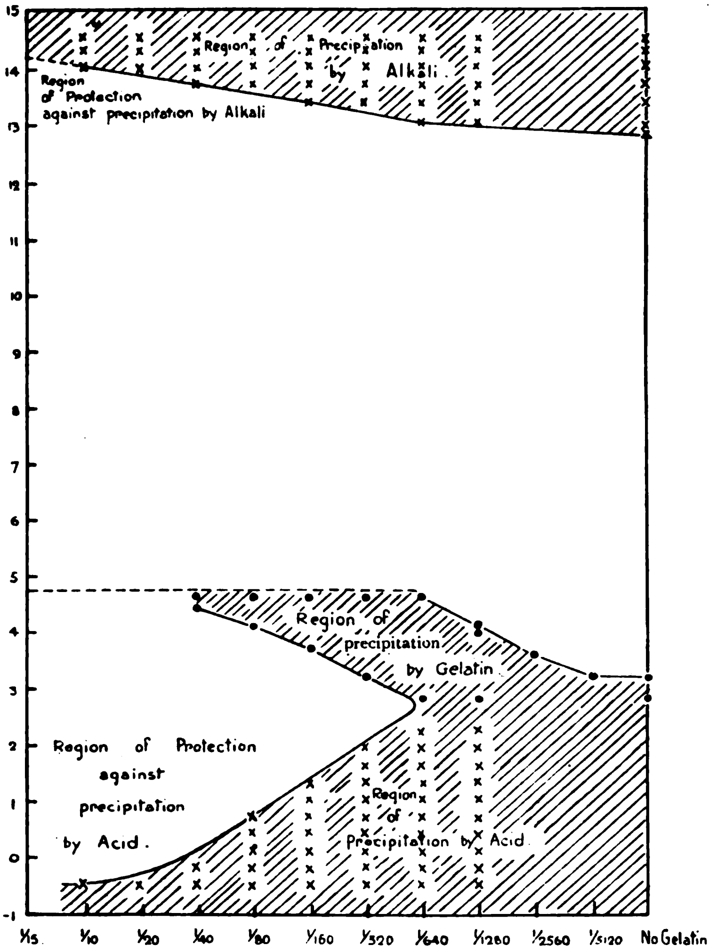


FIG. 4.—Illustrating the precipitation of 5 per cent. Cholesterol Sol by relatively concentrated acid and alkali, the protective effect of gelatin against such precipitation, and the continuity of the region of precipitation by acid with that of precipitation by gelatin.

● Readings from fig. 2.

× Readings from experiments on protection by gelatin against precipitation by alkali and by acid.

layer on the surface of the particles. This thin surface layer would afford a basis for the postulation of a membrane equilibrium which might be supposed to exist between this film and the surrounding fluid. Similar suggestions have been made previously (Loeb, 1924; J. A. Wilson, 1923). This may afford an explanation of the fact that the P.D. between the cholesterol particles and the medium does not increase indefinitely when the *pH* is made greater or less than 4·7.

It is of interest to calculate the depth which the gelatin sufficient to protect the cholesterol sol from precipitation would attain if it were distributed uniformly over the surface of the particles of cholesterol. Sufficient data are available to make such a calculation, as the amount of gelatin per cubic centimetre sufficient to give protection against 2·5 per cent. sodium chloride solution is known, and also the number and size of the cholesterol particles.

The sol contained 0·0005 gm. of cholesterol per cubic centimetre, and the radius of each particle was approximately 100 $\mu\mu$, *i.e.* 10^{-6} cm.

The number of particles present is therefore $5 \times 10^{-4} / \frac{4\pi}{3} \times 10^{-18}$ $= \frac{15}{4\pi} . 10^{14}$, and the surface area of these particles is $4\pi 10^{-12} \times \frac{15}{4\pi} \times 10^{14}$ $= 15 \times 10^{2}$ sq. cm.

Hence the depth of a gelatin layer spread over all the particles in a solution containing 1/40000 gm. or $2·5 \times 10^{-5}$ gm. of gelatin per cubic centimetre is $2·5 \times 10^{-5} / 15 \times 10^{2} = 1·6 \times 10^{-8}$ cm.

The gelatin present is sufficient to provide a layer approximately $2 . 10^{-8}$ cm. in thickness. This of the order of magnitude of atomic diameters.

In view of the well-known affinity of a gelatin molecule for water, which is probably great in comparison with that of a molecule of gelatin for another molecule of gelatin, a layer of gelatin forming over such a surface would tend to become as attenuated as possible, and the figure obtained suggests that when protection just takes place the layer has a thickness corresponding to a depth of from one to two atoms. This protective layer may therefore well be described as monomolecular, as in this case the molecule would naturally become an attenuated sheet containing in depth as few atoms as possible. Obviously, when precipitation takes place the amount of gelatin present is insufficient to form even a monomolecular layer, although there is sufficient partially to neutralise the electric charge on the cholesterol particle. The result of these conditions is naturally that the particles ceasing to repel one another and having part at least of their surfaces bare and cohesive, are precipitated. As has been shown in the results dealing with the precipitation of cholesterol sol by gelatin at various pH values and its relation to the P.D. of the particles, at a certain concentration of gelatin stability exists when there is a small P.D. between the particles and the medium either positive or negative, whereas, when by changing the pH the cataphoretic P.D. becomes less than a certain critical value (about 9 m.v.), precipitation takes place. On the other hand, at slightly greater concentrations of gelatin 1 in 40000 or 1 in 20000, in fact at precisely that concentration which has been shown above to be sufficient to form a monomolecular layer, the particles remain in suspension even although the P.D. is zero. At that point a monomolecular layer would appear to be formed and to prevent cohesion of the particles independently of the presence or absence of a cataphoretic P.D.

SUMMARY.

1. The properties of a colloidal sol of cholesterol have been investigated, more particularly in relation to the effect of pH in the presence of protein (gelatin) of various concentrations.

2. The cataphoretic P.D. between the particles of the sol and the medium has been determined in the presence of gelatin at various concentrations and at various pH values.

3. The effect of the presence of gelatin in preventing the precipitation of a cholesterol sol by sodium chloride at various pH values has been investigated. It has been shown that the amount of gelatin required to protect is approximately that required to cover the particles with a layer of gelatin from one to two atoms thick.

4. The mechanisms of precipitation and protection are discussed in the light of these results.

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X.—The Action of Salts with Multivalent Cations on Colloidal Solutions of Gold and Gum Benzoin.

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It has been pointed out in a previous paper (Wright and Kermack, 1923), that whereas the precipitation of colloidal gum benzoin by salts containing a uni- or bi-valent cation commences at a given concentration and continues to take place with increasing concentration, in the case of ferric chloride, there occurs with continuously increasing concentrations of this salt, first non-precipitation, second precipitation, then a range of concentrations where no precipitation occurs, and lastly precipitation again. Such a series of changes has been observed with other sols under the action of tervalent ions (*cf.* Burton, 1916). As, however, in a series of dilutions as normally carried out, there occur changes in the *pH* as well as in the concentration of the ferric chloride, it appeared to be of interest to study the effects of varying both *pH* and salt concentration and to extend the investigation to various tervalent ions and to other sols. The effects of the following salts were examined: ferric chloride, aluminium chloride, lanthanum chloride, and also calcium chloride and beryllium sulphate. It will be recalled that in the case of aluminium chloride (Wright and Kermack, *loc. cit.*), no zone of precipitation occurred, but that this salt appeared to behave like the salt of a univalent cation. It appears that the difference between its action and that of ferric chloride is merely accidental, and that a similar region of non-precipitation is obtained if the *pH* is suitably adjusted.

It may be pointed out that these results have a direct bearing on the reaction of amphoteric substances such as gelatin, with colloidal suspensions. It will be seen on comparing, for example, fig. 4 with fig. 2 of the previous paper (Wright and Kermack, *loc. cit.*), that the effect of salts which behave as amphoteric electrolytes on colloidal suspensions is very similar to that of proteins. There exists a region where the amphoteric electrolyte in a colloidal condition is able to confer on the lyophobe negatively charged colloid a positive charge, and the zone between this region of positive charge and the region of normal negative charge is one in which precipitation takes place.

EXPERIMENTAL.

The colloidal "benzoin" used was prepared as follows:—10 g. of powdered Sumatra gum benzoin were allowed to remain in contact with 100 c.c. of absolute alcohol for twenty-four hours at room temperature and then filtered. 0·3 c.c. of this alcoholic extract was added with vigorous agitation to 20 c.c. of distilled water at 65°, and the resulting suspension was filtered and cooled.

The gold sol was prepared by adding to 100 c.c. of distilled water 1 c.c. of a 1 per cent. solution of photographic gold chloride, 1 c.c. of 0·7 per cent. potassium carbonate (previously dried and ignited), heating to boiling point, and then adding with vigorous shaking, after removal of the flame, 1 c.c. of a 1 per cent. solution of neutralised 40 per cent. formalin. If the glass vessels used are clean, and the distilled water pure, no difficulty is experienced in obtaining bright red sols with practically no shimmer. Occasionally failure has been caused by contamination of the distilled water. The glassware used is thoroughly cleansed by immersion in sulphuric acid-bichromate mixture, followed by thorough washing.

The order of experimentation is to prepare a series of dilutions of the salt, and also a series of dilutions of alkali or acid. A block of test-tubes is then arranged in rows and columns, each column corresponding to one of the concentrations of the salt solutions, and each row to one of the dilutions of acid (or alkali). 1 c.c. of salt solution is then added to each test-tube in the corresponding column, and 1 c.c. of acid (or alkali) to each test-tube in the corresponding row, so that each test-tube contains 2 c.c. of water containing salt and acid (or alkali) of a definite concentration. 2 c.c. of the gold sol or "benzoin" sol are then added to each test-tube, the mixture shaken thoroughly and allowed to stand overnight. Readings are then made according to the following convention. "Benzoin" sol:—(0) no precipitation; (1) slight precipitation at bottom of tube; (2) partial precipitation; (3) almost complete precipitation; (4) complete precipitation. Gold sol:—In general two types of precipitation are observed, with the formation of a red or of a purplish-blue precipitate respectively. In either case (5) denotes complete precipitation, with a clear supernatant liquid; (0) denotes no change; whilst (1), (2), (3), and (4) denote intermediate degrees of precipitation, similar to the notation already described for the "benzoin" sol. In the case of those tubes in which the precipitate was red, note was made of this fact. The degree of precipitation having been observed, the *pH* of the supernatant fluid in those "precipitated" tubes which appeared to be important was determined colorimetrically.

TABLE I.—ALUMINIUM CHLORIDE AND COLLOIDAL GOLD.

	Conc. of $AlCl_3$:—	M/20.	M/40.	M/80.	M/160.	M/320.	M/640.	M/1280	M/2560.	M/5120.	M/10240.		M/40960.		M/163840.		M/655360.		M/2621440.	H_2O.
Concentration of HCl.	N/25	**5** 3·6	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...
	N/50	**5** 4·4	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** 3·0	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**4** ...	**5** ...	**4** 2·0
	N/100	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**3** 2·4	**3** 2·3	**5** 2·2	**3** ...
	N/200	**5** ...	**5** ...	**5** ...	**4** ...	**4** 3·6	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**4** ...	**5** 3·2	**2** ...	**1** ...	**0** ...
	N/400	**5** ...	**5** ...	**3** 4·4	**2** ...	**0** ...	**2** ...	**4** 4·4	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**4** ...	**5** 3·4	**0** ...	**0** ...	**0** ...
	N/800	**5** ...	**5** ...	**3** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** ..	**2** 5·4	**3** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**5** 4·2	**2** ...	**1** ...	**0** ...
	N/1600	**5** ...	**4** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** 5 8	**1** ...	**3** 4·4	**5** 4·4	**5** ...	**5** ...	**5** 4·1	**5** 4·2	**5** 4·0	**0** ...
	N/3200	**5** ...	**3** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** ...	**5** 4·0	**5** ...	**5** ...	**5** ...	**5** ...	**5** ...	**0** ...
	N/6400	**5** ...	**3** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** ...	**1** ...	**1** ...	**3** 5·0	**5** 5·2	**5** ...	**5** ...	**5** 5·6	**2** 5·6	**0** ...
	N/12800	**5** ...	**3** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** 6·4	**0** ...	**1** ...	**2** 6·2	**4** 5·4	**5** ...	**5** ...	**5** 5·8	**5** 6·0	**0** ...
	N/25600	**5** ...	**5** ...	**1** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** ...	**1** ...	**0** ...	**0** 6·4	**1** 6·2	**2** 6·2	**4** 5·8	**5** ...	**4** ...	**4** ...	**5** 6·4	**0** ...
	H_2O	**5** ...	**4** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** 5·2	**0** 5·4	**0** 6·8	**0** 6·6	**1** 6·2	**1** 6·2	**3** ...	**5** 6·2	**5** ...	**5** ...	**5** ...	**0** ...
Concentration of NaOH.	N/25600	**5** ...	**4** ...	**0** ...	**0** ...	**0** ...	**0** ..	**0** ...	**0** 5·2	**0** 5·4	**0** 5·8	**0** 6·6	**0** 6·6	**0** 6·4	**1** ..	**3** 6·4	**4** 6·6	**5** 6·8	**5** 7·0	**0** ...
	N/12800	**5** ...	**3** ..	**0** ...	**0** ...	**0** ...	**0** ...	**0** ..	**0** ...	**0** ...	**0** ...	**0** ...	**0** ...	**1** ...	**1** 6·8	**5** ...	**5** 6·8	... ...	... ...	**0** ...
	N/6400	**5** ...	**3** 6 4	**0** . .	**0** ...	**0** ...	**0** 5·0	**0** ...	**0** ...	**0** ...	**0** ...	**0** 6·8	**0** 7·0	**4** 6·6	**5** ...	**5** ...	... ...	... ...	... ...	**0** ...
	N/3200	**5** ...	**2** ...	**0** ...	**0** ...	**0** ...	**0** ...	**0** 5·0	**0** 5·4	**0** 6·4	**0** 6·5	**5** 7·2	**3** ...	**3** 7·4	**1** ...	**0** ...	... ...	... ...	... ...	**0** ...
	N/1600	**5** ...	**1** ...	**0** ...	**0** ...	**0** ...	**0** 5·0	**0** 5·4	**0** 6·4	**0** 6·6	**5** 7·2	**4** 7·4	**0** ...	**0** ...	**0** ...	**0** ...	... ...	... ...	... ...	**0** ...
	N/800	**5** ...	**1** ...	**0** ...	**0** ...	**0** 5·0	**0** 5·2	**0** 6·4	**0** 6·6	**5** 7·4	**4** 7·6	**0** ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	**0** ...
	N/400	**5** ...	**0** ...	**0** ...	**0** 4·8	**0** 5·0	**0** 5·2	**0** 6·0	**5** 8·2	**5** 8·4	**0** ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	**0** ...
	N/200	**5** ...	**0** ...	**0** ...	**0** 5·0	**0** 5·2	**5** 8·2	**5** 8·8	**2** ...	**0** ...	**0** ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	**0** ...
	N/100	**3** ...	**0** ...	**0** ...	**0** 5·0	**5** 8·8	**5** ...	**4** 9·8	**4** ...	**0** ..	**0** ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	**0** ...
	N/50	**5** ...	**4** ...	**3** 4·2	**5** 9·2	**5** ...	**5** ...	**5** ...	**3** 10·0	**2** ...	**0** ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	**0** ...
	N/25	**0** 4·4	**1** 4·4	**5** 9·6	**5** ...	**5** 9·8	**2** ...	**1** ...	**0** ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	**0** ...
	N/12·5	**5** 10·2	**5** ...	**5** ...	**4** ...	**3** 10·2	**2** ...	**0** ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	**0** ...
	N/6·25	**5** 12·8	**5** ...	**4** ...	**3** ...	**1** ...	**0** ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	**0** ...

(The numerals in heavy type denote the degree of precipitation and the other numerals the *pH* values.)

The results of an experiment with $AlCl_3$ and gold sol are shown in Table I, and to represent the important features more clearly the results have been transferred to fig. 8, the shaded regions representing precipitation, *i.e.* regions in which corresponding tubes show a precipitate of either 3, 4, or 5. The results in other cases are represented by figs. 1–7 and 9, the figures in each case being prepared from a table similar in principle to Table I. The actual tables we have not considered it necessary to submit, as practically all the important information is contained in the figures.

It may be noted that the important feature is the relative position of the various regions in the figure. Not much emphasis is to be laid on the exact value of the concentrations, etc., at which precipitation takes place, particularly as very slight changes in the preparation of the colloid in its concentration, etc., cause variations in the results, but the general relation of the negatively charged, the positively charged, and the precipitated regions are constant, but as will be seen vary from salt to salt. It may be noted that the limits at high concentrations of salt are vague as the *pH* varies there in large steps.

DISCUSSION.

Two very marked differences are at once apparent between the effect of bivalent and that of tervalent cations. Firstly, the tervalent ions precipitate in very much lower concentrations than the bivalent ions. For instance, taking the amount of salt required to precipitate "benzoin" sol at *pH* 5 for purposes of comparison, there is required in the case of calcium chloride, M/80; of beryllium sulphate, M/80; of lanthanum chloride, M/1280; of ferric chloride, M/10240; and of aluminium chloride, M/20480. Secondly, it is seen that with calcium and beryllium there is no positively charged region. With iron, aluminium, and lanthanum there is a region in which the gold or benzoin is positively charged. The result of this is the appearance with those salts of a zone phenomenon characteristic apparently of tervalent ions.

If we now restrict ourselves to the tervalent cations it would appear that the action of their salts in producing precipitation is related to their degree of hydrolysis. For example, at a concentration of M/160, ferric chloride precipitates "benzoin" at *pH* 2·7 approximately, but not at a slightly higher *pH*, aluminium chloride at *pH* 3·8, and lanthanum chloride at *pH* 5 6. For gold sol, which, in general, is more easily precipitated, the corresponding figures are 3, 3·8, and 6. The following figures showing the

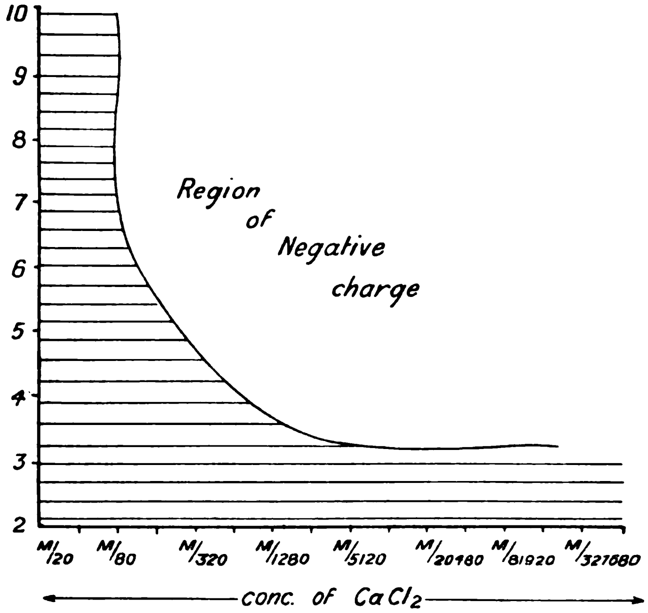

FIG. 1.—$CaCl_2$ and Colloidal Benzoin.

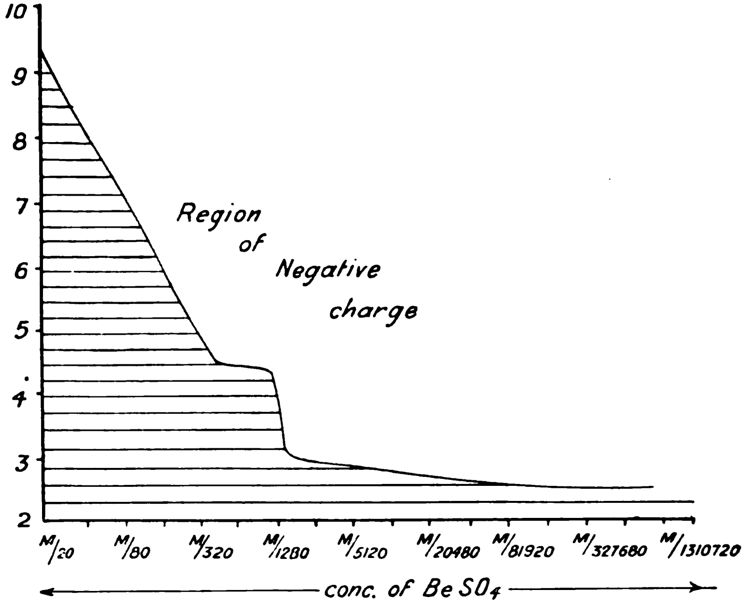

FIG. 2.—$BeSO_4$ and Colloidal Benzoin.

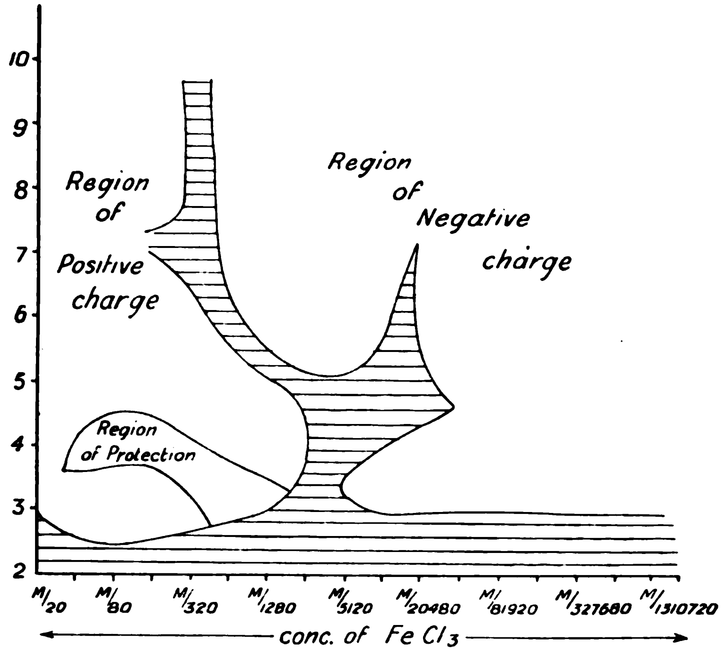

FIG. 3.—$FeCl_3$ and Colloidal Benzoin.

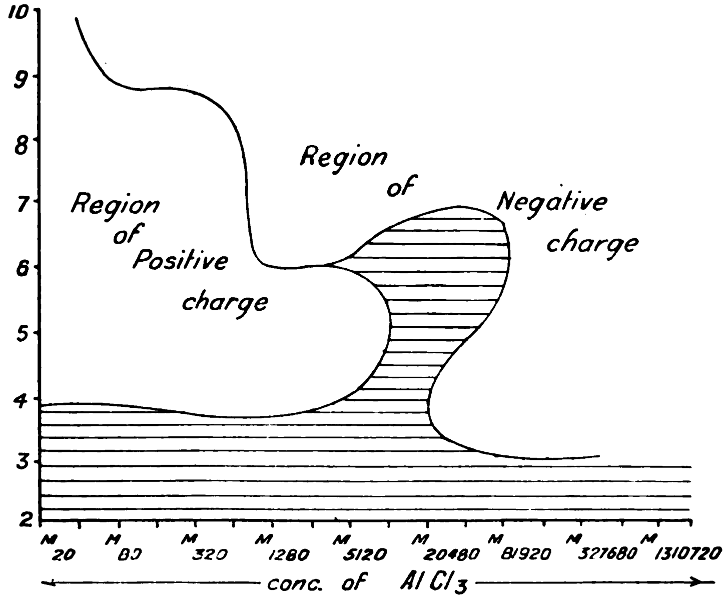

FIG. 4.—$AlCl_3$ and Colloidal Benzoin.

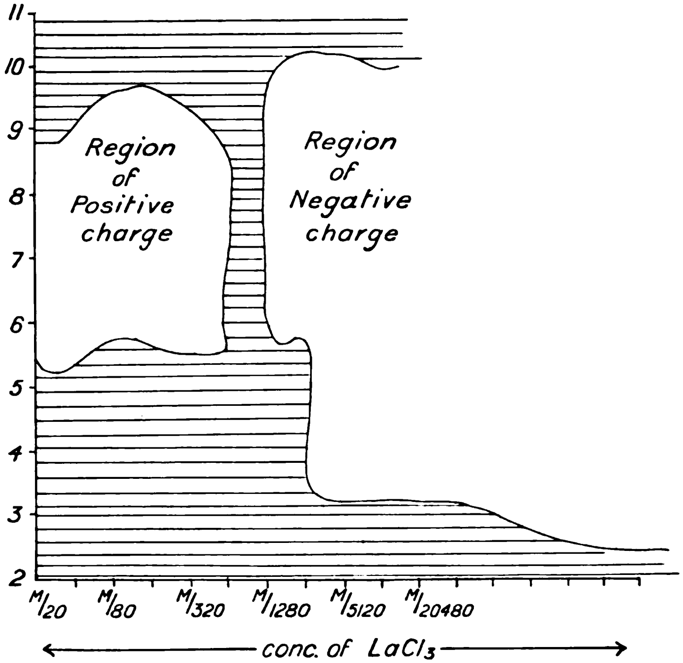

FIG. 5.—$LaCl_3$ and Colloidal Benzoin.

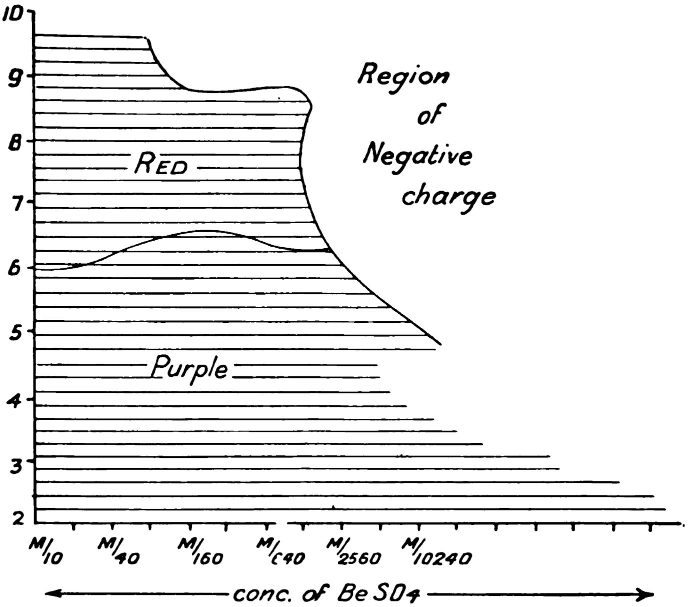

FIG. 6.—$BeSO_4$ and Colloidal Gold.

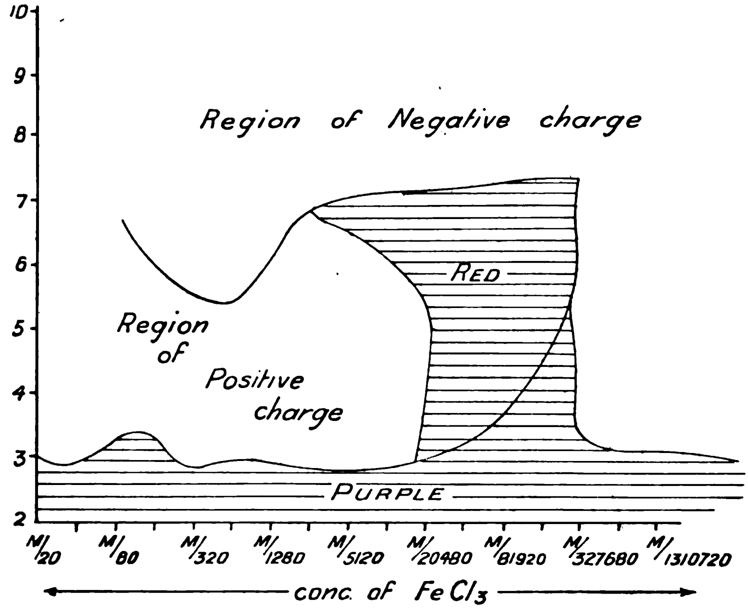

Fig 7.—$FeCl_3$ and Colloidal Gold.

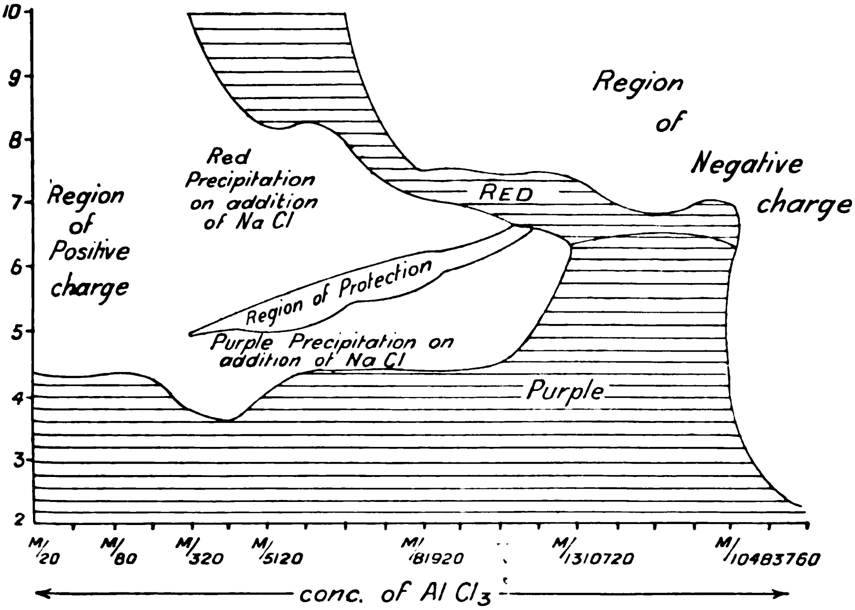

Fig. 8.—$AlCl_3$ and Colloidal Gold.

relation between the degree of hydrolysis and the dilution in the cases of ferric chloride, aluminium chloride, and lanthanum chloride are extracted from Landolt-Börnstein (1923):—

Salt.	Dilution.	Hydrolysis.
$FeCl_3$	33·34 1	37 per cent.
„	666·7 1	91 per cent.
$AlCl_3$	1024·0 1	4·5 per cent.
$LaCl_3$	32·0 1	0·00672 per cent.

The salt hydrolysed in greater degree exerts its power to confer a positive charge on the colloidal particles in more acid solution than does that hydrolysed in a less degree. It would seem, therefore, that this conferring of a positive charge is related to the formation of hydroxide. In the case of ferric chloride, hydroxide is formed in appreciable quantities in acid solution (*pH* 3). In the case of lanthanum chloride, hydroxide is formed only when the reaction is very near the neutral point *pH* 7, or on its alkaline side.

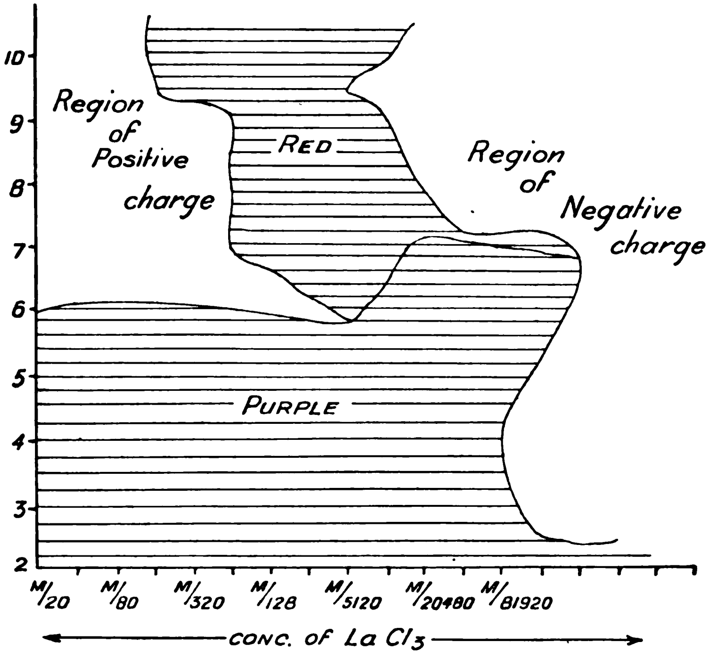

Fig. 9.—$LaCl_3$ and Colloidal Gold.

It would appear, however, that the presence of metallic hydroxide is not alone sufficient to account for the phenomena observed. If this were the case we would expect that on making the solutions more alkaline, the positive charge would remain, and in fact it might be expected to increase. However, considering solutions containing a sufficiently low concentration of salt, there is encountered, after leaving the positively charged region, first of all a zone of precipitation; and then in the case of aluminium or iron, a region where no precipitation occurs, and where the charge is negative. In the case of lanthanum this region is obscured as the insoluble lanthanum hydroxide is precipitated whenever the solution becomes alkaline, thus masking the state of the colloidal gold or "benzoin." It would appear, therefore, that a mere increase in the amount of hydroxide does not ensure stability of the colloidal solution, but that a decrease in stability may actually accompany this. As the amount of hydroxide increases, and with it the *pH*, the number of metallic ions will decrease. The decrease of stability would be accounted for if a certain number of metallic ions were necessary in order that stability should be maintained. The two conditions, then, under which salts of iron, aluminium or lanthanum confer a positive charge on the colloidal particles would appear to be (1) the presence of metallic hydroxide, and (2) the presence of an adequate number of positively charged metallic ions. It is possible that under these conditions a complex between the colloid and the hydroxide is formed, which is associated with a positively charged tervalent ion. The latter confers on the complex a positive charge. This would be analogous to what takes place in the formation of colloidal iron hydroxide, where the presence of the ferric ion would appear to be necessary to render the particles of ferric hydroxide positive, and so stabilise them. These complexes of hydroxide plus ion may be similar in nature to the complexes postulated by Pauli (1920), who suggests that in the case of colloidal iron, for example, a complex $x[\mathrm{Fe(OH)_3}].y[\mathrm{Fe\text{-}An.}]$ exists, where An. denotes the anion. Dissociation of the anion from this complex would give rise to a product containing hydroxide associated with positively charged ions similar to that suggested above. This view that a complex is formed between the colloidal particles and the metallic hydroxide is borne out by the fact that in the positively charged region the particles in suspension are larger than in the negatively charged region, as seen under ultramicroscopic observation, and that in the case of experiments with the gold sol, two types of precipitate were noted. In the more acid regions, as will be seen from the figures, the precipitates were purple, in the more alkaline regions red. In the case of aluminium the change takes place about *pH* 6, in the cases of iron and lanthanum between *pH* 3 and *pH* 5, and between *pH* 6 and *pH* 7 respectively. Again, these figures correspond to the order of the degree of hydrolysis of the salts. This change which occurs in the more dilute solutions of salt is seen to correspond to the zone of non-precipitation in the more concentrated solutions, which in fact separates the purple from the red precipitates in these regions.

It is of great interest to compare with these results those obtained, using salts with bivalent cations. Here there exists no region in which the charge is positive. Even in the case of beryllium, which in certain respects resembles aluminium more than lanthanum does, and also more than it itself resembles calcium, the resulting type of figure is essentially similar to that obtained with calcium. If our views of the explanation of these phenomena in the case of tervalent ions are correct, then either no association of hydroxide and colloid is possible in the case of bivalent ions, or else there occurs no stabilising association of the free cations with such a complex. It would appear to be quite probable that the second hypothesis is true, for instance it is possible by using an alkaline solution of beryllium sulphate to obtain a precipitation of gold, fig. 6. The explanation may be that tervalent ions are associated with a more powerful atomic field of force, and can form such associated complexes more easily than bivalent ions and with much greater ease than univalent ions. A similar cause may be responsible for the great precipitating power of tervalent ions, as the more powerful the atomic field, the more readily would such ions become adsorbed on surfaces, and hence reduce the charge of colloidal particles. It is well known that amongst univalent ions, those which can form complexes, *e.g.* silver, precipitate more readily than, say, the alkali metals.

In some instances, an equal volume of 2 per cent. sodium chloride solution was added to the non-precipitated positively charged colloidal solutions in order to ascertain whether or not they were protected. Slight protection was evident in the case of gold sol and aluminium chloride, and it is of interest to note that the precipitates on either side of this region were red and purple respectively (see fig. 8). A similar addition of sodium chloride was made in the case of "benzoin" sol and ferric chloride (see fig. 3), the only other combination investigated in this respect. According to our views this would be the region where the highest number of positively charged metallic ions exist simultaneously with sufficient hydroxide to form a surface layer. On the more acid side, there is not enough hydroxide to form such a surface layer, as the gold when precipitated separates in the blue condition, whilst on the more alkaline side the number of metallic ions will presumably decrease.

In conclusion, it will be observed that these solutions of amphoteric molecules derived from metallic salts bear considerable resemblance to solutions of gelatin and of similar proteins in their action on lyophobe colloids. In both cases there is a group possessing considerable affinity for water. In one case it is a polypeptide containing many—NHCO—groups, in the other case a trihydroxy derivative of a metal. Both molecules can acquire an electrical charge, in the one case by dissociation of one of its terminal carboxyl groups, or by the addition of a charged hydrogen to a terminal amino group, and in the other case probably by association with a charged ion.

The effect of all varieties of ions and molecules in modifying the properties of interfacial surfaces is of the greatest importance in the interpretation of the most diverse phenomena of colloidal chemistry, and the elucidation of the mechanism of such effects will undoubtedly prove of the greatest value for the understanding more particularly of biological colloid reactions.

CONCLUSIONS.

(1) Salts of tervalent cations are able to confer, under appropriate conditions of concentration and *pH*, a positive charge on the particles of a negatively charged colloidal sol.

(2) Separating the region where the sol particles are positively charged from the region where they are negatively charged is a zone of precipitation.

(3) These zónes are closely related to the degrees of hydrolysis of the salts.

(4) The zone of positive charge may be due to the existence of a molecule of metallic hydroxide which is able to establish itself on the surface between the sol particles and water, and to associate itself with a metallic ion.

(5) No such phenomena have been observed with bivalent ions.

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(*Issued separately April* 29, 1925.)

XI.—The Modes of Vibration of a Stretched Membrane with a Particular Law of Density. By Dr E. L. Ince, M.A.

(MS. received December 20, 1924. Read January 12, 1925.)

§ 1. In this paper it is proposed to study the problem of the vibrations of a uniformly-stretched membrane whose superficial density σ depends upon the distance r from a fixed point in the membrane in accordance with the law

$$\sigma = C(a^2 - r^2),$$

in which C and a are constants. The constant a will be known as the *critical radius* and the circle $r = a$ as the *critical circle.*

The equation of motion which determines the normal displacement ζ is, in polar co-ordinates referred to the fixed point as pole,

$$\frac{\partial^2\zeta}{\partial r^2} + \frac{1}{r}\frac{\partial\zeta}{\partial r} + \frac{1}{r^2}\frac{\partial^2\zeta}{\partial\theta^2} = \frac{\sigma}{T}\frac{\partial^2\zeta}{\partial t^2},$$

where T is the *tension* of the membrane.

It will be supposed that the membrane vibrates with the frequency $p/2\pi$, so that

$$\zeta = z(r, \theta)\sin(pt - \epsilon),$$

which leads to the equation

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2} + \frac{Cp^2}{T}(a^2 - r^2)z = 0.$$

The problem will be stated, not in its usual form, viz. to determine the possible modes of vibration and their frequencies when the physical constants of the membrane are known, but as follows: *to determine the dimensions of a membrane having the given law of density so that the membrane may admit of vibrations in a specified mode and of frequency $p/2\pi$.*

The factor Cp^2/T is of dimensions -4 in length and is to be regarded as known in terms of the unit of length; let the unit of length be chosen so that this factor has the value 4, a value which happens to be convenient. It is also convenient to write $n+1$ for a^2. The partial differential equation upon which the investigation depends is thus brought into the form

$$(1) \quad . \quad . \quad . \quad \frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2} + 4(n + 1 - r^2)z = 0.$$

Particular interest attaches to the case where the law of density is such that n is a positive integer.

CIRCULAR MEMBRANES.

§ 2. The simplest solution of the above equation, of a form appropriate to the discussion of membranes with a circular boundary, is

$$z = e^{-r^2} r^n \cos n(\theta - \theta_0),$$

where θ_0 is arbitrary. This suggests the more general solution

$$(2) \quad . \quad . \quad . \quad . \quad . \quad z = e^{-r^2} r^s \mathrm{R}_{n-s} \cos s(\theta - \theta_0),$$

in which, from physical considerations, s is necessarily a positive integer and R_{n-s} is a function of r alone, which satisfies the ordinary differential equation

$$(3) \quad . \quad . \quad . \quad \frac{d^2\mathrm{R}}{dr^2} + \left\{\frac{2s+1}{r} - 4r\right\}\frac{d\mathrm{R}}{dr} + 4(n-s)\mathrm{R} = 0.$$

The characteristic feature of this equation is that it has a regular singular point, with exponents 0 and $-2s$, at the origin, and an irregular singular point at infinity. It is, in reality, of the type of the confluent hypergeometric equation,* and, in Whittaker's notation,

$$(4) \quad . \quad . \quad . \quad . \quad . \quad \mathrm{R}_{n-s} = e^{r^2} r^{s-1} \mathrm{W}_{\frac{1}{2}n+\frac{1}{2},\, \frac{1}{2}s}(2r^2)$$

is a solution. It follows that the partial differential equation (1) is satisfied by

$$(5) \quad . \quad . \quad . \quad . \quad . \quad z = r^{-1} \mathrm{W}_{\frac{1}{2}n+\frac{1}{2},\, \frac{1}{2}s}(2r^2) \cos s(\theta - \theta_0).$$

In the particular case when $n-s$ is a positive *even* integer, say $2m$, the equation (3) possesses a solution which is a polynomial in r^2 of degree m. Thus, for example,

$$\mathrm{R}_0 = 1, \ \mathrm{R}_2 = 1 - \frac{2}{n-1}r^2,$$

$$\mathrm{R}_4 = 1 - \frac{4}{n-3}r^2 + \frac{4}{(n-2)(n-3)}r^4,$$

$$\mathrm{R}_6 = 1 - \frac{6}{n-5}r^2 + \frac{12}{(n-4)(n-5)}r^4 - \frac{8}{(n-3)(n-4)(n-5)}r^6,$$

and, in general,

$$(6) \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathrm{R}_{2m} = 1 + \sum_{i=1}^{m} \mathrm{A}_i (2r)^i,$$

where

$$\begin{aligned} \mathrm{A}_i &= (-1)^i {}_m\mathrm{C}_i / (s+1)(s+2) \ . \ . \ . \ (s+i) \\ &= (-1)^i {}_m\mathrm{C}_i / (n-2m+i)(n-2m+i-1) \ . \ . \ . \ (n-2m+1). \end{aligned}$$

The theory of algebraic equations shows that the equation

$$\mathrm{R}_{2m} = 0,$$

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* Whittaker and Watson, *Modern Analysis*, chap. xvi.

which determines the radii of the nodal circles, has m distinct positive (non-zero) real roots. Moreover, the greatest root is less than $\sqrt{\frac{n}{2}}$, and therefore all of the m nodal circles lie within the critical circle, whose radius is $\sqrt{(n+1)}$. Thus, when n is a positive integer and $n-s$ is even, there is a possible mode of vibration with s nodal diameters and $\frac{1}{2}(n-s)$ nodal circles.

§ 3. The following are examples of possible cases.

1° $s=n-2$.

The only possible circular boundary is given by

$$r=\sqrt{\frac{n-1}{2}} \qquad (n \geq 2).$$

2° $s=n-4$.

The radius of the larger possible boundary is

$$r=\sqrt{\{\tfrac{1}{2}n-1+\tfrac{1}{2}\sqrt{(n-2)}\}} \qquad (n \geq 4).$$

There is then an interior nodal circle of radius

$$r=\sqrt{\{\tfrac{1}{2}n-1-\tfrac{1}{2}\sqrt{(n-2)}\}}.$$

3° $s=n-6$.

The boundary and the two interior nodal circles are the roots of the equation $R_6=0$. This equation has three real and distinct roots which cannot be obtained explicitly in terms of n.

If the membrane were of uniform density $(n+1)C$, that is to say, if the density at the centre were maintained throughout, the solution corresponding to (2) would be

$$z=J_s\{2r\sqrt{(n+1)}\}\cos s(\theta-\theta_0).$$

The following table gives a numerical comparison between the laws in the particular case $n=6$.

Radii of Boundary and Nodal Circles.

s.	$\sigma=(n+1)C$.	$\sigma=C(n+1-r^2)$.
4	1·434	1·581
2	1·591 ·971	1·732 1·000
0	1·635 1·043 ·454	1·773 1·071 ·461

In the case of the uniform membrane, only those nodal circles are given which correspond, in order, with the possible nodal circles of the non-uniform membrane. The physical character of the non-uniform membrane restricts the number of nodal circles. It is only to be expected that the radii in the non-uniform case are greater than the corresponding radii in the uniform case.

§ 4. It is thus seen that, when n is a positive integer, precise results as to the possible modes of vibration may be obtained. The law of density is, however, of a very specialised form, and it is necessary to generalise the results obtained by removing the restriction that n is a positive even integer. The information which has been obtained in this restricted case will be of use in the treatment of the more general case.

When n, and therefore $n-s$, is unrestricted, the expression for R_{n-s} takes the form of an infinite series:

$$(7) \qquad \mathrm{R}_{n-s}=1-\frac{n-s}{1!\,(s+1)}r^2+\frac{(n-s)(n-s-2)}{2!\,(s+1)(s+2)}r^4-\frac{(n-s)(n-s-2)(n-s-4)}{3!\,(s+1)(s+2)(s+3)}r^6+\ldots$$

Though the problem of determining the precise location of the zeros of R_{n-s} is as yet unsolved, nevertheless a certain amount of information, of a qualitative character, is obtainable.

In the first place, if m is the integer next above $\frac{1}{2}(n-s)$, the terms up to and including the term in r^{2m} alternate in sign, and thereafter remain of one sign. Consequently the equation

$$\mathrm{R}_{n-s}=0$$

has at most m real positive roots if

$$2m \geq n-s > 2m-2,$$

and it has been shown that for that range of values of $n-s$ there are precisely m positive roots.*

There remains the question as to whether or not these m roots lie below the critical value $r=\sqrt{(n+1)}$. This question is attacked by an appeal to the Sturmian theory.†

By means of the transformation

$$\mathrm{R}=e^{r^2}r^{-s-\frac{1}{2}}w,$$

equation (3) is brought into the normal form

$$(8) \quad . \quad . \quad . \quad . \quad . \quad \frac{d^2w}{dr^2}+\left\{4n+4-4r^2-\frac{s^2-\frac{1}{4}}{r^2}\right\}w=0,$$

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* A. Milne, *Proc. Edin. Math. Soc.*, xxxiii (1915), p. 51.

† Bôcher, *Leçons sur les méthodes de Sturm*, chap. iii., especially § 14.

and, except for possible zeros at the origin, the zeros of R_{n-s} are identical with those of w_{n-s}. The coefficient of w in this equation steadily decreases as s increases, and for any fixed value of s ($s > \frac{1}{2}$) tends to the limit $-\infty$ as r tends to zero. Consequently if, for any given value of s, say s_1, r_i and r_{i+1} are consecutive zeros of w, then for any greater value of s, say s_2, there will not be more than one zero of w between r_i and r_{i+1}. Also if, when $s = s_1$, the positive zeros are arranged as an increasing sequence $r_1, r_2, r_3, \ldots$ and when $s = s_2$ the zeros are similarly arranged in the sequence $r_1', r_2', r_3', \ldots$ then

$$r_i' > r_i.$$

These facts may concisely be summed up in the statement that *n being fixed, an increase in s causes the real roots to move away from the origin.* When $s = n - 2m$ there are m positive zeros of R_{n-s} all within the range $r^2 < n + 1$. As s increases beyond this value the zeros steadily increase until when $s = n - 2m + 2$ the largest root becomes infinite and thus disappears, leaving $m - 1$ positive zeros all less than the critical value. This shows that when

$$n - 2m + 2 > s \geq n - 2m$$

the $m - 1$ smallest positive zeros of R_{n-s} remain within the critical circle $r^2 = n + 1$ and correspond to nodes which are physically possible, whilst at a certain value of s in the interval the largest zero passes out of the critical circle and thus ceases to be a value of r for which there is a physically possible node. It is not difficult to find, for fairly small values of n and m, the corresponding numerical value of s for which the critical circle is a nodal circle, but it has not been found possible to obtain even an approximately correct general formula indicating that ranges of values of s for which all of the m zeros correspond to actual nodes.

ELLIPTIC MEMBRANES.

§ 5. It will now be supposed that, the law of density being as before, the boundary of the membrane and the closed nodal lines are confocal ellipses. This problem is particularly interesting because in no previous case have definite results been obtained for an elliptic boundary. It is true that, as in the case of the circular boundary, the law of density has to be specially adjusted to give solutions which can be manipulated, that is to say solutions in a finite form, but the knowledge which can be obtained when there is a particular relationship between the constants in the law of density throws light upon the more general case. A point worthy of note is that in this case the nodes, which are confocal ellipses, do not coincide with the lines of equal density, which are concentric circles.

In Cartesian co-ordinates, with the scale chosen as before, the fundamental equation is

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}+4(n+1-x^2-y^2)z=0.$$

To bring this equation into a form suited to the study of membranes with elliptic boundaries, a change to confocal co-ordinates is made, by means of the substitution

$$\text{(9)} \quad . \quad . \quad . \quad . \quad . \quad . \quad \begin{cases} x=\sqrt{(\tfrac{1}{2}l)}\cosh u\cos v, \\ y=\sqrt{(\tfrac{1}{2}l)}\sinh u\sin v. \end{cases}$$

The equation then becomes

$$\text{(10)} \quad . \quad \frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}+\{(n+1)l(\cosh 2u-\cos 2v)-\tfrac{1}{8}l^2(\cosh 4u-\cos 4v)\}z=0.$$

If now

$$\text{(11)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad z=\mathrm{U}(u)\mathrm{V}(v),$$

there arises the pair of ordinary linear equations

$$\text{(12}a\text{)} \quad . \quad . \quad \frac{d^2\mathrm{U}}{du^2}+\{-\mathrm{A}+\tfrac{1}{8}l^2+(n+1)l\cosh 2u-\tfrac{1}{8}l^2\cosh 4u\}\mathrm{U}=0,$$

$$\text{(12}b\text{)} \quad . \quad . \quad \frac{d^2\mathrm{V}}{dv^2}+\{\mathrm{A}-\tfrac{1}{8}l^2-(n+1)l\cos 2v+\tfrac{1}{8}l^2\cos 4v\}\mathrm{V}=0,$$

in which the constant A has to be determined. The Floquet theory * of linear differential equations with periodic coefficients shows that, for an infinite set of discrete values of A, equation (12b) possesses the purely periodic solutions required by the physical problem. For any value of A in this set, but for no other values of A, equation (12a) admits of a solution which is one-valued in sinh u or in cosh u.

The equation (12b) appears incidentally in a paper by Whittaker † on the solution of linear differential equations by means of integral equations; a study of the conditions under which solutions of period π or 2π are possible was recently made by the present author. ‡

The following notation will be adopted. The periodic solutions of (12b) are even or odd. An even periodic solution of period 2π which reduces to a multiple of cos mv when $l=0$ will be denoted by $\mathrm{C}_n{}^m(v)$; the corresponding solution of (12a) will be written $\mathrm{Ch}_n{}^m(u)$, so that

$$\mathrm{Ch}_n{}^m(u)=\mathrm{C}_n{}^m(iu).$$

Similarly an odd solution of (12b), of period 2π, which reduces to a

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* *Ann. Éc. Norm.* (2), xiii (1883), p. 47.

† *Proc. Edin. Math. Soc.*, xxxiii (1914), p. 22.

‡ *Proc. Lond. Math. Soc.* (2), xxiii (1924), pp. 56–74.

multiple of sin mv when $l=0$, will be denoted by $S_n{}^m(v)$, and the corresponding solution of (12a) by $Sh_n{}^m(u)$, so that

$$Sh_n{}^m(u) = -iS_n{}^m(iu).$$

Except in the trivial case $l=0$, the equation (12b) cannot have at the same time an even and an odd periodic solution.* In fact, if it has a periodic solution, the second solution corresponding to it is definitely non-periodic.

The problem under consideration resolves itself into the determination of the zeros of U, that is of $Ch_n{}^m(u)$ or of $Sh_n{}^m(u)$ and of V, that is of $C_n{}^m(v)$ and of $S_n{}^m(v)$. The roots of the equation $U=0$ give rise to elliptic nodes, those of $V=0$ to hyperbolic nodes. An elliptic node, say $u=a$, will lie wholly within the critical circle $r=\surd(n+1)$ and therefore have a physical significance if, and only if,

$$(13) \quad . \quad . \quad . \quad . \quad . \quad \tfrac{1}{2}l\cosh^2 a < n+1.$$

The functions $C_n{}^m(v)$ and $S_n{}^m(v)$ really fall into four distinct classes, each having its own characteristic features, as follows:

(a) $S_n{}^m(v)$, m odd. The solution is expressible as a series which contains only sines of odd multiples of v. The major axis, $v=0$, of the confocal system is a node.

(b) $C_n{}^m(v)$, m odd. The series-solution involves only cosines of odd multiples of v. The minor axis, $v=\frac{1}{2}\pi$, is a node.

(c) $S_n{}^m(v)$, m even. The series-solution involves only sines of even multiples of v. Both axes are nodes.

(d) $C_n{}^m(v)$, m even. The solution proceeds as a series of cosines of even multiples of v. Neither axis is, in general, a node.

§ 6. The cases of special interest, when the solutions are of a finite form arise when m and n are integers of the same parity and $m \leq n$. Thus when n is an odd positive integer, say $2p-1$, the p solutions

$$C_n{}^1(v),\ C_n{}^3(v),\ .\ .\ .\ ,\ C_n{}^n(v)$$

are expressible in a finite form; let the corresponding values of A be

$$A_n{}^1,\ A_n{}^3,\ .\ .\ .\ ,\ A_n{}^n.$$

The p solutions

$$S_n{}^1(v),\ S_n{}^3(v),\ .\ .\ .,\ S_n{}^n(v)$$

are of a finite form; the corresponding values of A will be denoted by

$$B_n{}^1,\ B_n{}^3,\ .\ .\ .,\ B_n{}^n.$$

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* Ince, *loc. cit.*, p. 65.

In the same way, when n is an even positive integer, say $2p$, the $p+1$ solutions

$$C_n^0(v),\ C_n^2(v),\ \ldots,\ C_n^n(v)$$

corresponding to the characteristic constants

$$A_n^0,\ A_n^2,\ \ldots,\ A_n^n,$$

and the p solutions

$$S_n^2(v),\ S_n^4(v),\ \ldots,\ S_n^n(v)$$

corresponding to

$$B_n^2,\ B_n^4,\ \ldots,\ B_n^n,$$

respectively, are of a finite form.

With each such function $C_n^m(v)$ or $S_n^m(v)$ is associated the corresponding function $Ch_n^m(u)$ or $Sh_n^m(u)$ respectively. The problem is to determine firstly the number of the zeros of the latter functions, and then to settle whether or not these zeros correspond to physically-possible nodes.

§ 7. The least integer value of n which is of any interest is 2. There are then two cases in which even solutions of a finite form arise, viz.:

$$\text{(i)}\ A_2^0 = 2 - 2\sqrt{(1+l^2)},$$
$$Ch_2^0(u) = e^{-\frac{1}{2}l\cosh 2u}[l + \{1 - \sqrt{(1+l^2)}\}\cosh 2u];$$
$$\text{(ii)}\ A_2^2 = 2 + 2\sqrt{(1+l^2)},$$
$$Ch_2^2(u) = e^{-\frac{1}{2}l\cosh 2u}[l + \{1 + \sqrt{(1+l^2)}\}\cosh 2u].$$

$Ch_2^2(u)$ has no real zero; no possible elliptic membrane corresponds to this case. $Ch_2^0(u)$, on the other hand, has one real and positive zero defined by

$$\cosh 2u = \{1 + \sqrt{(1+l^2)}\}/l.$$

But since it is necessary that the parameter α of any actual elliptic node (or boundary) should satisfy the inequality

$$l\cosh^2\alpha \leq 2(n+1)$$

it follows that

$$l \leq \tfrac{60}{11}.$$

The number l depends upon the distance between the foci of the confocal nodal system, thus, if S and S_1 are the foci

$$SS_1^2 = 2l.$$

Also, if e is the eccentricity, then $e = \operatorname{sech}\alpha$.

Thus when $l \leq \frac{60}{11}$, there is a mode of vibration, in the given frequency, where the boundary is an ellipse the squares of whose semi-axes major and minor are respectively $\frac{1}{4}\{1+l+\sqrt{(1+l^2)}\}$ and $\frac{1}{4}\{1-l+\sqrt{(1+l^2)}\}$. The square of the eccentricity is $1+l-\sqrt{(1+l^2)}$, and therefore the maximum eccentricity is $\sqrt{\frac{10}{11}} = {\cdot}95$.

When $n=2$, there is one odd solution expressible in a finite form, viz. when $B_2^2=4$,

$$\mathrm{Sh}_2^2(u)=e^{-\frac{1}{2}l\cosh 2u}\sinh 2u,$$

but it is of no physical interest.

§ 8. The case $n=3$ gives some idea of the vibrations which correspond to solutions involving odd multiples of the argument. There are two even and two odd solutions in a finite form. The even solutions are

$$\text{(i}a\text{)}\quad A_3^1=5+l-2\sqrt{(4-2l+l^2)},$$
$$\mathrm{Ch}_3^1(u)=e^{-\frac{1}{2}l\cosh 2u}[3l\cosh u+\{4-l-2\sqrt{(4-2l+l^2)}\}\cosh 3u]\,;$$
$$\text{(ii}a\text{)}\quad A_3^3=5+l+2\sqrt{(4-2l+l^2)},$$
$$\mathrm{Ch}_3^3(u)=e^{-\frac{1}{2}l\cosh 2u}[3l\cosh u+\{4-l+2\sqrt{(4-2l+l^2)}\}\cosh 3u].$$

$\mathrm{Ch}_3^3(u)$ has no real zero; $\mathrm{Ch}_3^1(u)$ has one real positive zero $u=a$ given by

$$\cosh 2a=\{2+\sqrt{(4-2l+l^2)}\}/l.$$

The necessary condition

$$\cosh^2 a\leq 8$$

implies

$$l\leq\tfrac{96}{13}$$

and the maximum eccentricity is $\sqrt{\tfrac{12}{13}}=\cdot 96$. In general the square of the eccentricity is $\tfrac{1}{3}\{2+l-\sqrt{(4-2l+l^2)}\}$.

Thus there is a mode of vibration in which the boundary is the ellipse $u=a$, the only node is the minor axis.

The two odd solutions are

$$\text{(i}b\text{)}\quad B_3^1=5-l-2\sqrt{(4+2l+l^2)},$$
$$\mathrm{Sh}_3^1(u)=e^{-\frac{1}{2}l\cosh 2u}[3l\sinh u+\{4+l-2\sqrt{(4+2l+l^2)}\}\sinh 3u]\,;$$
$$\text{(ii}b\text{)}\quad B_3^3=5-l+2\sqrt{(4+2l+l^2)},$$
$$\mathrm{Sh}_3^3(u)=e^{-\frac{1}{2}l\cosh 2u}[3l\sinh u+\{4+l+2\sqrt{(4+2l+l^2)}\}\sinh 3u].$$

$\mathrm{Sh}_3^3(u)$ has no real zero except $u=0$; $\mathrm{Sh}_3^1(u)$ has, as well as $u=0$, the real positive zero $u=a$ given by

$$\cosh 2a=\{2+\sqrt{(4+2l+l^2)}\}/l,$$

and this zero has a physical significance when

$$l\leq\frac{32}{5}.$$

In this case the square of the eccentricity is $2+l-\sqrt{(4+2l+l^2)}$. The maximum possible eccentricity is $\sqrt{\tfrac{4}{5}}=\cdot 89$.

The node of this mode of vibration is the major axis.

§ 9. In the case $n=4$, the three characteristic values of A, namely A_4^0,

A_4^2, and A_4^4, which lead to solutions in a finite form, are the three roots of the cubic equation

$$\eta^3 - 20\eta^2 + (64 - 16l^2)\eta + 192l^2 = 0.$$

This equation cannot be solved explicitly in terms of l, but only numerically for given values of l. As a particular instance, let $l = 1$. Then

$$\text{(i)}\quad A_4^0 = -2{\cdot}057.$$

$Ch_4^0(u)$ has two real positive zeros, both of which are admissible: these are given by

$$\cosh 2a_1 = 7{\cdot}457 \qquad (e = {\cdot}486),$$
$$\cosh 2a_2 = 1{\cdot}726 \qquad (e = {\cdot}857).$$

$C_4^0(v)$ has no real zeros. In this mode of vibration $u = a_1$ is the elliptic boundary and $u = a_2$ is a confocal elliptic node. There are no other nodes.

$$\text{(ii)}\quad A_4^2 = 5{\cdot}712.$$

$Ch_4^2(u)$ has one real positive zero, given by

$$\cosh 2a = 5{\cdot}778 \qquad (e = {\cdot}543),$$

and $C_4^2(v)$ has one zero β in the range $0 \leq v < \pi$, given by

$$\cos 2\beta = -{\cdot}560.$$

This mode of vibration has the elliptic boundary $u = a$ and the confocal hyperbolic node $v = \beta$.

$$\text{(iii)}\quad A_4^4 = 16{\cdot}345.$$

$Ch_4^4(u)$ has no real zero; there is consequently no possible elliptic boundary.

When $n = 4$ there arise two cases where an odd solution is expressible in a finite form. These cases are as follows:

$$\text{(i)}\quad B_4^2 = 10 - 2\sqrt{(9 + l^2)},$$
$$Sh_4^2(u) = e^{-\frac{1}{2}l\cosh 2u}\left[2l\sinh 2u + \{3 - \sqrt{(9 + l^2)}\}\sinh 4u\right];$$
$$\text{(ii)}\quad B_4^4 = 10 + 2\sqrt{(9 + l^2)},$$
$$Sh_4^4(u) = e^{-\frac{1}{2}l\cosh 2u}\left[2l\sinh 2u + \{3 + \sqrt{(9 + l^2)}\}\sinh 4u\right].$$

$Sh_4^4(u)$ has no real zero except $u = 0$ and therefore has no physical reality. $Sh_4^2(u)$, on the other hand, has one real positive zero $u = a$, given by

$$\cosh 2a = \{3 + \sqrt{(9 + l^2)}\}/l,$$

and this zero corresponds to an admissible elliptic boundary, the square of whose eccentricity is $\frac{1}{3}\{l + 3 - \sqrt{(9 + l^2)}\}$, when

$$l \leq \tfrac{140}{17}.$$

The only real zeros of $S_4^2(v)$ are $v=0$ and $v=\frac{1}{2}\pi$; these correspond to nodes along the major and minor axes respectively.

§ 10. When $n=5$ there are three determinations of A to which correspond even solutions, and three of B to which correspond odd solutions expressible in a finite form. In the case of the even solutions, $\eta=\mathrm{A}$ satisfies the equation

$$\eta^3-(35+3l)\eta^2+(259+102l-13l^2)\eta-(225+675l-205l^2-15l^3)=0.$$

It is not possible to obtain A explicitly in terms of l; as an illustration take, as before, $l=1$. Then

$$\text{(i)}\ \mathrm{A}_5^1=2{\cdot}686.$$

$\mathrm{Ch}_5^1(u)$ has two real positive zeros, namely $u=\alpha_1$ and $u=\alpha_2$ where

$$\cosh 2\alpha_1=9{\cdot}244 \qquad (e={\cdot}442),$$
$$\cosh 2\alpha_2=2{\cdot}412 \qquad (e={\cdot}766).$$

$\mathrm{C}_5^1(v)$ has no zero except $v=\frac{1}{2}\pi$. If $u=\alpha_1$ be taken as the fixed boundary of the membrane, there are two nodes, viz. the confocal ellipse $u=\alpha_2$ and the minor axis $v=\frac{1}{2}\pi$.

$$\text{(ii)}\ \mathrm{A}_5^3=10.$$

$\mathrm{Ch}_5^3(u)$ has one real positive zero only, viz. $u=\alpha$ where

$$\cosh 2\alpha=7{\cdot}873 \qquad (e={\cdot}475),$$

giving the elliptic boundary. The nodes are the minor axis and the confocal hyperbola $v=\beta$, where β is the one zero of $\mathrm{C}_5^3(v)$ in the range $0\leq v<\pi$, and

$$\cos^2\beta={\cdot}5635.$$

$$\text{(iii)}\ \mathrm{A}_5^5=25{\cdot}314.$$

$\mathrm{Ch}_5^5(u)$ has no real zeros; there is no possible elliptic boundary.

In the case of the odd solutions, $\eta=\mathrm{B}$ satisfies the same equation as $\eta=\mathrm{A}$ above, but with the sign of l reversed. When $l=1$.

$$\text{(i)}\ \mathrm{B}_5^1=-2{\cdot}696.$$

$\mathrm{Sh}_5^1(u)$ has two real positive zeros α_1 and α_2 where

$$\cosh 2\alpha_1=10{\cdot}120 \qquad (e={\cdot}424),$$
$$\cosh 2\alpha_2=\ 3{\cdot}228 \qquad (e={\cdot}688).$$

$\mathrm{S}_5^1(v)$ has no zero but $v=0$. The elliptic boundary may be taken to be $u=\alpha_1$, there is then the confocal elliptic node $u=\alpha_2$ and a node along the major axis.

$$\text{(ii)}\ \mathrm{B}_5^3=9{\cdot}384.$$

$\mathrm{Sh}_5^3(u)$ has one real positive zero $u=\alpha$, where

$$\cosh 2\alpha=7{\cdot}998 \qquad (e={\cdot}471).$$

When $u=a$ is taken as the boundary, the nodes are the zeros of $S_5^3(v)$, that is to say, the confocal hyperbolic node $v=\beta$, where

$$\cos^2\beta=\cdot155$$

and the major axis $v=0$.

$$\text{(iii)}\ B_5^5=25\cdot312.$$

$Sh_5^5(u)$ has no real zero, and consequently no membrane is possible.

§ 11. In the following table are tabulated the results in the particular case where $l=1$. The symbols a, b, and e have the significance usually attributed to them in the geometry of the central conics and relate in columns 3–5 to the bounding ellipse, and in column 6 to the nodal ellipses or hyperbolæ.

m.	*n.*	*a.*	*b.*	*e.*	Nodes.
0	2	·924	·595	·765	None.
1	3	1·087	·826	·650	Minor axis.
1	3	1·190	·955	·595	Major axis.
0	4	1·454	1·271	·486	Ellipse $a=\cdot825$, $b=\cdot426$, $e=857$.
2	4	1·302	1·092	·543	Hyperbola $a=\cdot302$, $b=\cdot639$, $e=2\cdot337$.
2	4	1·339	1·136	·528	Major and minor axes.
1	5	1·601	1·436	·442	Ellipse $a=\cdot923$, $b=\cdot594$, $e=\cdot766$ and minor axis.
3	5	1·489	1·311	·475	Hyperbola $a=\cdot531$, $b=\cdot467$, $e=1\cdot332$ and minor axis.
1	5	1·667	1·510	·424	Ellipse $a=1\cdot028$, $b=\cdot746$, $e=\cdot688$ and major axis.
3	5	1·500	1·323	·471	Hyperbola $a=\cdot278$, $b=\cdot650$, $e=2\cdot540$ and major axis.

These figures may be compared with the radii of the circular boundary and concentric nodes in the case $l=0$. The dimensions in these cases are given in the following table:—

n.	*r.*	Nodes.
2	·707	None.
3	1·000	A diameter.
4	1·306	Concentric circle $r=\cdot541$.
4	1·225	Two perpendicular diameters.
5	1·542	Concentric circle $r=\cdot796$ and a diameter.
5	1·414	Three diameters equally spaced.

§ 12. The above particular examples indicate the nature of the vibrations which may be expected to occur in more general cases. The ease with which these simple cases can be manipulated arises out of the possibility of expressing the solutions in a finite form when n and m are positive integers of the same parity. When m is a positive integer, but n is not an integer of the same parity as m, the functions $Ch_n^m(u)$ and

$\mathrm{Sh}_n{}^m(u)$ are apparently expressible only in an infinite form. In such cases as these, quantitative results cannot yet be hoped for. But by the application of the Sturmian theory some descriptive results are obtainable.

For small values of l, at least, $\mathrm{Ch}_n{}^m(u)$ and $\mathrm{Sh}_n{}^m(u)$ have real positive zeros equal in number to the integer next above $\frac{1}{2}(n-m)$, and $\mathrm{C}_n{}^m(v)$ and $\mathrm{S}_n{}^m(v)$ have m zeros in the interval $0 \leq v < 2\pi$. The former zeros correspond to elliptic nodes, which will all be physically possible for sufficiently small values of l; the latter correspond to hyperbolic nodes, including, it may be, the major and minor axes.

Let m be fixed, thus fixing the number of possible hyperbolic nodes. Then, according to the Sturmian theory, an increase in n causes the positive zeros of $\mathrm{Ch}_n{}^m(u)$ and $\mathrm{Sh}_n{}^m(u)$ to diminish, and the corresponding nodes to contract, the eccentricity meanwhile increasing. At a certain point, which depends upon l, a zero at infinity appears and travels, for increasing n, towards the origin. This zero corresponds to a theoretical nodal ellipse which for a certain value of n, depending upon l, is wholly within the critical circle and therefore has a physical reality.

RECTANGULAR MEMBRANES.

§ 13. The case in which the boundary of the membrane is a rectangle deserves mention on account of its simplicity. In the equation in Cartesian co-ordinates

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}+4(n+1-x^2-y^2)z=0 \tag{14}$$

let $z=\mathrm{X}(x)\mathrm{Y}(y)$, then

$$\frac{d^2\mathrm{X}}{dx^2}+4(p+\tfrac{1}{2}-x^2)\mathrm{X}=0, \tag{15a}$$

$$\frac{d^2\mathrm{Y}}{dy^2}+4(n-p+\tfrac{1}{2}-y^2)\mathrm{Y}=0, \tag{15b}$$

where p is any number, and therefore a solution is

$$z=\mathrm{D}_p(2x)\mathrm{D}_{n-p}(2y),$$

where D represents a parabolic-cylinder function* which is finite at the origin.

When p is a positive even integer, $\mathrm{D}_p(2x)$ has $\frac{1}{2}p$ positive and $\frac{1}{2}p$ negative zeros; when p is odd $\mathrm{D}_p(2x)$ has $\frac{1}{2}(p-1)$ positive, $\frac{1}{2}(p-1)$ negative zeros, and the zero $x=0$. If x_1 and y_1 are the greatest zeros of $\mathrm{D}_p(2x)$ and $\mathrm{D}_{n-p}(2y)$, the condition $x_1{}^2+y_1{}^2<n+1$ is satisfied whenever p and $n-p$ are positive integers. All vibrations which arise theoretically in these cases are practically possible.

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* Whittaker and Watson, *Modern Analysis*, § 16·5.

§ 14. As an illustration, consider the case $n=6$. When p is integral, $D_p(2x)$ has no zero, except $x=0$, unless $p \geq 2$. The only forms to be considered are

$$\text{(i)}\ z=D_2(2x)D_4(2y) \quad \text{and} \quad z=D_4(2x)D_2(2y);$$
$$\text{(ii)}\ z=D_3(2x)D_3(2y).$$

In the first case, $D_2(2x)$ has two zeros, namely, $x=\pm\frac{1}{2}$, and $D_4(2y)$ has four zeros, namely, $y=\pm\frac{1}{2}\sqrt{(3\pm\sqrt{6})}$, that is

$$y=\pm 1{\cdot}167,\ y=\pm{\cdot}371.$$

The bounding rectangle can be taken as $x=\pm\frac{1}{2}$, $y=\pm 1{\cdot}167$; and there are then two internal nodes, $y=\pm{\cdot}371$. In the second case, the boundary is $x=\pm\frac{1}{2}\sqrt{3}$, $y=\pm\frac{1}{2}\sqrt{3}$, and the nodes are $x=0$, $y=0$.

To take a more advanced case, let $n=9$. The following distinct cases then arise:

$$\text{(i)}\ z=D_2(2x)D_7(2y).$$

Boundary: $x=\pm\frac{1}{2}$, $y=\pm 1{\cdot}87$. Nodes: $y=\pm 1{\cdot}18$, $y=\pm{\cdot}58$, $y=0$.

$$\text{(ii)}\ z=D_3(2x)D_6(2y).$$

Boundary: $x=\pm\frac{1}{2}\sqrt{3}$, $y=\pm 1{\cdot}66$. Nodes: $x=0$, $y=\pm{\cdot}94$, $y=\pm{\cdot}31$.

$$\text{(iii)}\ z=D_4(2x)D_5(2y).$$

Boundary: $x=\pm 1{\cdot}17$, $y=\pm 1{\cdot}43$. Nodes: $x=\pm{\cdot}37$, $y=\pm{\cdot}68$, $y=0$.

§ 15. When p and $n-p$ are positive, but not integral, it is convenient to replace the functions D by one or other of the functions* E or O, where

$$E_p(2x)=e^{-x^2}\left\{1-\frac{p}{2!}(2x)^2+\frac{p(p-2)}{4!}(2x)^4-\frac{p(p-2)(p-4)}{6!}(2x)^6+\ \ldots\right\},$$

$$O_p(2x)=e^{-x^2}\left\{2x-\frac{p-1}{3!}(2x)^3+\frac{(p-1)(p-3)}{5!}(2x)^5-\frac{(p-1)(p-3)(p-5)}{7!}(2x)^7+\ldots\right\}.$$

E or O is taken according as a node is not or is desired along the line of symmetry.

When p is an even integer, $2k$, E_p has k positive and k negative zeros. An increase in p causes each zero to travel towards the origin (and the corresponding nodes to approach the axes), and an additional pair of zeros appears. For a value of p just above $2k$, the modulus of the new zeros is large, as p increases the modulus diminishes. When p has increased to $p+1$, this new pair of zeros has progressed so far towards the origin as to correspond to nodes within the critical circle $r^2=n+1$. Thus θ being a definite number between 0 and 1, dependent upon k, when $2k \leq p < 2k+\theta$,

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* Milne, *Proc. Edin. Math. Soc.*, xxxiii (1915), p. 59.

$E_p(2x)$ has $2k$ zeros which are relevant (*i.e.* correspond to nodes or boundary within the critical circle). When $2k+\theta<p\leq 2k+2$, there are $2k+2$ relevant zeros.

Precisely similar results hold with regard to O_p.

Thus for any given value of n there is an infinite set of rectangular membranes which vibrate in the given frequency. They vary from the square membrane which corresponds to the solution

$$z=E_{\frac{1}{2}n}(2x)E_{\frac{1}{2}n}(2y)$$

to the elongated membrane whose corners are on the critical circle, and which correspond to

$$z=E_a(2x)E_{n-a}(2y),$$

where a is a definite number dependent upon n and $0<a<1$. These membranes have no nodes along the axes. Such nodes are introduced by replacing one or both of the functions E by a function O.

§ 16. Solutions of the partial differential equations (1), (11), and (14), of a general character, are given by the present author in a paper recently published by this Society (*Proceedings*, vol. xliv, pp. 242–247). The present paper arose through a desire to obtain a physical application of the class of equations therein discussed.

(*Issued separately May* 15, 1925.)

XII.—A Survey of Clyde Plankton. By Sheina M. Marshall, B.Sc., Assistant Naturalist, Millport Marine Biological Station. *Communicated by* Professor J. GRAHAM KERR, F.R.S.

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INTRODUCTION.

THE tow-nettings on which this survey is based were taken mostly from Keppel Pier. The tide as a rule runs down the channel strongly for several hours after high water, and the current is sufficiently strong to hold out the tow-nets more or less horizontal. Owing to some peculiarity in the configuration of the sea-floor at Keppel, the downward current is much the stronger and often begins an hour or several hours before high water. The flood-tide is never strong enough nor constant enough to hold out a tow-net for any length of time, and so most of the observations are based on tow-nettings taken on the ebb. Tow-nettings were taken almost every day and the tow-nets were usually left out for an hour.

Comparisons made of tow-nettings taken at the pier and elsewhere show that the former are fairly representative, not only of the water in the channel outside, but also of the surface water in the open parts of the Clyde area generally. They cannot, however, be taken as representative of any particular depth.

The tow-nets used were of two types, one with thirty meshes to the linear inch and the other with one hundred and twenty meshes. Although the finer tow-net is not fine enough to retain the smallest organisms, a certain proportion of these are generally found in it, probably entangled among the diatoms of the catch. The tow-nettings were supplemented by water-samples taken at the same time and centrifuged according to the method described by Dr Lebour, 1917 (1).

There has been little continuous work done on the plankton of the Firth of Clyde. In 1902 and part of 1901 a series of weekly tow-nettings were taken from off Keppel Pier, and from these were prepared the "Report on the Medusæ of the Firth of Clyde," by E. T. Browne, and the "Report on the Free-swimming Crustacea found in the Firth of Clyde," by T. Scott, both of which appeared in 1905. In a second short paper E. T. Browne gives notes of some of the larger planktonic organisms apart from the Medusæ, but as the tow-nets used were of comparatively coarse mesh (50–80 strands to the inch) probably most of the smaller forms were lost. Diatoms are noticed as being "very thick" in July, and abundant in June and August. The seasonal distribution of the larger animals agrees well with the present results. Some observations were also made by Sir John Murray during his cruises in the "Medusa," 1884–1892, and by Sir William Herdman in Loch Fyne and the Kilbrennan Sound, 1907–1910.

A summary of the plankton month by month in 1923 is given at the end of this paper. The general course of events was the same in both years—1923, 1924—and is similar to that at other marine stations. There is a winter minimum followed in early spring by a large increase in the number of diatoms and of larval forms (*e.g.* copepod, cirripede, molluscan). This is followed in May and June by a period poor in diatoms but very rich in crustacean life, particularly in *Calanus finmarchicus*. July, August, and September are the months richest in variety during the year—diatoms are abundant, peridinians reach their maximum, larvæ of all kinds are numerous, as are also some of the larger metazoa such as Medusæ and Chætognatha. The autumnal diatom maximum is in August or September, and the number and variety of planktonic organisms thereafter decrease.

There were several noteworthy differences between 1923 and 1924. In July 1923 the diatom *Rhizosolenia fragillima* was very numerous and seemed to form a summer maximum between the spring and autumn rises of *Skeletonema* and *Thalassiosira*. In 1924, although present at the same time, it was much less numerous. In August, however, *Chætoceros sociale* became very abundant and can be said to have formed the autumn maximum, for although *Skeletonema* and *Thalassiosira* increased in numbers in the last half of September they were much less important.

Among the peridinians, *Peridinium ovatum* was much more common all through 1924 than in 1923, and *Ceratium furca*, which had been comparatively unimportant in 1923, appeared in large numbers in the end of August and September and remained throughout the winter.

In the autumn of 1924 radiolarians were very scarce. This was possibly due in part to the stormy weather, for they were slightly more numerous in deep water outside.

It is interesting to compare these observations with the results of plankton work elsewhere. The succession of forms throughout the year is in general closely similar. Judging from the two years surveyed here, the spring diatom maximum appears to come earlier and to disappear earlier than at the other British stations. The summer months, July and

August, which at Port Erin are poor in diatoms, are here rich; Plymouth also is poor, although the poverty in summer is apparently less marked there. The peridinian maximum occurs in summer at all three stations, although it is later here than elsewhere.

Calanus is at its maximum here in May and June, but at Port Erin not until July and August, while at Plymouth it is common off and on all through the summer.

Certain species occur at all three stations although in varying abundance, *e.g.* *Skeletonema* is a rare visitor at Port Erin, but here and at Plymouth it is one of the most important diatoms which occur. It has been stated that it prefers water of low salinity, and this may account for its abundance in the Clyde. It is the most important diatom at Kiel where again the salinity is low.

At Port Erin the smaller Peridiniales have not been investigated, but it is interesting to find that several of the new species described by Dr Lebour from Plymouth are found also in the Clyde. Many of the small forms described by Lohmann from Kiel appear here too, some of which occur at Plymouth while others do not. A distinguishing feature of the Clyde winter plankton is the presence of considerable numbers of radiolarians.

I. SYSTEMATIC.

DIATOMS.

Paralia sulcata.—Occurs in small quantities throughout the winter and spring but is never abundant.

Skeletonema costatum.—The most important diatom here. It occurs in enormous quantities in spring (at the end of March and beginning of April) and again in autumn. In 1923 the period of abundance was much longer than in 1924. In the autumn of 1924 it increased considerably in numbers but never rose to such abundance as in the previous year. It is present all the year round but is very scarce in winter and early summer.

Thalassiosira.—This has much the same seasonal distribution as *Skeletonema*, appearing in large quantities in spring and autumn. It remains longer in both cases, being more numerous in June and late autumn.

T. nordenskioldi and *T. gravida* are the more important species.

Lauderia borealis.—Occurs during the spring and autumn maxima but is never abundant. It is present in small numbers throughout the summer.

Leptocylindricus danicus.—Occurs at the spring maximum and again in late summer. Its maximum in 1924 was in September.

Guinardia flaccida.—Appears at the spring maximum and has much the same seasonal distribution as *Lauderia.*

Hyalodiscus stelliger.—Occurs in small numbers during the winter.

Coscinodiscus.—Essentially a winter type and has its maximum then. *C. concinnus* and *C. radiatus* are the commonest species. Both are present in small numbers during the summer and *C. concinnus* is common in July. *C. granii* is rare and occurs only in winter.

Rhizosolenia.—This is a summer diatom and its species follow one another in definite order.

R. setigera is the first to appear in early April. It is scarce during the summer and may reappear again in small numbers in autumn.

R. fragillima appears in June and increases to its maximum in July. In 1923 it was very abundant for three weeks, but in 1924 it was not so numerous. The species as it occurs here is smaller than the type, measuring only 28–45μ in length and 8–10μ in breadth.

R. shrubsolei appears at the same time as *R. fragillima* but is commoner in August than in July.

R. semispina is common in July and August.

R. stolterforthii is common in August and September.

R. farœnsis and *R. delicatula* occur in summer but are rare.

Corethron criophilum.—Rare, in April and September.

Chætoceros.—This is an important genus and helps to form the spring and autumn increases. Its maximum, however, occurs in August. Many small species which have not been identified occur in small numbers in summer.

C. criophilum is present in small numbers all the year round.

C. boreale, C. decipiens, and *C. didymum* occur from March till October.

C. sociale was very abundant in August 1924 and formed the autumn diatom maximum.

C. debile was the commonest *Chætoceros* in the summer of 1923 but was unimportant in 1924.

C. breve, C. constrictum, C. contortum, and *C. tortissimum* occurred in small numbers in summer.

Eucampia zoodiaca.—Occurs in spring and occasionally up till July. Rare in September.

Cerataulina bergoni.—Appears during the spring maximum and remains throughout most of the summer. In 1924 it was common at the end of June.

Biddulphia.—Entirely a winter diatom. The three forms, *B. mobiliensis*, *B. regia*, and *B. sinensis*, are all common. They are found from November till May, with their maximum in December or January.

Ditylium brightwelli.—Appears in March and, though never in large numbers, is fairly common till the middle of April.

Fragilaria oceanica.—Fairly common at the end of May and in June.

Thalassiothrix nitzschioides.—Not very common. Occasional in spring and summer.

Asterionella.—*A. japonica* appears at the spring maximum and is present in small numbers until June.

A. bleakleyi appears occasionally.

Navicula.—Various species occur all the year round. They are most numerous in early summer.

Pleurosigma.—Various species occur chiefly in winter and spring.

Nitzschia.—This genus is present most of the year but is most abundant in late summer and autumn.

N. closterium is the only species in winter and is rare then. It is commonest in August. Two forms are found: one the typical form with curved ends, and the other, which is much commoner here, with finer and longer straight ends.

N. seriata occurs during the late summer but is never very numerous.

N. delicatissima is present from March till October and is sometimes numerous.

Bacillaria paradoxa.—Occurs off and on in small numbers throughout the year.

Dinoflagellata.

The Dinoflagellates have not yet been worked out completely, but notes are given of those species which have been identified.

Exuviella marina.—Present in small numbers most of the year but is scarce in early summer.

Prorocentrum micans.—Scarce or absent from April till July but is present in small numbers during the rest of the year, with a maximum in September when it is fairly common.

Dinophysis.—Occurs all the year round. It is rare from November till April and is fairly common all summer, with its maximum in July or

August. *D. acuta* and *D. acuminata* are the commonest species, the latter occurring only in summer.

Phalacroma rotundata.—Occurs with *Dinophysis* but is generally less common. Its maximum is in July and August.

P. pulchella was seen once or twice.

Glenodinium bipes.—Appears in spring, and then becomes scarce in early summer. It increases again, attaining its maximum in August or September.

G. trochoideum.—Common at times during the summer, with its maximum in July.

Several small thecate forms, common from June till August, are possibly to be referred to this genus.

Gonyaulax.—*G. triacantha* occurs rarely in April and May. *G. spinifera* is common in June and present throughout the summer.

Diplopsalis lenticula.—Occurs from May or June till October. Various minute species of *Diplopsalis* or allied genera appeared in summer, but were not identified.

Diplopeltopsis minor.—Occurs in June and July in small numbers.

Peridiniopsis asymmetrica.—Appears in August and is present in small numbers till November. It was taken from deep water in January also.

Heterocapsa triquetra.—Fairly common all summer, with a maximum in June. It is present up till October.

Peridinium.—The genus is present almost all the year round, although it was absent during the winter, 1922–1923. In 1923 there were two distinct maxima, a small one in May, in which all the species increased, and a larger one in August. In 1924 there was only one, in August, and *Peridinium* as a whole was not so abundant as in the previous summer.

P. cerasus.—Occasional during the summer, June till November.

P. ovatum occurs all the year round but is commoner from April to October than during the winter. The maximum is in July or August. It was the most abundant *Peridinium* in 1924.

The form found here differs from those examined by Dr Lebour (1923) from Plymouth and elsewhere in the arrangement of the dorsal epithecal plates. The second anterior intercalary is variable in position. In the majority of cases it is asymmetrical in so far as it is further towards the left side, but in most cases it does not touch the third precingular. Out of 103 specimens examined in early summer 60 had the second anterior intercalary pushed to the left side to a varying extent, 17 were apparently completely symmetrical, and in 26 the plate touched both third and fourth precingulars. The specimens varied considerably in size from 66–88μ across, but the size showed no correlation with the plate arrangement. In July a further series were examined, and these were found to be on the average smaller in size and to have a much larger percentage of asymmetrical plates. Apparently the plate arrangement in peridinians is not an absolutely constant feature but may vary about a mean position.

P. pellucidum has much the same seasonal distribution as *P. ovatum* but is not so common.

P. depressum.—This was much the commonest species in 1923, with its maximum in the end of July and beginning of August. It remained in small numbers during the winter, 1923–1924.

P. pentagonum occurs from April till October but never in large numbers.

P. conicum occurs during the summer in small numbers.

P. catenatum.— Occasional during the summer.

P. brevipes.—Occasional during the summer.

Ceratium.—The genus is present all the year round but is scarcest in winter and early spring. It was more abundant in 1924 than in 1923.

C. tripos appears in August and is fairly common during the autumn. It remains through the winter in small numbers.

C. furca.—Not very common in 1923, but in the latter half of August 1924 it was numerous, and reached a maximum in September when large swarms occurred.

C. fusus.—Present from June throughout the summer and autumn but is never very numerous.

C. longipes.—One of the commonest species here. It is present during the summer and has its maximum in August. In 1924 it decreased rapidly in the middle of August, being replaced by *C. furca.* It sometimes appeared in large and very local swarms. On 22nd July it was very abundant off the south-west coast of Bute although tow-nettings several miles distant showed no unusual numbers. Three weeks later at Strachur in Loch Fyne it was even more abundant. A half-hour's surface tow-netting was filled with a thick brown mass of this species. Ten days later this local abundance had completely disappeared. Owing to lack of intervening observations it is impossible to say whether this was one large bank of *Ceratium* travelling northwards or whether the abundance was produced locally.

C. intermedium.—Common during the summer and autumn.

C. macroceros.—Rare, in August and September.

Amphidinium.—Never common, but occurs in small numbers during the summer.

A. crassum occurs in water-samples during the summer. It agrees with the species as figured by Dr Lebour from Plymouth.

A. rotundatum.—Rare, during the summer.

Gymnodinium.—Various species are found, many of which are minute and unidentified. They occur in varying numbers all summer and are rarer in winter.

G. rhomboides.—This is perhaps the commonest species, found during most of the year but commonest in spring and summer.

G. filum.—Common during the spring but scarce at other times.

G. gracile.—Occasional from June till August.

G. vestifici.—A form resembling this was occasional in spring.

Gyrodinium.—Follows much the same course as *Gymnodinium.*

G. glaucum.—The commonest species here. It occurs off and on all the year round but is commonest in spring and summer.

G. spirale appears occasionally during the summer. A species resembling this but much smaller (30–50μ) is commoner, with a maximum in April.

G. britannia.—Rare, in September 1924.

G. crassum.—Fairly common in September.

G. viridescens.—One specimen obtained in January at ten fathoms. This species as described by Kofoid and Swezy is one of the sand-dwelling forms, and its presence in the tow-net must therefore be exceptional.

Cochlodinium.—Rare, occurring most frequently in summer. Two species are found, one being *C. pulchellum* which is very rare, and the other a small species resembling *C. faurei*, although considerably smaller in size.

Torodinium.—*T. robustum* occurs during most of the year, least frequently in summer.

Polykrikos.—*P. schwartzi* appears in August and is present in small numbers until October.

Nematodinium.—*N. armatum* was absent in 1923 but fairly common for a week or two in July 1924. It varied greatly in appearance.

Pouchetia.—Several forms occurred in summer, particularly in July 1924. Among these was one resembling *P. maculata.*

P. parva.—I refer to this species a small form found occasionally all the year round. When examined it has usually lost all trace of furrows and is seen as a small oval body, yellowish in colour, containing a large ocellus with a concentrated red pigment spot and a well-integrated lens. It measures 20–30μ in length and a little less in breadth, and the ocellus may be as much as 10–15μ in length. It is not encysted.

Proterythropsis vigilans.—Frequent at the end of March and beginning of April.

Pyrocystis lunula.—Rare, from August till October. Round cysts containing *Gymnodinium* individuals were not uncommon in August 1924.

Ellobiopsis chattoni.—Occasionally found epizoic on the antennæ and mouth parts of copepods.

Syndinium sp.—Fairly common in *Calanus* in summer.

FLAGELLATA.

Phæocystis pouchetii.—Comes at the diatom maximum and is common in March and April, disappearing in May. It reappears in small quantities in October.

Dinobryon sp.—Frequent in water-samples in summer.

Halosphæra viridis.—Was common in January and February 1923 but was quite absent in 1924.

Trochiscia.—*T. clevei* was frequent in water-samples in June 1924. *T. vanhoeffeni* and *T. dictyon* are very rare.

Calycomonas sp.—Appears in summer and autumn in the guts of other animals, *e.g. Calanus, Oikopleura,* but is never seen free in the plankton, perhaps owing to its minute size.

Pontosphæra huxleyi.—This is a very small species and generally missed in the water-samples. Judging from its appearance in the guts of other animals, it occurs from April to August, with a maximum in July.

Distephanus speculum.—Rare except during its maximum in September. It was frequent in April 1923 but not in 1924, and seems to be more numerous during the winter than during the summer.

Dictyocha fibula.—Very rare until October 1924 when it was frequent in water-samples and was occasional throughout the following winter.

Ebria tripartita.—Occurs from June till October, with a maximum in July when it is common in the water-samples.

RHIZOPODA.

Amœba sp.—Common in water-samples in summer.

Acanthonia mülleri.—This is a winter form appearing rarely in October, increasing in numbers throughout November, and having its maximum in December or January. It disappears in April or May. It was much scarcer in the spring and winter of 1924 than in 1923.

Plectanidæ.—A form belonging to this family appears occasionally, but is rare.

Heliozoa.—A small form with radiate pseudopodia, resembling *Acanthocystis,* occurs rarely in June.

CILIATA.

Tontonia appendiculariformis.—Occasional specimens seen in summer and autumn. They are much smaller than the type, measuring only $36 \times 21\mu$, as compared with Fauré-Frémiet's $140 \times 80\mu$, and differ also in possessing green chromatophores. The long contractile tentacle resembles that of the type in structure.

Tintinnus subulatus.—Fairly common from the middle of June till the end of September.

T. steenstrupii.—Found only in Loch Fyne where it was obtained in summer and autumn.

Tintinnopsis campanula.—Present in small numbers throughout the year but is rare from November till January. Fairly common from June till September.

T. ventricosa occurs throughout the year and is commonest in October and November.

T. beroidea.—Much the same as *T. ventricosa* though not so common.

Cittarocyclis serrata.—Fairly common from July till October and abundant in August.

Mesodinium sp.—Occasional throughout the year.

Tiarina fusus.—Fairly common in summer and autumn.

Unidentified Infusoria were common during the summer.

CŒLENTERATA and CTENOPHORA.

Hybocodon prolifer.—Occasional in June 1923.

Sarsia tubulosa.—Rare, in May.

Corymorpha nutans.—Occurs most of the year but is less common in winter.

C. aurata occurs from late summer till spring but is never common.

Purena gemmifera.—Common in July and August.

Rathkea blumenbachii.—In 1924 it was common in July and frequent till September.

Leuckartiara octona.—Common at the end of August and frequent in September.

Bougainvillea britannica.—Occasional in the end of April and in May.

Nemopsis crucifera.—Once in August 1924.

Melicertidium octocostatum.—Occasional in early summer.

Podocoryne carnea.—Occasional in early summer.

Willia stellata.—Frequent in October, occasional in November and December.

Obelia.—Medusæ occur from April till November, being commonest in August and September.

Phialidium hemisphericum.—This is the commonest Medusa here and is found all the year round though not common in winter. It is most numerous from June till September.

Ephyræ.—One or two swarms in February.

Arachnactis larva.—Rare, in April.

Pleurobrachia pileus.—Fairly common in early summer. Browne reports it as occurring frequently in the autumn of 1901.

Bolina infundibulum.—Common from September till December and occasionally at other times of the year.

PLATYHELMINTHES.

Pharyngora bacillaria.—Seen several times inside the umbrella of *Melicertidium octocostatum* in June.

Hemiurus appendiculatus.—Occurred in *Acartia clausii* and in *Calanus.* Rare.

A Turbellarian appeared in July and was present in small numbers till the end of August.

NEMATODA.

Ascaris sp.—Larva occasional in *Sagitta.*

Nematode unidentified occurred in *Calanus.*

CESTODA.

Plerocercoides æquorea.—Found occasionally in *Calanus* during the winter; large numbers present in one host.

Cestode larvæ unidentified were occasional in *Sagitta* and in *Calanus.*

ANNELIDA.

Pelagobia.—Larva rare. Found in February.

Polynoid larvæ.—Rare. Found in October.

Terebellid larvæ.—Frequent during the summer.

Polychæte larvæ.—Occurred all the year round, being least common from August till December: Spionid larvæ were the commonest.

Antolytus sp.—Occasional.

Mitraria.—Occasional from March till December, but never common.

Tomopteris helgolandica.—Occasional from October till February. One specimen taken in June.

CHÆTOGNATHA.

Sagitta bipunctata.—Occurs all the year round, with its maximum in July. More frequent in autumn than in spring.

Spadella cephaloptera.—Rare. Found in May.

POLYZOA.

Cyphonautes larva.—Occurs all the year round, but is not common except in September and October.

PHORONOIDEA.

Actinotrocha larva.—Rare in April and May, occasional in August.

ROTIFERA.

Synchæta sp.—Fairly common from the end of June till September.

CLADOCERA.

Podon intermedius.—Occurs from April to September, and is common from July.

Evadne nordmanni.—Common in May and June, fairly common till September and frequent in October.

COPEPODA.

Calanus finmarchicus.—Occurs all the year round, with its maximum in May and June. It is scarcest in autumn, and is commoner in January and February than at any other time except during its maximum.

Paracalanus parvus.—Occurs rarely in autumn. This species, however, was recorded by Scott (1905) as fairly common in spring.

Pseudocalanus elongatus.—Occurs all the year round, with its maximum in the winter. In 1923 it was also common in May and June, but in 1924 it was present only in small numbers.

Undinopsis bradyi.—Occasional throughout the year.

Centropages typicus.—Once in December.

Centropages hamatus.—Occurs from May till October and is common from June till August. In 1924 it appeared in March.

Temora longicornis.—Occurs all the year round, but less abundantly than the other common copepods. Commonest in May and September.

Anomalocera patersoni.—Occasional from April to October and numerous some days in June.

Labidocera wollastoni.—Seen once in October.

Acartia clausii.—This had much the same seasonal distribution as *Pseudocalanus*, both in 1923 and 1924. It was commonest in the winter months.

Monstrilla danæ.—Once in June.

Oithona helgolandica.—Occurs during most of the year. It is rare during the summer and most abundant in November.

Harpacticids.—Occasional throughout the year.

Caligus rapax.—Occasional from March to July and October to December. They are sometimes found free in the plankton and carrying eggs.

Nauplii.—Occur all the year round, most commonly from January to May and from October to November. They occur in enormous numbers at the end of February and in March.

CIRRIPEDIA.

Larvæ occur during most of the year. Nauplii are very common in February and March. Metanauplii appear about the middle of April, and cypris stages about the end. All three stages were found in May and June. Nauplii appeared again in September, and were fairly common till November.

AMPHIPODA.

Caprella sp.—Occasional May, June, and November.

Metopa sp.—Occasional in January and March, frequent in October and November. It is found very often living inside the bell of *Phialidium*, and will make its way there if these Medusæ are in the plankton.

Amphipods unidentified. Occasional throughout the year, mostly in tow-nettings taken at night.

ISOPODA.

Idotea sp.—Occasional throughout the year. Most frequent in early summer.

Eurydice sp.—Several times in May.

Microniscus sp.—Occasional on copepods from September till February.

CUMACEA.

Cumacids unidentified. Occasional during the year, almost all in night tow-nettings.

EUPHAUSIACEA.

Nyctiphanes sp.—Adult rare. In May, and often from deep water outside.

Euphausid eggs and larvæ appear at the end of March and are fairly common in April. The later stages are found till August.

DECAPODA.

Zoeæ.—Occur during the whole year, but are commonest in summer, especially in July and August.

Megalopæ.—Frequent from July till October.

Nephrops sp.—Larva occasional from May till August.

MOLLUSCA.

Larval gasteropods and pelecypods are present all the year round, being very common from March till May and in November and December.

ECHINODERMATA.

Ophiopluteus.—Occur from the end of March till November, but are commonest in the summer months, especially August. Larvæ of *Ophiothrix fragilis* and *Ophiura albida* were recognised.

Echinopluteus.—Fairly common in July and August.

Spatangopluteus.—Fairly common in the end of August and September. *Echinocardium cordatum* was the commonest species.

Bipinnaria and Brachiolaria.—Occasional in June, frequent in July, and rare in August.

APPENDICULARIÆ.

Oikopleura dioica.—Occurs from May till October, sometimes present in large numbers. In both 1922 and 1923 *Oikopleura* disappeared completely from November till the following May, but Browne (1905) (1) records it as occurring every month of the year 1902, although rare in the winter.

Fritillaria furca.—Common from October till May, absent in June and most of July, and rare in August and September. It attains its maximum in early April. In 1924 it disappeared early, and there were no appendicularians in April and May.

PISCES.

Fish eggs were common from the end of March till August, and disappeared during September.

II. THE PLANKTON MONTH BY MONTH.

JANUARY.

DIATOMS.—The commonest diatom this month was *Coscinodiscus*, *C. concinnus* and *C. radiatus* being the commonest species. *Biddulphia* (*B. mobiliensis*, *B. sinensis*, *B. regia*) came next in numbers. There were always a small number of *Chætoceros* present, and of various other diatoms such as *Paralia*. *Skeletonema* although absent from the tow-nets was found in very small numbers in the water-samples.

DINOFLAGELLATA.—Peridinians were scarce. In 1923 *Peridinium* was entirely absent during the winter months, but in 1924 it was present in small numbers, *P. depressum* being the commonest species and *P. pellucidum* rarer. *Ceratium tripos*, *C. fusus*, and *C. furca* are generally to be found, particularly the first named.

FLAGELLATA. — Silicoflagellates were rare. *Halosphæra viridis* appeared in considerable numbers in 1923 but not in 1924.

RHIZOPODA.—Radiolarians (*Acanthonia mülleri*) were common. In 1923 they were often as numerous as the diatoms, but in 1924 they were much fewer and decreased considerably in numbers towards the end of the month.

CŒLENTERATA.—*Phialidium* was rare.

MOLLUSCA.—Larval molluscs were fairly common.

CRUSTACEA. — Copepod nauplii were common. Of the copepods *Pseudocalanus elongatus* was the commonest and *Acartia clausii* next. *Calanus finmarchicus* was more abundant in 1924 than in 1923. *Oithona helgolandica* was fairly common and *Temora longicornis* rare.

APPENDICULARIÆ.—*Fritillaria furca* was common.

FEBRUARY.

DIATOMS.—*Coscinodiscus* and *Biddulphia* remained the most abundant diatoms, although with an increasing number of other forms, *Skeletonema costatum, Thalassiosira gravida,* and *T. nordenskioldi. Guinardia flaccida, Lauderia borealis, Ditylium brightwelli,* and *Chætoceros* appeared in small quantities.

DINOFLAGELLATA.—Peridinians still scarce, *Ceratium* being the only form constantly present.

FLAGELLATA.—*Halosphæra* still present.

RHIZOPODA.—*Acanthonia* common in 1923.

CŒLENTERATA.—*Phialidium* rare. Ephyræ of *Aurelia* appeared on the 15th and 28th. *Bolina* rare.

ANNELIDA. — Larval polychætes became increasingly common. *Tomopteris helgolandica* occasional.

CHÆTOGNATHA.—*Sagitta bipunctata* occasional.

MOLLUSCA.—Larval forms fairly common, gasteropod larvæ being more numerous than pelecypod.

CRUSTACEA.—Copepod nauplii very common and cirripede nauplii became so at the end of the month. Zoeæ appeared though not very numerous.

The copepods remained much the same as in January, except that *Temora* had disappeared and *Oithona* was decreasing in numbers.

APPENDICULARIÆ.—*Fritillaria* still common.

MARCH.

DIATOMS.—Up till the 26th (1923) things remained much as in February. On that day enormous quantities of *Skeletonema* and *Thalassiosira* appeared in the nets, the former predominating greatly. *Chætoceros* was also abundant. *Lauderia, Paralia, Guinardia, Rhizosolenia, Ditylium,* and *Asterionella* were usually present in small numbers and were much commoner after the 26th. The change on this day was remarkably sudden and is perhaps partly accounted for by the weather conditions. Previous to the 25th there had been six weeks of dry sunny weather with an east wind. On the 25th the wind changed to the south and some rain fell.

DINOFLAGELLATA.—Peridinians became more frequent. *Ceratium* was still the commonest. The naked forms were fairly common in the water-samples. *Gymnodinium rhomboides* and *G. filum* along with a small form of *Gyrodinium spirale* were the commonest.

FLAGELLATA.—*Phæocystis* appeared and was common at the end of the month. *Distephanus* was frequent in water-samples in 1923 but rare in 1924.

RHIZOPODA.—*Acanthonia* was fairly common up till the 26th, but its numbers fell after that.

CŒLENTERATA —Medusæ were scarce, *Phialidium* being the most frequent.

ANNELIDA.—Polychæte larvæ were common.

POLYZOA.—*Cyphonautes* larvæ were frequent.

MOLLUSCA.—Larval molluscs were common.

ECHINODERMATA.—Larvæ appeared at the end of the month.

CRUSTACEA.—Copepod and cirripede nauplii appeared in large numbers at the end of the month. Euphausid eggs and nauplii appeared then too, and zoeæ increased in number. *Calanus* was scarce, *Pseudocalanus* and *Acartia* common, *Temora* rare, and *Oithona* decreasing.

Podon intermedius appeared in the middle of the month.

APPENDICULARIÆ.—*Fritillaria* was common.

PISCES.—Fish eggs appeared at the end of the month.

APRIL.

DIATOMS.—Diatoms were abundant all month, *Skeletonema* being the commonest and *Thalassiosira* next. *Chætoceros* (*C. boreale* and *C. decipiens* predominating, with *C. contortum*, *C. criophilum*, and *C. didymum* also present) was also very common, while *Biddulphia* had begun to decrease. *Rhizosolenia* was fairly common. *Lauderia*, *Paralia*, *Guinardia*, *Cerataulina*, *Ditylium*, and *Asterionella* were frequent.

DINOFLAGELLATA.—*Ceratium* was always present though not in large numbers. *Peridinium* appeared in 1923, *P. depressum* being the commonest species while *P. pellucidum* and *P. ovatum* were scarcer. *Glenodinium bipes* appeared and was common in the water-samples. The naked forms were much the same as in March. Both in 1923 and 1924 *Proterythropsis vigilans* appeared just after the diatom increase and was frequent in the water-samples for a week or two.

FLAGELLATA.—*Phæocystis* common all month, increasing towards the end. *Distephanus* occasional.

RHIZOPODA.—*Acanthonia* was still present, but in small numbers only. *Amœbæ*, some with irregular lobate pseudopodia and a few with radiate pseudopodia, were frequent in the water-samples.

CILIATA.—*Tintinnopsis ventricosa* was frequent.

CŒLENTERATA.—*Phialidium* occasional.

ANNELIDA.—Polychæte larvæ were common.

MOLLUSCA.—Larval molluscs were common.

ECHINODERMATA.—Larvæ were fairly common, especially Ophiopluteus.

CRUSTACEA.—Cirripede nauplii were very common at the beginning of the month and cypris stages at the end. Copepod nauplii were common and euphausid fairly common.

Calanus, *Pseudocalanus*, and *Acartia* were common, *Temora* was increasing slightly in numbers and *Oithona* still decreasing. *Anomalocera patersoni* was occasional.

APPENDICULARIÆ.—*Fritillaria* was still common and was present in large numbers at the beginning of the month.

PISCES.—Fish eggs were fairly common.

MAY.

DIATOMS.—Diatoms were much scarcer this month. *Thalassiosira* was now more abundant than *Skeletonema*. *Coscinodiscus* was always present in small numbers, but *Biddulphia* had disappeared. *Chætoceros* and *Rhizosolenia* were always present in small numbers also.

DINOFLAGELLATA.—A large swarm of *Peridinium*, chiefly *P. depressum*, appeared from 8th to 11th (1923), and they were numerous all month. *Ceratium* was scarcer, *Glenodinium bipes* was common all month in water-samples, and *Dinophysis* was fairly common.

The naked forms were common. *Amphidinium crassum*, *Gymnodinium rhomboides*, *G. filum*, *Gyrodinium glaucum*, and *G. spirale* were the predominant forms. Pouchetiidæ were scarce.

FLAGELLATA.—*Phæocystis* was common at the beginning but had disappeared by the end of the month.

RHIZOPODA.—*Acanthonia* still present in dwindling numbers. *Amœbæ* frequent in the water-samples.

CILIATA.—*Tintinnopsis ventricosa* was common.

CŒLENTERATA.—*Phialidium* frequent at the end of the month, but other medusæ rare.

ANNELIDA.—Polychæte larvæ common.

CHÆTOGNATHA.—*Sagitta* frequent.

POLYZOA.—*Cyphonautes* frequent.

MOLLUSCA.—Larval molluscs common.

ECHINODERMATA.—Larvæ common, especially Ophiopluteus.

CRUSTACEA.—Cirripede larvæ of all stages, and copepod nauplii, were common, and so were zoeæ. *Calanus* was beginning its maximum and sometimes occurred in enormous shoals. *Pseudocalanus*, *Acartia*, and *Temora* were common, and *Anomalocera* and *Centropages hamatus* occasional. *Podon* and *Evadne nordmanni* were common.

APPENDICULARIÆ.—*Fritillaria* was common at the beginning but fell off at the end of the month, while *Oikopleura dioica* appeared at the beginning and gradually increased in numbers.

PISCES.—Fish eggs were common.

JUNE.

DIATOMS.—The diatoms were still decreasing in quantity. At the beginning of the month *Thalassiosira* and *Skeletonema* were the most abundant, but at the end *Chætoceros* had replaced them. *Rhizosolenia*, particularly the small species *R. fragillima* and *R. delicatula*, were fairly common. *Fragilaria oceanica* was common at the end of the month.

DINOFLAGELLATA.—*Peridinium* was common all month, but *Ceratium* was scarce and the other peridinians and gymnodinians decreased from the beginning to the end of the month.

RHIZOPODA.—*Acanthonia* had disappeared. In 1924 it disappeared much earlier, being very scarce in April and absent in May. *Amœbæ* were scarcer this month than last.

CILIATA.—*Tintinnopsis ventricosa* and some *T. campanula* were still common.

CŒLENTERATA.—*Phialidium* and *Corymorpha nutans* were common, and other medusæ appeared occasionally.

ANNELIDA.—Polychæte larvæ were common.

POLYZOA.—*Cyphonautes* was scarcer this month than last.

MOLLUSCA.—Larval molluscs very common.

ECHINODERMATA.—Very young stages common in the first half of the month. Later larvæ also common.

CRUSTACEA.—Cirripede nauplii and cypris stages and copepod nauplii were common. Euphausid larvæ were common at the beginning. *Calanus* was at its maximum and occurred in immense shoals. *Pseudocalanus* was also abundant, and *Acartia, Temora,* and *Centropages hamatus* were common. *Anomalocera* was occasional. *Oithona* had practically disappeared by the end of the month.

Podon was occasional and *Evadne* common.

APPENDICULARIÆ.—*Oikopleura* was very common all month and had entirely replaced *Fritillaria.*

PISCES.—Fish eggs were common most of the month.

JULY.

DIATOMS.—*Skeletonema* and *Thalassiosira* had practically disappeared and diatoms were scarce until the 6th (1923), when enormous quantities of *Rhizosolenia fragillima* appeared in the tow-nets. This abundance lasted for about three weeks and by the end of the month *Chætoceros* (chiefly *C. debile* and *C. densum*) predominated. Other species of *Rhizosolenia* (*R. shrubsolei, R. stolterforthii*) were also fairly common. *Coscinodiscus concinnus* was fairly common at the end of the month.

DINOFLAGELLATA.—*Peridinium* was increasing towards its maximum as was *Ceratium* also. *P. depressum* was the commonest species and occurred in large quantities at the end of the month. Naked forms were not very numerous.

FLAGELLATA.—*Distephanus* was rare, but *Ebria* was fairly common in the water-samples.

CILIATA.—Tintinnoids were increasing, *Cittarocyclis serrata* being the commonest species at the end of the month.

CŒLENTERATA.—*Phialidium* and *Corymorpha* were the commonest medusæ, and *Purena gemmifera* was also common. *Pleurobrachia* was frequent.

ROTIFERA.—Rotifers were generally present.

ANNELIDA.—Polychæte larvæ always present but not numerous.

CHÆTOGNATHA.—Small specimens of *Sagitta* were common.

POLYZOA.—*Cyphonautes* was absent at the beginning but appeared again at the end of the month.

MOLLUSCA.—Larval molluscs were much less common than in June.

ECHINODERMATA.—Larvæ common, especially at end of month.

CRUSTACEA.—Nauplii, both copepod and cirripede, and zoeæ were common. Euphausid larvæ were occasional. *Calanus* was common, though in much smaller numbers than in June. *Centropages* was also common, and *Pseudocalanus, Acartia,* and *Temora* fairly common.

Podon and *Evadne* were fairly common.

APPENDICULARIÆ.—*Oikopleura* was very common and *Fritillaria* appeared once or twice at the end of the month.

PISCES.—Fish eggs were common.

AUGUST.

DIATOMS.—*Chætoceros* was the predominant diatom this month, various species being very common, especially *C. debile*. *Rhizosolenia shrubsolei* was also very common, and *R. fragillima* gradually disappeared while *R. stolterforthii* increased. *Skeletonema* and *Thalassiosira* were generally present in very small numbers, and *Coscinodiscus* was fairly common.

DINOFLAGELLATA.—*Peridinium* and *Ceratium* were at their maximum, *Peridinium* at the beginning and *Ceratium* at the end of the month. *Dinophysis* was common all month, and various other forms, *Gonyaulax*, *Diplopsalis*, *Pyrocystis*, etc., were occasional. The Gymnodinioidæ were not common. *Polykrikos schwartzii* appeared in the middle of the month and was always present in small numbers afterwards.

FLAGELLATA.—*Distephanus* was rare and *Ebria* frequent in the water-samples. *Phæocystis* was occasional.

CILIATA.—*Cittarocyclis serrata* appeared in large quantities during the first week, while *Tintinnopsis campanula* and *Tintinnus subulatus* were frequent.

CŒLENTERATA.—*Phialidium* and *Purena gemmifera* were common all month, and *Leuckartiara octona* became common at the end.

ROTIFERA.—Rotifers were fairly common.

ANNELIDA.—Polychæte larvæ were common.

CHÆTOGNATHA.—*Sagitta* was frequent.

ECHINODERMATA.—Ophiopluteus and Echinopluteus larvæ common all month, and Bipinnaria rare.

MOLLUSCA.—Larval forms were fairly common.

CRUSTACEA.—Copepod nauplii were common and so were zoeæ. Megalopæ were frequent. *Calanus* was scarcer than in July, but *Pseudocalanus*, *Acartia*, and *Centropages* were common. *Temora* was decreasing in numbers and *Oithona* increasing. *Podon* and *Evadne* were fairly common.

APPENDICULARIÆ.—*Oikopleura* was common and *Fritillaria* rare.

PISCES.—Fish eggs were numerous in the first half of the month.

SEPTEMBER.

DIATOMS.—For the first twelve days of September 1923, *Chætoceros* and *Rhizosolenia* were very common, but there followed a period when diatoms were rather scarce. On the 20th, *Skeletonema* and *Thalassiosira* appeared in very large numbers again, and were abundant after that. *Coscinodiscus* was fairly common most of the month.

DINOFLAGELLATA.—*Peridinium* and *Ceratium* were always present but were never abundant. *Glenodinium bipes* appeared towards the end of the month, and *Dinophysis*, along with *Heterocapsa triquetra*, was fairly common then. The Gymnodinioidæ all increased towards the end of the month.

FLAGELLATA.—*Distephanus* reached its maximum this month, and from the middle to the end was fairly common in the water-samples. *Ebria* was also frequent.

RHIZOPODA.—A single specimen of *Acanthonia* was seen on the 18th.

CILIATA.—*Tintinnopsis ventricosa* was increasing towards the end of the month, but all the other Tintinnoids were decreasing.

CŒLENTERATA.—*Phialidium* and *Purena gemmifera* were fairly common, and *Leuckartiara* and *Obelia* frequent. *Bolina* was sometimes abundant.

ANNELIDA.—Polychæte larvæ were fairly common.

POLYZOA.—*Cyphonautes* larvæ were frequent and increased towards the end of the month.

ECHINODERMATA.—Larvæ fairly common.

MOLLUSCA.—Larval forms fairly common.

CRUSTACEA.—Copepod and cirripede nauplii were always present, zoeæ frequent, and megalopæ occasional. *Calanus*, *Pseudocalanus*, *Acartia*, *Centropages*, and *Oithona* were all fairly common. *Temora* was generally present and *Anomalocera* occasional. *Podon* and *Evadne* were fairly common.

APPENDICULARIÆ.—*Oikopleura* was fairly common and *Fritillaria* rare.

PISCES.—Fish eggs were frequent at the beginning of the month, absent at the end.

OCTOBER.

DIATOMS.—The diatoms decreased greatly from the beginning to the end of the month. *Skeletonema* and *Thalassiosira* were abundant at the beginning but decreased greatly later, and *Chætoceros* did the same. *Rhizosolenia* had disappeared altogether by the end, but *Coscinodiscus* was fairly common all the time.

DINOFLAGELLATA.—*Peridinium* and *Ceratium* were always present in small numbers, while *Diplopsalis, Gonyaulax, Dinophysis,* and *Phalacroma* were occasional. *Heterocapsa* was common all month. The Gymnodinioidæ were fairly common. *Gymnodinium rhomboides, Gyrodinium glaucum, G. spirale, G. crassum,* and *Torodinium robustum* being the commonest forms. *Polykrikos* disappeared during the month.

FLAGELLATA.—*Distephanus* was frequent at the beginning and rare at the end of the month, and *Ebria* was rare all the month.

RHIZOPODA.—*Acanthonia* appeared rarely.

CILIATA.—*Tintinnopsis ventricosa* was common all the month and *T. campanula* at the beginning. *Tontonia appendiculariformis* was seen twice.

CŒLENTERATA.—*Phialidium* was the only common medusa, although *Obelia* was frequent. *Bolina* was generally present.

ANNELIDA.—Polychæte larvæ were fairly common, although decreasing in numbers. *Tomopteris* was occasional.

CHÆTOGNATHA.—*Sagitta* was frequent.

POLYZOA.—*Cyphonautes* was fairly common.

ECHINODERMATA.—Larvæ were frequent.

MOLLUSCA.—Larval forms were fairly common at the beginning but decreased greatly towards the end of the month.

CRUSTACEA.—Nauplii, both cirripede and copepod, were common, and the latter were abundant on the 23rd (1923). Zoeæ were fairly common. *Calanus* occurred only in small numbers, *Pseudocalanus* and *Acartia* were abundant, *Centropages* and *Oithona* fairly common, and *Anomalocera* occasional.

APPENDICULARIÆ.—*Oikopleura* and *Fritillaria* changed places again, the former decreasing and the latter increasing in numbers throughout the month.

NOVEMBER.

DIATOMS.—Diatoms were not abundant. *Coscinodiscus* and *Biddulphia* were increasing in numbers but the others (*Skeletonema, Thallassiosira, Chætoceros*) were all decreasing. *Paralia sulcata* was present in small numbers.

DINOFLAGELLATA.—Peridinians were also decreasing, *Peridinium depressum* being the commonest. *C. tripos* and *C. furca* were the commonest species of *Ceratium.* The Gymnodinioidæ were scarce except for *Torodinium,* which was frequent in the water-samples.

FLAGELLATA.—*Distephanus* was rare, and *Ebria* had disappeared.

RHIZOPODA.—*Acanthonia* was scarce at the beginning and fairly common at the end of the month.

CILIATA.—*Tintinnopsis ventricosa* was common most of the month.

CŒLENTERATA.—*Phialidium* was fairly common. *Bolina* was frequent.

ANNELIDA.—Polychæte larvæ were common again. *Tomopteris* was occasional.

POLYZOA.—*Cyphonautes* was frequent at the beginning and scarce at the end of the month.

ECHINODERMATA.—Larvæ appeared rarely.

MOLLUSCA.—Larval forms, especially larval gasteropods, were fairly common all the month and increased towards the end.

CRUSTACEA.—Cirripede nauplii were fairly common and copepod nauplii common. Zoeæ were frequent.

Pseudocalanus, *Acartia*, and *Oithona* were abundant. *Calanus* was present in small numbers, *Temora* was infrequent.

APPENDICULARIÆ.—*Fritillaria* was common all the month, but *Oikopleura* had disappeared by the end.

DECEMBER.

DIATOMS.—*Coscinodiscus* and *Biddulphia* were the most abundant diatoms and were common all the month, *Biddulphia* being more numerous than *Coscinodiscus*.

DINOFLAGELLATA.—Peridinians were scarce, and occurred much as they did in November.

FLAGELLATA.—*Distephanus* was rare. *Dictyocha* appeared once.

RHIZOPODA.—*Acanthonia* increased all the month and was more abundant than *Biddulphia* by the end of it.

CILIATA.—*Tintinnopsis ventricosa* decreased all the month.

CŒLENTERATA.—*Bolina* was often present in large numbers, but none of the medusæ were common except *Phialidium*.

ANNELIDA.—Polychæte larvæ were generally present in small numbers. *Tomopteris* was rare.

MOLLUSCA.—Larval gasteropods were abundant all the month and larval pelecypods fairly common.

CRUSTACEA.—Nauplii were much scarcer, and zoeæ only occasional. *Pseudocalanus*, *Acartia*, and *Oithona* were still abundant. *Temora* was rare, and *Calanus* always present in small numbers.

APPENDICULARIÆ.—*Fritillaria* was fairly common.

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XIII.—The Relation of Sea-Growth and Spawning Frequency in *Salmo salar*. By **W. L. Calderwood**, Inspector of Salmon Fisheries of Scotland.

(Read February 9, 1925. MS. received March 19, 1925.)

THE seasons at which the Atlantic salmon ascends rivers differ to a considerable extent in different countries.

In northern Norway and eastern Canada the severe winter conditions compel salmon to ascend from the sea only after the rivers have become free from ice. In one or two of the large rivers of northern Siberia it has been noticed that the salmon are well up into the upper reaches when the ice breaks up, from which it has been argued that they ascend under the ice.

In the rivers which enter the Baltic, as in those of Germany, France, and northern Spain, the runs are in the summer and autumn, but are no doubt now modified a good deal by the action of man.

In Britain, and especially in Scotland, we have the spring run as well as the later runs of other countries. In all the larger rivers of the east coast fish ascend from the sea, in a highly nourished state, before the fish of the previous year have spawned out, and from that time onwards, through the spring, summer, and autumn, successive runs of fish occur. There is no month of the year during which one may not find salmon ascending from the sea.

In small rivers the same does not hold good; and in the case of the west coast rivers, even of the largest rivers, no fresh salmon ascend before the month of March, most of the west coast rivers being summer and autumn streams, although numbers of clean sea-trout may be found in the spring by those who know where to look for them.

The scales show the cessation of feeding, when the salmon leaves the sea, as well as the difference between summer and winter feeding in the sea; and in the systematic study of scales carried on by the Fishery Board for Scotland, fish are classified in age groups according to the number of winters shown on the scales.

The time of greatest growth, or of most rapid growth, is unusually late in the salmon, owing to the period of river-life before the first migration to the sea, during which the fry, and parr, and smolts live like small trout.

The great growth takes place during the first summer of sea-feeding, often continued during the second summer in the sea; and it is this period of early sea-life to which I wish specially to call attention.

Two years commonly elapse before the first seaward migration, whereas in the case of most creatures the period of most rapid growth is then far past. In mammals it appears to be usually before birth, and even in invertebrates to be at a much earlier stage.

The youngest fish to return are the grilse, which have spent only a year and a varying number of months in the sea. On this account they form group 1+. In the period of their short sea-life they have grown about three times the length they were when they descended as smolts.

The next group consists of the small spring fish. They are ascending when the spawned grilse are descending, but have spent a second winter in the sea. They form group 2, and are a very numerously represented group in rivers like the Dee, Beauly, Conon, Helmsdale, etc.

Fish which ascend during the following summer, and which show on their scales, therefore, two winter bands plus some later feeding, form group 2+.

Similarly, the members of group 3 are the large spring fish, now commonly 18 or 20 lb., which, having spent three years in the sea and two of early life in the river, are five years old.

Groups 3+, 4, and 4+ form the still later and older members of the series. These fish have not lost weight by leaving their feeding-grounds and spawning, and are therefore fish of great size. I may say, however, that all fish over 50 lb. are males, indeed practically all fish over 35 lb. are males. I have known only two cases of females over 40 lb. in my experience of Scottish salmon fisheries during the last twenty-five years, and both came from the Lochy; one was 43 and the other 48 lb.

By the time those sea-feeding fish have grown to their great size, a large number of fish of the same age have spawned and returned to the sea, but since we never find fish coming into our rivers older than eight or at the most nine years, the presumption is that this represents the limit of the reproductive period.

We have evidence from Canada that, in the case at least of fish from rivers which enter the Gulf of St Lawrence, the reproductive period is considerably longer.

The scales of a Moisie fish which had been a three years' smolt showed that it had spent three other years in the sea, and had then spawned on four consecutive seasons. Allowing for the times not marked on the scale during which the fish was in fresh water, this is taken to represent a total age of fourteen years and a fraction. Similarly, another fish with four spawning marks on its scale has been estimated as being twelve years old. Those ages are unknown with us, and possibly may be exceptional in Canada, but it is well to recollect that we are dealing with a plastic species capable of great modification of habit.

In the scales read in this country, it has to be understood that the summer feeding begins to show in the month of May, and occasionally in late April. So reference to a winter's growth carries the age on to at least the end of April.

In the study of the smolt part of the scale it has recently been found that, while the two-year-old smolt is the usual one, the three-year-old and four-year-old smolt is not so uncommon as was supposed.

For instance, in 2323 samples of scales from Spey fish there were 25 per cent. of three-year-old smolts, or fish which had been three-year-old smolts; fourteen fish had been four-year-old smolts, and one had been a five-year-old smolt.

Now, of this 25 per cent. of three-year-old smolts, it was found that the great majority had returned to spawn as grilse; or, taking it the other way, of all the grilse in the 1+ group 47 per cent. had been three-year-old smolts.

It was noticed further that no three-year-old smolts were to be found in the old groups, and only a few in the groups which one may describe as of middle life. In other words, the oldest smolts were found to be coming back to spawn sooner than the others.

In a collection of 2232 Dee scales the same was found.

Mr Menzies, the Assistant Inspector of Salmon Fisheries, has therefore given a great deal of attention to the smolt conditions, to the smolt calculated lengths, and to the actual lengths of large numbers of smolts caught in the Dee, and has shown conclusively that the early river-life has a definite influence upon the later sea-life.

The well-grown smolts are those which leave the river first, and the first year's growth is, apparently, the dominating period.

Smolts of different rivers are of slightly different sizes. This is easily understood by the different conditions of feeding. We know, for instance, that trout in a stream draining a limestone country are always better grown than trout in a stream flowing off granite or through peat, and the trout's food is the same as the smolt's food.

But feeding in the sea varies to a greater extent, yet the influence of the early river-feeding appears to exercise a greater effect upon the subsequent habits of the fish than the sea-feeding.

Why is it, for instance, that the salmon which enter the streams of our outlying islands such as Lewis, Mull, Skye, etc., as well as the streams of small and rocky catchment areas on the mainland, are always small? They all feed in the sea, and it is impossible that there should always be poor sea-feeding just off those islands or small catchment basins.

The explanation seems to lie in the poor river-feeding for the parr and smolts in those island and rocky barren grounds. It would appear, therefore, that if people wish to increase the size of these island fish, they must arrange to augment the food supplies for the little fish in their early life.

It is like the Alaskan Indian who starves his dogs in order that they may have small stomachs and eat less. He gets the small stomachs, but he also gets small dogs.

But another feature which we notice in those small Highland streams is that the salmon return to spawn more frequently than do the fish of the large rivers.

There are fewer fish, but they spawn more frequently. In all probability the same explanation applies.

In the little river Add in Argyll, where we have marked salmon and taken scales for several successive springs, we have always found a uniform class of fish; we have always caught a few of the fish we had marked on the previous year; and we have always found, when we examined the scales, that about 50 per cent. had previously spawned. We have also found scales from Add fish with three and four spawning marks.

On one occasion we caught a kelt with four spawning marks—a fish which, therefore, had spawned five times. This instance is, I think, unique, so far.

We do not find these conditions in the large rivers of the east coast.

In all the scale collections which have been made by various investigators, only eight scales showing four spawning marks exist. Two I have already referred to as from the St Lawrence. The others are: four from the Add, including the kelt which had spawned five times; one from the Conon, already described in *The Field* by Mr Menzies; and one from Loch Maree obtained by Mr G. H. Nall, and described in *The Salmon and Trout Magazine* for January 1925.

Turning now to the period of sea-life as shown by the scales of the various groups, it is possible by calculating the age-lengths for each year to see what variation exists as between ages.

2323 Spey fish have been compared with 3285 Dee fish, and at all ages the Spey fish are the larger, yet the same ratio exists as between the ages. This has been found also in fish from the Findhorn, Don, Forth, and Tweed.

There may be seasons in which the feeding and growth is greater than in others, but the numbers now examined are sufficiently large to establish the fact that the young groups—the grilse, small spring fish, and small summer fish,—which form by far the most numerous spawners in most of our spring rivers, have relatively small sea-growth in their first sea-year. Conversely, the fish which have remained for several years in the sea without returning to fresh water have a relatively great growth in their first sea-year, and frequently also in the later years of their sea-life.

Taking as an illustration a table of figures arrived at by Mr Menzies * representing the Spey series of 2323 fish, we have the following table of averages:—

Age Group.	Calculated Lengths in Cm.			
	1st Year.	2nd Year.	3rd Year.	4th Year.
1+	41·7	...	...	...
2	44	72·9	...	...
2+	44·2	72·9	...	...
3	45·8	74·5	92·9	...
3+	48·4	77·5	97·9	...
4	54·5	78·9	96·1	113·2
4+	42·5	71·25	87·25	98·25

The two last lines may be not very trustworthy, since we are here dealing with small numbers of very large fish; but if we take the table up to the 3+ stage only, you will notice that in the fish of average total length of 97·9 the first year's growth was 48·4, as compared with only 41·7 of the grilse or 1+ group. It will be noticed also that throughout the older groups the same holds good.

The longer the fish has stayed in the sea, the more does it show large growth in its early sea-years.

This is not the case with purely marine fish as shown by their scales.

It is as if we were dealing with the balancing of the two chief instincts, feeding and reproduction. In these large rivers of the east coast it appears that when a normal two-year-old smolt enters the sea and experiences a summer of abundant feeding, it continues to remain in the sea to the detriment of reproduction. If, on the other hand, the river-feeding has been prolonged before the little fish has attained the condition of a migrating smolt, the feeding in the sea produces only a moderate growth, and the

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* *Fisheries*, *Scotland*, *Salmon Fish*, *1921*, No. II, p. 30.

fish returns to fresh water at an early age in response to the reproductive instinct.

These early returning fish are the most numerous spawners in the rivers such as the Dee, Beauly, Conon, etc. They were found to form three-quarters of the total catch of the Dee in the years 1921 and 1922. At the same time our sampling of over 3000 Dee fish showed that only 3 per cent. had previously spawned, while the sampling of over 2000 from another season showed that 1·7 per cent. had previously spawned.

In the west coast river Add, on the contrary, 50 per cent. of the fish caught in our marking operations had previously spawned, and it is rare to find fish which have remained several consecutive years feeding in the sea. I received scales of one such fish which had been caught on the rod, and constitutes the record fish for the river. It was on its first return to fresh water after five years in the sea, yet it was only 26 lb.

By contrast a Deveron fish killed last October, which had been four years in the sea, was 61 lb.

The contrast which occurs amongst the spawning fish of opposite types of river is of another nature. In the large east coast rivers we have a large number of young fish spawning only once in their lives. In a river like the Tay we have a large number of group 3 spawning only once.

In the west we not infrequently find a small stock of fish spawning every year. I am not prepared to say that in large rivers of the west coast like the Lochy and Awe and Ewe annual spawning is the rule. We have not yet made sufficient study of the fish of those rivers to make any positive statement; but, since in those large rivers there exists ample areas of water in which the parr and smolts may obtain good feeding, and since one knows that in those rivers numbers of large fish are captured annually, I should be surprised if we found a number of fish which spawn annually.

Those rivers do not have the same runs of spring fish as the eastern rivers; groups 2 and 3 are practically absent. But they have all the summer runs, with a good percentage of large fish ascending as early as March and April, the largest fish being commonly caught in May.

It seems fair to estimate therefore that they will show, in the matter of spawners, the same conditions as do the large east coast rivers, minus the spring runs.

Apart from the effects of the early feeding in the fish of small rivers, to which I have referred already, it is clear, of course, that no fish which spawns two or three times in its life can possibly attain to the weight of fish which adopt the long feeding habit in the sea. Each time it enters fresh water it leaves its feeding-grounds, and steadily drops weight during its sojourn in the river. At spawning time, if it is a female, it drops about 23 per cent. of its weight.

The conditions which I have described are so opposite that it is practically impossible to give what could be regarded as a life curve of the Atlantic salmon. Quite a series of curves would be necessary in order to show the conditions which exist, and any combination of these would rob the curve of the important contrasts.

But with regard to the particular point which is emphasised in this short paper, a sigmoid curve representing the fish of long feeding habit would show the line of weight or length compared to time in the sea, rising steeply in the early sea-years and continuing to a high level; whereas the fish of spawning habit would show a curve much less steep in the early sea-years, and rising only to a moderate level.

(*Issued separately May* 8, 1925.)

XIV.—The Minimum System of Two Quadratic Forms in n Variables.

By **Professor H. W. Turnbull**, M.A., and **J. Williamson**, M.A., Lecturer in the University of St Andrews.

(MS. received February 14, 1925. Read March 9, 1925.)

INTRODUCTION.

THE following pages deal with the simultaneous system of two general quadratic forms in n homogeneous variables. It is a special case of *Gordan's Theorem* * which proves such systems to be finite, for the general projective group of linear transformations. While several works have dealt with the cases when $n=2$, 3,† or 4,‡ nothing seems to have been written on the general case except a memoir in the year 1908.§ We continue, and simplify, the results there obtained, and now establish that

(1) *All rational integral concomitants of two quadratic forms in* n *variables and any number of sets of linear variables may be expressed as rational integral functions of* (3n + 1) *concomitants, and forms derivable by polarisation.*

(2) *These* (3n + 1) *forms, called the H system, constitute a strictly irreducible system.*

This system is exhibited in § 7 as

$$\mathrm{H} \quad \left\{ \begin{array}{ll} n+1 & \textit{invariants} \\ n & \textit{quadratic covariants} \\ n-1 & \textit{functional determinants} \\ 1 & \textit{covariant of order } \mathrm{n}, \end{array} \right.$$

together with polars.

The work is divided into three chapters: I §§ 1–4 is introductory notation, II §§ 5–17 provides a proof of these theorems, while III §§ 18–21 gives the non-symbolic and canonical forms of the results.

I.—NOTATION.

§ 1. *Variables.*—Let $(x) \equiv x^{(1)}, x^{(2)}, \ldots x^{(n)}$, (1)

denote a set of n independent variables, and suppose that (x) denotes point co-ordinates in space of $(n-1)$ dimensions.

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* Hilbert, *Math. Ann.*, xxxvi (1890), p. 473 ; Grace and Young, *Algebra of Invariants.*

† *Ibid.*, p. 161, also pp. 274–286.

‡ Turnbull, *Proc. London Math. Soc.* (2), xviii (1919), pp. 69–94.

§ Turnbull, *Trans. Camb. Phil. Soc.*, xxi (1909), pp. 197–240.

Let (ξ_1), (ξ_2), . . . (ξ_{n-1}) each be sets of n variables cogredient with (x). (2)

Let $$(u) \equiv u^{(1)}, u^{(2)}, \ldots u^{(n)}, \qquad (3)$$

denote a set of n independent variables contragredient to (x). Then (u) may be regarded as co-ordinates of a space of $(n-2)$ dimensions—the reciprocals of the point co-ordinates.

Further, let (u_1), (u_2), . . . (u_{n-1}) each be sets of n variables cogredient with (u) and therefore contragredient to (x). (4)

Co-ordinate System.—A linear space S_{n-r} is the intersection of $(r-1)$ linear spaces S_{n-2}, and therefore should be completely defined by $(r-1)$ sets of variables (u). It is easily proved to have co-ordinates given by the quantities $\pi = (u_1 u_2 \ldots u_{r-1})$, $\binom{n}{r-1}$ in number, which are the minors formed from the first $(r-1)$ columns in the determinant $(u_1 u_2 \ldots u_{n-1} a)$
$$= | u_1^{(1)} u_2^{(2)} \ldots u_{n-1}^{(n-1)} a^{(n)} |.$$

§ 2. *Dual Co-ordinate Systems.*—We combine these with an alternative dual system developed by the $(n-1)$ symbols ξ_1, ξ_2, . . . ξ_{n-1}, as follows:—

$$\left.\begin{array}{rcccccl} u_1 & = & \pi_1 & = & u & = & (\xi_1\xi_2 \ldots \ldots \xi_{n-1}), \\ (u_1 u_2) & = & \pi_2 & & & = & (\xi_1\xi_2 \ldots \xi_{n-2}), \\ \cdots & & \cdots & & & & \cdots \\ \cdots & & \cdots & & & & \cdots \\ (u_1 u_2 \ldots u_{n-2}) & = & \pi_{n-2} & & & = & (\xi_1\xi_2), \\ (u_1 u_2 \ldots \ldots u_{n-1}) & = & \pi_{n-1} & = & x & = & \xi_1. \end{array}\right\} \quad . \quad (1)$$

The number of co-ordinates in the set π_r is $\binom{n}{r}$, $r = 1, 2, \ldots (n-1)$.

These sets give a basis for a complete investigation * of concomitants of n-ary ground forms. To take less than $n-1$ sets ξ would lead to an incomplete result: to take more than $n-1$ sets would lead to only one further concomitant, the outer product $(\xi_1\xi_2 \ldots \xi_n)$.

Let $$\sum_{ij=1}^{n} a_{ij} x^{(i)} x^{(j)} = f, \qquad \{a_{ij} = a_{ji}\}, \quad . \quad . \quad . \quad . \quad (2)$$

be a quadratic form in the n variables $x^{(1)}, x^{(2)}, \ldots x^{(n)}$.

Let f be denoted symbolically by

$$a_x^2 = (ax)^2 = (a \mid x)^2 = \left[\sum_{i=1}^{n} a_i x^i \right]^2, \quad . \quad . \quad . \quad . \quad (3)$$

where the symbolic product $a_i a_j$ is equivalent to the actual coefficient a_{ij}.

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* Clebsch, *Göttinger Nachrichten*, xvii (1872); Weitzenböck, *Invariantentheorie*, pp. 148–159, Groningen (1923).

Thus,
$$a_{ij} = a_i a_j = a_j a_i = a_{ji}. \qquad (4)$$

Similarly let
$$b_x{}^2 = (bx)^2 = (b \mid x)^2 = \sum_{i,j=1}^{n} b_{ij}\, x^{(i)} x^{(j)} = f', \qquad (5)$$
be a second quadratic form in the same n variables.

§ 3. FUNDAMENTAL THEOREM.*—*Any rational integral concomitant of two quadratics is expressible as an aggregate of terms whose factors are of two kinds only—outer and inner products.*

An *outer product* is a determinant of the n^{th} order consisting of n symbols either of type c or of type u, c standing for either of the symbols a or b, and u for the variables. It is denoted by
$$(c_1 c_2 \ . \ . \ . \ c_i u_1 u_2 \ . \ . \ . \ u_j), \quad i + j = n. \qquad (6)$$

An *inner product* is denoted by various alternatives,
$$u_x = (ux) = (-)^{n-1}(xu) = (u \mid x) = (x \mid u) = \sum_{i=1}^{n} u_i\, x^{(i)}. \qquad (7)$$

If, in the outer product (6) $j = n - 1$, we obtain the inner product (cx), so that an inner product is in reality a special case of an outer product.

§ 4. ABBREVIATED NOTATION FOR OUTER PRODUCTS.—When in an outer product (6), r of the symbols c_1, c_2, . . . denote a's belonging to f, and s denote b's belonging to f', where $r + s = i$, we write
$$(c_1 c_2 \ . \ . \ . \ c_i\, u_1 u_2 \ . \ . \ . \ u_j) = (a_1 a_2 \ . \ . \ . \ a_r\, b_1 \ . \ . \ . \ b_s\, u_1 \ . \ . \ . \ u_j) = (a_r \beta_s\, u_1 \ . \ . \ . \ u_j). \qquad (8)$$

Thus a_r is a symbol of *currency* r or an *r-fold* symbol, and stands for the $\binom{n}{r}$ minors in the first r columns of the determinant
$$(a_1 a_2 \ . \ . \ . \ a_r\, b_1 \ . \ . \ . \ b_s\, u_1 \ . \ . \ . \ u_j).$$

By means of the dual co-ordinate system (1) there is an alternative way of writing a symbolic factor which involves variables.

Let θ_k be a symbolic convolution of k symbols c, so that $(\theta_k \pi_{n-k})$ is the outer product $(c_1 c_2 \ . \ . \ . \ c_k u_1 u_2 \ . \ . \ . \ u_{n-k})$.

We define $(\theta_k \xi_1 \xi_2 \ . \ . \ . \ \xi_k)$, or more specifically
$$(\theta_k \mid \xi_1 \xi_2 \ . \ . \ . \ \xi_k) \quad \text{as} \quad \Sigma(c_1 \xi_1)(c_2 \xi_2) \ . \ . \ . \ (c_k \xi_k), \qquad (9)$$
summed for all $k!$ determinantal permutations (alternations) of either, $c_1 c_2 \ldots c_k$ with the order of $\xi_1 \xi_2 \ . \ . \ . \ \xi_k$ unchanged, or $\xi_1 \xi_2 \ . \ . \ . \ \xi_k$ with the order of $c_1 c_2 \ . \ . \ . \ c_k$ unchanged.

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* Clebsch, *Crelle*, lix (1861); Weitzenböck, *Wiener Berichte*, cxxii (1913); Grace and Young, *Algebra of Invariants*.

This last is denoted by these alternative devices :—

$$\left.\begin{array}{lll}\Omega_c(c_1\xi_1)(c_2\xi_2)\ldots(c_k\xi_k) & \text{or} & (\dot{c}_1\xi_1)(\dot{c}_2\xi_2)\ldots(\dot{c}_k\xi_k),\\ \Omega_\xi(c_1\xi_1)(c_2\xi_2)\ldots(c_k\xi_k) & \text{or} & (c_1\dot{\xi}_1)(c_2\dot{\xi}_2)\ldots(c_k\dot{\xi}_k).\end{array}\right\} \quad . \quad . \quad (10)$$

where Ω, or the dot, denotes the operation of alternation.

LEMMA.—*Any outer product involving variables is expressible as an aggregate of inner products.*

In fact $(c_1c_2 \ldots c_ku_1u_2 \ldots u_{n-k}) \equiv \Omega_c(c_1\xi_1)(c_2\xi_2) \ldots (c_k\xi_k)$ by what has already been explained.

In what follows we shall write U^k or V^k for a k-fold convolution of variables, which are any k of the $(n-1)$ ξ's and not necessarily the particular group π_{n-k}.

FURTHER NOTATION.—Symbols α, β, $\alpha^{(i)}$, $\beta^{(j)}$ with no lower suffix, denote symbols of currency $(n-1)$, *i.e.* symbols cogredient with (ξ).

In any symbolic product, g_i denotes the i^{th} bracket factor.

The symbol $[(a\alpha)^2]^r$ denotes $(a_1\alpha^{(1)})^2(a_2\alpha^{(2)})^2 \ldots (a_r\alpha^{(r)})^2$.

Throughout, use has been made of the fundamental identity,

$$(a_1a_2 \ldots a_n)b_x = \Omega_a(a_1a_2 \ldots a_{n-1}b)(a_nx). \quad . \quad . \quad . \quad (11)$$

The notation $P \equiv R$ means that $P-R$ is reducible.

II.—COMPLETE SYSTEM OF TWO QUADRATICS $f=a_x^2$ AND $f'=b_x^2$.

§ 5. We distinguish between the Complete System and the Irreducible System thus:—

If the Complete System has N members, and if the Irreducible System has M members, then $N \geq M$. In fact the Complete System is sufficient, while the Irreducible System is also necessary.

A system is complete if any rational integral concomitant can be expressed rationally and integrally in terms of its members. It is irreducible when its size is reduced to a minimum.

There are three possible types of Complete Systems

$$\left.\begin{array}{lll}\text{(i) } G & = & G(a, b, u_1, u_2 \ldots u_{n-2}\, x),\\ \text{(ii) } H & = & H(a, b, \xi_1, \xi_2, \ldots \xi_{n-2}, \xi_{n-1}),\\ \text{(iii) } K & = & K(a, b, \pi_1, \pi_2 \ldots \pi_{n-2}, x),\end{array}\right\} \quad . \quad . \quad (1)$$

where each contains coefficients a, b of the two quadratic ground forms and $(n-1)$ independent sets of variables as indicated. Geometrically K is the most desirable, since in G and H the intermediate variables $\pi_2, \ldots \pi_{n-2}$ do not occur explicitly.

The system G, which has been worked out already in a very complicated form, is quoted in § 6.

The system H is here found to be very simple and to consist of four definite types. The system K has yet to be determined.

§ 6. THE G SYSTEM.—We now quote the G system as already established [*Trans. Camb Phil. Soc.*, xxi (1909), p. 232], namely,

The G system is such that every concomitant can be expressed as a sum of terms Ξ, F, X, *where*

$$\left.\begin{array}{l} \Xi \textit{ denotes a product of } \mathrm{q} \textit{ factors } (\theta\beta_s \mathrm{U})(\theta \mathrm{V}\underline{b})\underline{b}_x, \\ \mathrm{F} \textit{ denotes } \left(\dfrac{\beta_{s_1}\beta_{s_2}\ldots\beta_{s_q}}{b_1 b_2 \ldots b_i}\underline{ab}\mathrm{V}\right),\ q>i \\ \mathrm{X} \textit{ denotes } \underline{a}_x\underline{b}_x b_{1_x} b_{2_x} \ldots b_{i_x}, \end{array}\right\} \quad . \quad . \quad (2)$$

with the exception of those of weight two,

$$(\theta \mathrm{V}\underline{b})(\theta \mathrm{U}\underline{b'})\underline{b}_x b'_x. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Here the *underline* means that the letter above may or may not occur; while, in F, β_{s_i} means $b_i\beta_{(s-1)i}$ and $\dfrac{\beta_{si}}{b_i}$ means either β_{s_i} or $\beta_{(s-1)i}$.

§ 7. THE H SYSTEM.—*This consists of four types*:—

i. *The* n+1 *Invariants*

$$(a_r\beta_{n-r})^2, \quad r=0,\ 1,\ \ldots\ n; \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or, non-symbolically, *the* n+1 *coefficients of* λ^r *in*

$$|a_{ij}+\lambda b_{ij}| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

corresponding to the well-known Δ, Θ, Θ′, Δ′, when $n=3$.

ii. *The* n *Quadratic Covariants* f, f′ *and*

$$(a_r\beta_{n+1-r}\,\xi)^2, \quad r=2,\ 3,\ \ldots\ (n-1), \quad . \quad . \quad . \quad . \quad (6)$$

(or, non-symbolically, *the* n *coefficients* of λ^r *in the reciprocal of the quadric* $\sigma+\lambda\sigma'$ where σ (or σ') is the reciprocal of f, (or f')) *together with their polars with respect to any of the variables* (ξ_1), (ξ_2), . . . (ξ_{n-2}).

iii. *Functional Determinants*

$$(a_i\beta_{n-i})(a_i\mathrm{V}^i)(\beta_{n-i}\mathrm{U}^{n-i}), \quad i=1,\ 2,\ \ldots\ n-1, \quad . \quad . \quad (7)$$

or, non-symbolically,

$$\left|\frac{\partial f}{\partial\eta_1^{(1)}}\ \ldots\ \frac{\partial f}{\partial\eta_i^{(i)}}\ \ \frac{\partial f'}{\partial\eta_{i+1}^{(i+1)}}\ \ldots\ \frac{\partial f'}{\partial\eta_n^{(n)}}\right|, \quad . \quad . \quad . \quad . \quad (8)$$

where (η_i) denotes any (ξ) as long as no two are the same either in the f or the f' columns.

iv. *One Covariant of Degree* n,

$$\left[\prod_{i=2}^{n-1}(a_i\beta_{n-i-1}b_i)(a_i\beta_{n-i-1}b'_i)(b'_ix)\right](b_2b_3\ \dots\ b_{n-1}ab)a_xb_x, \quad . \quad . \qquad (9)$$

or, non-symbolically, *the Jacobian of the* n *quadratic covariants given in* ii. *This covariant equated to zero represents the* n-*cornered polyhedron self conjugate with respect to the two quadratics* f *and* f′.

§ 8. THEOREM.—*The* H *system is complete.*

We now proceed to prove this, with the help of several Lemmas (§§ 8–14). The results given in the G system form a starting-point; so it is necessary to replace the variables (u) of G by equivalent aggregates of variables (ξ) as defined in § 1 (2).

Consider first the concomitants of weight two:—

$$(\theta\,\mathrm{U}^i\underline{b})(\theta\,\mathrm{V}^i\underline{b}')b_x\underline{b}'_x.$$

If $i \neq 0$ the concomitant is reducible as it can be expressed as a sum of terms

$$\Pi(c_k\xi_k)(c_k\xi_j)\underline{b}_\eta\underline{b}'_\zeta\underline{b}_x\underline{b}'_x$$

where each factor $(c_k\xi_k)(c_k\xi_j)$ is itself a concomitant.

Again since each bracket factor which does not contain any variables, $(i=0)$, must contain exactly n symbols, it is impossible for both (θ) and (θb) to be bracket factors. Hence the only concomitants of weight two are

$$\left.\begin{array}{l}\text{(i)}\ \ \theta^2 \text{ where } \theta=(a_i\beta_{n-i}),\\ \text{(ii)}\ \ (\theta b)(\theta b')b_xb'_x \text{ where } \theta=(a_j\beta_{n-j-1}).\end{array}\right\} \quad . \quad . \quad . \qquad (10)$$

In (i) $i=0, 1, \dots n$. In (ii) $j=2, 3, \dots (n-1)$, excluding the reducible cases, $j=1$ or n.

Also (ii) is equivalent to

$$(a_j\beta_{n-j+1}x)^2, \qquad j=2,\ 3,\ \dots\ (n-1). \quad . \quad . \quad . \qquad (11)$$

§ 9. Now consider in a similar manner the bracket factors in the product Ξ, where $(\theta\beta_s\mathrm{V}^k)(\theta\underline{b}\mathrm{U}^j)\underline{b}_x$ is a typical factor P say.

If $k \neq 0$, then $k \geq s$; and P may be expressed as a series of terms each of which involves a reducible concomitant factor of type $(\theta\mathrm{U}^{k-s})(\theta\mathrm{V}^{k-s})$. Therefore $k=0$.

If $j \neq 0$, $(\theta b\mathrm{U}^j)b_x$ is reducible, since it may be expressed as a sum of terms each involving a concomitant factor $(b\eta)b_x$.

The types of Ξ factors accordingly are

$$\left.\begin{array}{l}(1)\ \ (\theta\beta_s)(\theta\mathrm{V}^{n-s}),\\ (2)\ \ (\theta b)(\theta b')b_x.\end{array}\right\} \quad . \quad . \quad . \qquad (12)$$

§ 10. Consider next the factor F

$$=\left[\frac{\beta_{s_1}\beta_{s_2}\ .\ .\ .\ \beta_{s_q}}{b_1 b_2\ .\ .\ .\ b_i}\underline{ab}\mathrm{V}^k\right]\underline{a}_x\underline{b}_x$$

where $q>i$.

If $k\neq 0$ neither a nor b may occur, else F may be expressed as a sum of terms each involving reducible factors of the type $(a\eta)a_x$ or $(b\eta)b_x$.

Further, if $k=0$, a must occur in F, else any concomitant containing F would be reducible mod $b_\beta{}^2$.

F may therefore be one of three types:—

$$\left.\begin{array}{ll}(1) & \mathrm{F}=\left(\dfrac{\beta_{s_1}\ .\ .\ .\ \beta_{s_q}}{b_1\ .\ .\ .\ b_i}\mathrm{V}^k\right). \\ (2) & \mathrm{F}=\left(\dfrac{\beta_{s_1}\ .\ .\ .\ \beta_{s_q}}{b_1\ .\ .\ .\ b_i}a\right)a_x, \\ (3) & \mathrm{F}=\left(\dfrac{\beta_{s_1}\ .\ .\ .\ \beta_{s_q}}{b_1\ .\ .\ .\ b_i}ab\right)a'_x b'_x\end{array}\right\}\quad .\ \ .\ \ .\ \ .\ \ .\ \ (13)$$

§ 11. Consider now the concomitants in order of ascending weight.

Those of weight zero are the quadratics themselves $a_x{}^2$ and $b_x{}^2$ together with forms derived from them by polarisation. There are none of weight one and those of weight two have already been considered.

Weight Three.—Such a concomitant contains one Ξ factor of either type given by (12) and one F factor given by (13).

We obtain four types:—

$$\begin{array}{ll}(1) & (\theta\beta_s)(\theta\mathrm{V}^{n-s})(\beta_s\mathrm{V}^s), \\ (2) & (\theta\beta_s)(\theta\mathrm{V}^{n-s})(\beta_s a)a_x, \\ (3) & (\theta\beta_s)(\theta\mathrm{V}^{n-s})(\beta_s ab)a_x b_x, \\ (4) & (\theta b)(\theta b')b'_x b_\xi.\end{array}$$

Type (3) *reduces.*—Here $s=n-2$: therefore write $\theta=(cd)$ and $\mathrm{V}^2=(\xi'\xi'')$. Bracket the b from b_x in g_1, and (3) is equivalent to two terms, which have the concomitant factors $(c\xi')c_x$ and $(d\xi'')d_x$ respectively.

In *type* (1) $\theta=a_i\beta_k$ where $i+k=n-s$ and $i>0$. Therefore *type* (1) becomes

$$\begin{array}{ll} & (a_i\beta_k\beta_{n-i-k})(a_i\beta_k\mathrm{U}^{i+k})(\beta_{n-i-k}\mathrm{V}^{n-i-k}), \\ \equiv & \Sigma(a_i\beta_{n-i})(a_i\mathrm{V}^i)(\beta_{n-i}\mathrm{U}^{n-i}) \text{ say.}\end{array}$$

In *type* (4) $\theta=(a_i\beta_{n-1-i})$ and accordingly *type* (4) is equivalent to $(a_i\beta_{n+1-i}x)(a_i\beta_{n+1-i}\xi)$.

Type (2) is only a particular case of this, where $i=2$.

Concomitants of Weight Three are then of types:—

$$\left.\begin{array}{lll}(1) & (a_i\beta_{n-i})(a_i\mathrm{V}^i)(\beta_{n-i}\mathrm{U}^{n-i}), & \\ (2) \text{ and } (4) & (a_i\beta_{n+1-i}\xi)(a_i\beta_{n+\ -i}x), & i=2,3,\ .\ .\ .\ n-1.\end{array}\right\}\quad .\quad (14)$$

§ 12. CONCOMITANTS OF WEIGHT GREATER THAN THREE.—If, in F, $k \neq 0$ the concomitant reduces because it may be expressed as a sum of terms each involving a factor of type

$$(\theta\beta_s)(\theta V^{n-s})(\beta_s U^s).$$

LEMMA.—*Any concomitant of weight greater than three, which has a factor* $(\theta\beta_s)$ *where* $s>1$, *is reducible.*

Proof.—Let such a concomitant be $(\theta\beta_s)(\theta V)a_x M$.

Bracketing a from a_x in g_1 we either increase the number of a's convolved in g_1 and so reduce the concomitant, or else turn out another a, say a'. In the second case since a' occurs again in g_2 the concomitant contains the factor $a'_x a'_{\xi}$ and is therefore reducible.

Accordingly if the weight of a concomitant be greater than three then, in each bracket factor $(\theta\beta_s)$, s must be unity.

Now the F bracket factor contains n symbols one of which must be a, and so it must either be

$$(b_1 b_2 \ldots b_{n-1} a) \text{ or } (b_1 b_2 \ldots b_{n-2} a b),$$

where, since $s_i = 1$ for all values of i, $b_1, \ldots b_{n-1}$, denote $\beta_{s_1}, \ldots \beta_{s_{n-1}}$.

There are two such types of concomitants:—

$$\left.\begin{array}{ll} (1) & \prod\limits_{i=1}^{n-1}(\theta_i b_i)(\theta_i b'_i)b'_{i_x}(b_1 b_2 \ldots b_{n-1} a)a_x, \\ (2) & \prod\limits_{i=2}^{n-1}(\theta_i b_i)(\theta_i b'_i)b'_{i_x}(b_2 b_3 \ldots b_{n-1} a b)a_x b_x. \end{array}\right\} \quad . \quad . \quad . \quad (15)$$

§ 13. LEMMA.—*If* $\theta_i = \theta_s$ *then either of type* (15) *reduces.*

Proof.—Since in each type (15) $b_1 b_2 \ldots b_{n-1}$ are convolved in one bracket we may write each in the form

$$(\underline{b}_1 b_2 \ldots b_{n-1} \mid \underline{\theta}_1 \theta_2 \ldots \theta_{n-1})\underline{(\theta_1 b'_1)} \ldots (\theta_{n-1} b'_{n-1})M.$$

If $\theta_i = \theta_s$ we may write this

$$\begin{aligned} &\equiv (\underline{b}_1 b_2 \ldots b_{n-1} \mid \underline{\theta}_1 \theta_2 \ldots \theta_{n-1})(\theta_i \theta_s \mid b'_i b'_s)(b'_i x)(b'_s x)N \\ &\equiv 0 \end{aligned}$$

since

$$(\theta_i \theta_s \mid b'_i b'_s)(b'_i x)(b'_s x) = 0.$$

§ 14. LEMMA.—*If* $\theta_i = (a\beta_{n-2})$ *the concomitant reduces.*

Proof.—The concomitant may be written

$$(a\beta_{n-2} b_i)(a\beta_{n-2} b'_i)(b'_i x)M = E.$$

Bracketing the b'_i from $(b'_i x)$ in g_1 we have

$$(n-1)E \equiv (b'_i \beta_{n-2} b_i)(a\beta_{n-2} b'_i)a_x M + (a\beta_{n-2} b'_i)^2 (b_i x)M.$$

The first term is reducible mod b_β^2 and the second mod a_β^2.

There can therefore be only one irreducible concomitant of type (15), viz., that in which each θ is of type $(a_i\beta_{n-i-1})$ $i=2, \ldots (n-1)$, and all the θ's are different.

This irreducible concomitant is

$$\left[\prod_{i=2}^{n-1}(a_i\beta_{n-i-1}b_i)(a_i\beta_{n-i-1}b'_i)(b'_i x)\right](b_2b_3 \;.\;.\;.\; b_{n-1}ab)a_xb_x \;.\qquad . \quad (16)$$

Wherefore the types of irreducible concomitants are completely included in i–iv, *consisting of* (3n+1) *members together with forms derived therefrom by polarisation.*

§ 15. It is obvious that types i, ii, and iii are symmetrical in the symbols of both quadratics.

To prove that type iv *can be transformed into a form in which the symbols* a *and* b *are interchanged.*

$$\prod_{i=2}^{n-1}(a_i\beta_{n-i-1}b_i)(a_i\beta_{n-i-1}b'_i)(b'_i x)(b_2b_3 \;.\;.\;.\; b_{n-1}ab)a_xb_x,$$

$$\equiv \lambda\prod_{i=2}^{n-1}(a_{i-1}\beta_{n-i-1}b_ib'_i)(a_i\beta_{n-i-1}b'_i)(a_ix)(b_2b_3 \;.\;.\;.\; b_{n-1}ab)a_xb_x$$

$$\text{mod } (a_i\beta_{n-i-1}b'_i)^2 \text{ where } a_i = (a_{i-1}a_i) = (a_{i-2}a'_i a_i),$$

$$\equiv \mu\prod_{i=2}^{n-1}(a_{i-2}a'_i\beta_{n-i-1}b_i\, b'_i)(a_{i-2}a_i\beta_{n-i-1}b_i\, b'_i)(a_ix)(a'_2 \;.\;.\;.\; a'_{n-1}ab)a_xb_x$$

$$\text{mod } (a_{i-2}a'_i\beta_{n-i-1}b_i\, b'_i)^2,$$

$$\equiv \nu\prod_{i=2}^{n-1}(\beta_{n-i+1}a_{i-2}a'_i)(\beta_{n-i+1}a_{i-2}a_i)(a_ix)(a'_2 \;.\;.\;.\; a'_{n-1}ab)a_xb_x,$$

where λ, μ, ν are numerical and non-zero.

Putting $(n-i+1)=k$ and writing for a_{n+1-k}, a_k* the concomitant is equivalent to

$$\prod_{k=2}^{n-1}(\beta_k a_{n-k-1}a'_k)(\beta_k a_{n-k-1}a_k)(a_kx)(a'_2a'_3 \;.\;.\;.\; a'_{n-1}ab)a_xb_x.$$

This is the same type as that with which we started except that a and b are interchanged.

§ 16. THEOREM.—*The H System is Irreducible.*

Proof.—The system contains four types of concomitants given by i, ii, iii, and iv.

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* This is permissible as the suffix is simply a distinguishing mark and does not denote currency.

i. *The* (n + 1) *Invariants* $(a_r\beta_{n-r})^2$. $r=0\ 1, \ldots n$.

If $(a_r\beta_{n-r})^2$ is reducible it must be expressible as a sum of terms $(a_s\beta_{n-s})^2$ and this is impossible unless $r=s$. Therefore the $(n+1)$ invariants are irreducible.

ii. *The* n *Quadratic Covariants* a_x^2, b_x^2, $(a_r\beta_{n-r+1}x)^2$.

If $(a_r\beta_{n-r+1}x)^2$ is reducible it must be identically equal to

$$\lambda a_x^2(a_{r-1}\beta_{n-r+1})^2 + \mu b_x^2(a_r\beta_{n-r})^2, \quad . \quad . \quad . \quad . \quad (17)$$

where λ and μ are numerical. For these are the only possible reducible terms of degree $(r, n-r+1)$. This is true for all values of x and therefore is true for $x=a$, and $x=\beta$.

Now $(a_r\beta_{n-r+1}a) = \underset{a_r}{\Omega} a_{1_a}(a_{r-1}\beta_{n-r+1})$, $[a_r=(a_1a_{r-1})]$ and so $(a_r\beta_{n-r+1}a)^2$ contains the factor a_a^2, which is not a factor of the second term of (17). Accordingly μ vanishes.

Similarly it may be proved, by substituting $x=\beta$, that λ must vanish. But $(a_r\beta_{n-r+1}x)^2$ is not identically equal to zero, and therefore our original assumption is false, and $(a_r\beta_{n-r+1}x)^2$ is irreducible, $r=2, 3, \ldots (n-1)$.

iii. *The Functional Determinants* $(a_r\beta_s)(a_rU^r)(\beta_sV^s)$.

The concomitant $(a_r\beta_s)(a_rU^r)(\beta_sV^s)$ cannot be expressed as a sum of terms $(a_p\beta_q)(a_pU^p)(\beta_qV^q)$ unless $r=p$ and $s=q$.

If it contained an invariant factor this factor would be $(a_r\beta_s)^2$, and the other factor would be the bracket factor (U^rV^s), which is identically equal to zero, since at most there are $n-1$ independent sets (u). Since $(a_r\beta_s)(a_rU^r)(\beta_sV^s)$ is not identically equal to zero, the concomitant does not contain an invariant factor.

The only other possibility is, that it might be a sum of products of r forms polarised from a_x^2, and s from b_x^2. But such a product would contain $2(r+s)$ variables (ξ), and the original concomitant contains only $(r+s)$: in other words the weights of the two would disagree.

Accordingly $(a_r\beta_s)(a_rU^r)(\beta_sV^s)$ is irreducible.

iv. *The Covariant J of order* n *in* x.

If this covariant is reducible it must be an aggregate of terms, each of which is a product of invariant factors and the quadratic covariants f_{r+1}. If n is odd this is impossible. If n is even let it equal $2m$ say.

Then $J = \Sigma \prod_{i=1}^{m} I_i f_i$ where f_i denotes either a_x^2, b_x^2 or any of f_{r+1}, and I is a product of invariants. J is of degree $\frac{n(n-1)}{2}$ in the coefficients of each quadratic a_x^2 and b_x^2.

In I_i the total degree of the coefficients is 0 mod n.

In f_i the total degree of the coefficients is 1 mod n. Therefore the total degree of the coefficients on the right-hand side is m mod n, which contradicts the fact that the degree of the coefficients in J is 0 mod n. Therefore J is irreducible.

§ 17. ALGEBRAICAL INDEPENDENCE OF INVARIANTS AND COVARIANTS.—The number of algebraically independent invariants and covariants is $2n+1$, as is seen by counting constants* in the ground forms and the relations which state a general linear transformation (*cf.* Burnside and Panton, *Theory of Equations* (Dublin, 1904), ii, p. 292). This is fully explained by the syzygy which reduces the square of the Jacobian.†

In fact J^2 is a form of order n in the variables

$$f_1, f_2, \ldots f_n$$

with rational functions of the $(n+1)$ fundamental invariants for its coefficients. So the binary and ternary cases are typical of the general case.

III.—NON-SYMBOLIC REPRESENTATION OF THE CONCOMITANTS.

§ 18. Let

$$0 = u_a^2 = [\Sigma a_i u^{(i)}]^2 = \Sigma a_{ij} u^{(i)} u^{(j)} \quad . \quad . \quad . \quad . \quad (1)$$

be the tangential equation of the quadratic a_x^2.

Now

$$u_a^2 = -(n-1)! \begin{vmatrix} 0 & u^{(1)} & u^{(2)} & u^{(3)} & \ldots & u^{(n)} \\ u^{(1)} & a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ u^{(2)} & a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ u^{(n)} & a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix} \quad . \quad . \quad (2)$$

On expanding this determinant we see that the coefficient of $u^{(i)}u^{(j)}$ in u_a^2 is equal to $(n-1)!$ times the cofactor of a_{ij} in Δ where

$$\Delta = | a_{11} a_{22} \ldots a_{nn} | \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the discriminant of f.

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* This method is only safe after proving that a canonical form exists which involves the same number of constants. Otherwise it may lead to a downright error. We here assume the well-known result that the two general quadratic forms can be expressed simultaneously each as the sum of n squares.

† Gilham, *Proc. London Math. Soc.* (2), xx (1921), p. 326.

Thus

$$\left.\begin{array}{l} \alpha_{ij}=(n-1)!\ \mathrm{A}_{ij} \\ \beta_{ij}=(n-1)!\ \mathrm{B}_{ij} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

where B_{ij} is the cofactor of b_{ij} in Δ' where

$$\Delta'=|\,b_{11}b_{22}\ .\ .\ .\ b_{nn}\,| \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

If in the same way we build the point form of the tangential form $u_\alpha{}^2$, we get a similar determinant with x's in place of the u's and $(n-1)$ equivalent symbols a in place of the a's; thus,

$$(\alpha^{(1)}\alpha^{(2)}\ .\ .\ .\ \alpha^{(n-1)}x)^2=-(n-1)!\begin{vmatrix} 0 & x^{(1)} & x^{(2)} & . \ . \ . & x^{(n)} \\ x^{(1)} & \alpha_{11} & \alpha_{12} & . \ . \ . & \alpha_{1n} \\ x^{(2)} & \alpha_{21} & \alpha_{22} & . \ . \ . & \alpha_{2n} \\ . & . & . & . \ . \ . & . \\ . & . & . & . \ . \ . & . \\ x^{(n)} & \alpha_{n1} & \alpha_{n2} & . \ . \ . & \alpha_{nn} \end{vmatrix} \quad . \qquad (6)$$

where

$$(\alpha^{(1)}\alpha^{(2)}\ .\ .\ .\ \alpha^{(n-1)}x)=\underset{\alpha}{\Omega}(a_1\alpha^{(2)})(a_2\,\alpha^{(3)})\ .\ .\ .\ (a_{n-2}\alpha^{(n-1)})(a_{n-1}x)$$

and

$$\alpha^{(1)}=(a_1a_2a_3\ .\ .\ .\ a_{n-1}).$$

We now build the point form of the quadratic $u_\alpha{}^2+\lambda u_\beta{}^2$ which is equal to

$$\left.\begin{array}{l} -(n-1)!\begin{vmatrix} 0 & x^{(1)} & x^{(2)} & . \ . \ . & x^{(n)} \\ x^{(1)} & \alpha_{11}+\lambda\beta_{11} & \alpha_{12}+\lambda\beta_{12} & . \ . \ . & \alpha_{1n}+\lambda\beta_{1n} \\ x^{(2)} & \alpha_{21}+\lambda\beta_{21} & \alpha_{22}+\lambda\beta_{22} & . \ . \ . & \alpha_{2n}+\lambda\beta_{2n} \\ . & . & . & . \ . \ . & . \\ . & . & . & . \ . \ . & . \\ x^{(n)} & \alpha_{n1}+\lambda\beta_{n1} & \alpha_{n2}+\lambda\beta_{n2} & . \ . \ . & \alpha_{nn}+\lambda\beta_{nn} \end{vmatrix} \\ =-[(n-1)!]^n\begin{vmatrix} 0 & x^{(1)} & x^{(2)} & . \ . \ . & x^{(n)} \\ x^{(1)} & \mathrm{A}_{11}+\lambda\mathrm{B}_{11} & \mathrm{A}_{12}+\lambda\mathrm{B}_{12} & . \ . \ . & \mathrm{A}_{1n}+\lambda\mathrm{B}_{1n} \\ x^{(2)} & \mathrm{A}_{21}+\lambda\mathrm{B}_{21} & \mathrm{A}_{22}+\lambda\mathrm{B}_{22} & . \ . \ . & \mathrm{A}_{2n}+\lambda\mathrm{B}_{2n} \\ . & . & . & . \ . \ . & . \\ . & . & . & . \ . \ . & . \\ x^{(n)} & \mathrm{A}_{n1}+\lambda\mathrm{B}_{n1} & \mathrm{A}_{n2}+\lambda\mathrm{B}_{n2} & . \ . \ . & \mathrm{A}_{nn}+\lambda\mathrm{B}_{nn} \end{vmatrix} \\ =[(n-1)!]^n\mathrm{D}',\text{ say} \\ =[(n-1)!]^n(\mathrm{L}_0+\mathrm{L}_1\lambda+\ .\ .\ .\ \mathrm{L}_r\lambda^r+\ .\ .\ .\ +\mathrm{L}_{n-1}\lambda^{n-1}). \end{array}\right\} \quad . \qquad (7)$$

But the point form of this quadratic is denoted symbolically by

$$(\alpha^{(1)}\alpha^{(2)}\ .\ .\ .\ \alpha^{(n-1)}x)^2+\binom{n-1}{1}(\alpha^{(1)}\ .\ .\ .\ \alpha^{(n-2)}\beta x)^2\,\lambda+\ .\ .\ .$$

$$+\binom{n-1}{r}(\alpha^{(1)}\ .\ .\ .\ \alpha^{(n-r-1)}\beta^{(1)}\ .\ .\ .\ \beta^{(r)}x)^2\,\lambda^r+\ .\ .\ .\ +(\beta^{(1)}\beta^{(2)}\ .\ .\ .\ \beta^{(n-1)}x)^2\,\lambda^{(n-1)} \qquad (8)$$

Thus by equating coefficients of powers of λ we have

$$\binom{n-1}{r}(a^{(1)}a^{(2)}\ .\ .\ .\ a^{(n-r-1)}\beta^{(1)}\beta^{(2)}\ .\ .\ .\ \beta^{(r)}x)^2=[(n-1)!]^n\,\mathrm{L}_r, \qquad . \quad (9)$$

$$r=0,\ 1,\ .\ .\ .\ (n-1).$$

These, however, are not in the symbolic form $(a_{r+1}\beta_{n-r}x)^2$ given by (§ 7 (6)). The connection is in fact

$$(a^{(1)}a^{(2)}\ .\ .\ .\ a^{(n-r-1)}\beta^{(1)}\ .\ .\ .\ \beta^{(r)}x)^2=\mathrm{M}_r\,(a_{r+1}\beta_{n-r}x)^2 \qquad . \quad . \quad (10)$$

where

$$\mathrm{M}_r=\frac{[(n-1)\,!]^{(n-1)}}{(r+1)(n-r)}\Delta^{n-r-2}\Delta'^{r-1}.$$

Proof.—Let

$$(a^{(1)}a^{(2)}\ .\ .\ .\ a^{(i)}\beta^{(1)}\beta^{(2)}\ .\ .\ .\ \beta^{(r)}x)=(a^{(1)}a^{(2)}\ .\ .\ .\ a^{(i)}\theta)^2,\ i+r=n-1.$$

Put

$$a^{(1)}=(a_1a_2\ .\ .\ .\ a_{i-1}a_{n-i})$$
$$a^{(2)}=(a'_1a'_2\ .\ .\ .\ a'_{i-1}a'_{n-1})$$

so that

$$(a^{(1)}a^{(2)}\ .\ .\ .\ a^{(i)}\theta)=\underset{a}{\Omega}(a_1a^{(2)})(a_2a^{(3)})\ .\ .\ .\ (a_{i-1}a^{(i)})(a_{n-i}\theta),$$

a series of $\binom{n-1}{i-1}(i-1)\,!$ terms; and in the same way it is equal to a series with $a^{(2)}$ and $a^{(1)}$ interchanged.

Since a's are equivalent symbols, the product of those two is a series all of whose terms are equal, thus writing

$$\lambda=\left[\binom{n-1}{i-1}(i-1)\,!\right]^2,$$

$$\left.\begin{aligned}
(a^{(1)}a^{(2)}\ .\ .\ .\ a^{(i)}\theta)^2 &= -\lambda(a_1a^{(2)})(a_2a^{(3)})\ .\ .\ .\ (a_{i-1}a^{(i)})(a'_1a^{(1)})(a'_2a^{(3)})\ .\ .\ .\ (a'_{i-1}a^{(i)})(a_{n-i}\theta)(a'_{n-i}\theta),\\
&=\lambda n^{2-i}(a'_2a^{(3)})^2(a'_3a^{(4)})^2\ .\ .\ .\ (a'_{i-1}a^{(i)})^2(a_1a'_1a_2\ .\ .\ .\ a_{i-1}a'_{n-i})(a_1a'_1\ .\ .\ .\ a_{i-1}a_{n-i})(a_{n-i}\theta)(a'_{n-i}\theta),\\
&=\frac{\lambda n^{2-i}}{\binom{n}{i}}[a_a{}^2]^{i-2}(a_1a'_1a_2\ .\ .\ .\ a_{i-1}a_{n-i})^2(a'_{n-i}\theta)^2,\\
&=\lambda n^{1-i}[a_a{}^2]^{i-1}\frac{(n-i)\,!\ i\,!}{(n-1)\,!}(a_{n-i}\theta)^2,\\
&=\lambda\frac{(n\,!)^{i-1}(n-i)\,!\ i\,!}{n^{i-1}(n-1)\,!}\Delta^{i-1}(a_{n-i}\theta)^2,\ [\text{for } a_a{}^2=n\,!\ \Delta],\\
&=[(n-1)\,!]^i\frac{i\,!}{(n-i)\,!}\Delta^{i-1}(\beta_1\ .\ .\ .\ \beta_ra_{n-1}x)^2,\\
&=[(n-1)\,!]^i\frac{i\,!}{(n-i)\,!}\Delta^{(i-1)}\frac{[(n-1)\,!]^r}{(n-r)\,!}r\,!\ \Delta'^{(r-1)}(a_{n-i}\beta_{n-r}\,x)^2.
\end{aligned}\right\}\ (11)$$

Putting $i=n-1-r$ we have the result (10).

But

$$\binom{n-1}{r}(a^{(1)} \ldots a^{(n-r-1)}\beta^{(1)} \ldots \beta^{(r)}x)^2 = [(n-1)!]^n \mathrm{L}_r.$$

Therefore

$$(a_{r+1}\beta_{n-r}\, x)^2 = \Delta^{(r+2-n)}\Delta'^{(1-r)}(r+1)!\ (n-r)!\ \mathrm{L}_r. \qquad . \qquad . \quad (12)$$

This proof would apply in exactly a similar manner were (x) replaced by any two of the variables $(\xi_1) \ldots (\xi_{n-1})$. So by this method all the quadratic covariants and their polars with respect to any of the variables $(\xi_1) \ldots (\xi_n)$ can be expressed non-symbolically.

For let

$$(a_{r+1}\beta_{n-r}\, x)^2 = f_{r+1}, \quad . \quad . \quad . \quad . \quad . \quad (13)$$

then the n quadratic covariants are obtained by giving r in succession the values $0, 1, \ldots (n-1)$.

Incidentally it may be noted that

$$\left.\begin{aligned} f_1 &= n!\ a_x^2\Delta', \\ f_n &= n!\ b_x^2\Delta. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

§ 19. THE FUNCTIONAL DETERMINANTS E.—To obtain a non-symbolic expression for the concomitants of type

$$(a_i\beta_{n-i})(a_i\mathrm{V}^i)(\beta_{n-i}\mathrm{U}^{n-i}) = \mathrm{E}$$

we write

$$\mathrm{E} = |\, a_1^{(1)}a_2^{(2)} \ldots a_i^{(i)}b_{i+1}^{(i+1)} \ldots b_n^{(n)}\,|\,(a_1a_2 \ldots a_i \,|\, \xi_1\xi_2 \ldots \xi_i)(b_{i+1} \ldots b_n \,|\, \xi_1 \ldots \xi_{n-i})$$

where

$$a_1, a_2, \ldots a_i, b_{i+1}, \ldots b_n$$

are two sets of equivalent symbols referring to f and f' respectively, so that, for example, a_p and a_q are equivalent. Hence

$$\mathrm{E} = i!\ (n-i)!\ |\, a_1^{(1)} \ldots a_i^{(i)}b_{i+1}^{(i+1)} \ldots b_n^{(n)}\,|\,(a_1\xi_1) \ldots (a_i\xi_i)(b_{i+1}\xi_1) \ldots (b_n\xi_{n-i}). \quad (1)$$

Now multiply the k^{th} column of the determinant by the k^{th} ξ factor $(c_k\xi_l)$ where $c=a$ or b, according as $k \leq$ or $> i$, and let g denote the corresponding f or f'.

The k^{th} column then becomes :—

$$c_k^{(1)}(c_k\xi_l) = c_{11}\xi_l^{(1)} + c_{12}\xi_l^{(2)} + \ldots + c_{1n}\xi_l^{(n)} = \frac{1}{2}\,\frac{\partial g}{\partial \xi_l^{(1)}},$$

$$c_k^{(2)}(c_k\xi_l) = c_{21}\xi_l^{(1)} + c_{22}\xi_l^{(2)} + \ldots + c_{2n}\xi_l^{(n)} = \frac{1}{2}\,\frac{\partial g}{\partial \xi_l^{(2)}},$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$c_k^{(n)}(c_k\xi_l) = c_{n1}\xi_l^{(1)} + c_{n2}\xi_l^{(2)} + \ldots + c_{nn}\xi_l^{(n)} = \frac{1}{2}\,\frac{\partial g}{\partial \xi_l^{(n)}}.$$

Hence

$$\mathrm{E} = \frac{i!\ (n-i)!}{2^n}\left|\frac{\partial f}{\partial \xi_1^{(1)}} \ldots \frac{\partial f}{\partial \xi_i^{(i)}}\,\frac{\partial f'}{\partial \xi_1^{(i+1)}} \ldots \frac{\partial f'}{\partial \xi_{n-i}^{(n)}}\right|, \quad . \quad . \quad (2)$$

where $i=1, \ldots (n-1)$. The polarised forms are made by replacing the ξ suffixes $(1, 2, \ldots i;\ 1, 2, \ldots n-i)$ by suffixes $1, 2, \ldots (n-1)$ in any order, provided no two are the same in the f or the f' columns.

THEOREM.—*The covariant of degree* n, *given by* § 7 (9), *is the Jacobian of the* n *quadratic covariants* f_r, $r=1, \ldots n$.

Symbolically we write it

$$J=\left[\prod_{i=2}^{n-1}(a_i\beta_{n-i-1}b_i)(a_i\beta_{n-i-1}b'_i)(b'_ix)\right](b_2 \ldots b_{n-1}ab)a_xb_x.$$

Consider

$$(\beta_{n-i-1}a_ix)(\beta_{n-i+1}a_i\theta)=T$$

where

$$\beta_{n-i+1}=(b_ib'_i\beta_{n-i-1}).$$

Writing Ω_b for an operation on all b's, β's to the right of itself, we have

$$\begin{aligned}T&=(\beta_{n-i+1}a_i\theta)\,\Omega_b(b_i\beta_{n-i-1}a_i)(b'_ix),\\ &=(n-i+1)(\beta_{n-i+1}a_i\theta)(b_i\beta_{n-i-1}a_i)(b'_ix),\\ &=(n-i+1)(b_i\beta_{n-i-1}a_i)(b'_ix)\Omega_b(b_i\beta_{n-i-1}a_i)(b'_i\theta),\\ &\equiv-(n-i+1)(n-i)(b_i\beta_{n-i-1}a_i)(b'_i\beta_{n-i-1}a_i)(b'_ix)(b_i\theta),\end{aligned}$$

where the other term involves the factor $(b_i\beta_{n-i-1}a_i)^2$.

If now $(b_i\theta)$ denote the determinant $(b_2 \ldots b_i \ldots b_{n-1}ab)$, and $(\beta_{n-i+1}a_i)$ be denoted by ϕ_i, where $(\phi_ix)^2=f_i$, then

$$\left.\begin{aligned}J&\equiv(-)^{n-2}\prod_{i=2}^{n-1}\frac{(\phi_ix)}{(n-i)(n-i+1)}(\phi_2\phi_3 \ldots \phi_{n-1}ab)a_xb_x,\\ &\equiv\prod_{i=2}^{n-1}\frac{1}{(n-i)(n-i+1)}(a\phi_2\phi_3 \ldots \phi_{n-1}b)a_xb_x,\\ &\equiv\frac{1}{2^n}\prod_{i=2}^{n-1}\frac{1}{(n-i)(n-i+1)}\left|\frac{\partial f}{\partial x}\frac{\partial f_2}{\partial x}\frac{\partial f_3}{\partial x} \ldots \frac{\partial f_{n-1}}{\partial x}\frac{\partial f'}{\partial x}\right|,\\ &\equiv\frac{1}{2^n}\frac{1}{(n-1)!\ (n-2)!}\left|\frac{\partial f}{\partial x}\frac{\partial f_2}{\partial x} \ldots \frac{\partial f_{n-1}}{\partial x}\frac{\partial f'}{\partial x}\right|,\end{aligned}\right\} \quad . \quad . \quad (3)$$

where

$$f=\frac{f_1}{n!\ \Delta'} \quad \text{and } f'=\frac{f_n}{n!\ \Delta}.$$

§ 20. CANONICAL FORMS.—Suppose

$$\left.\begin{aligned}(A \mid X^2)&=\Sigma\, A_iX_i^2=F,\\ \text{and} \qquad (B \mid X^2)&=\Sigma\, B_iX_i^2=F',\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

be taken as the canonical forms of the two quadratics.

Then

$$\left.\begin{aligned}\Delta&=A_1A_2 \ldots A_n,\\ \text{and} \qquad \Delta'&=B_1B_2 \ldots B_n,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

The cofactor of A_j in Δ is Δ/A_j and the cofactor of B_j in Δ' is Δ'/B_j, so that

$$D' = \sum_{i=1}^{n} X_i^2(\Delta/A_i + \lambda\Delta'/B_i)^{-1}\prod_{j=1}^{n}(\Delta/A_j + \lambda\Delta'/B_j), \quad . \quad . \quad (3)$$

and

$$L_r = \Delta^{n-r-1}\Delta'^r\sum_{i=1}^{n} X_i\Sigma^i\frac{1}{B_1B_2 \ . \ . \ . \ B_rA_{r+1} \ . \ . \ . \ A_n}, \quad . \quad . \quad (4)$$

where Σ^i stands for the sum of all possible terms involving r symbols B and $(n-r-1)$ symbols A, in which neither B_i nor A_i occurs.

Accordingly,

$$\left.\begin{aligned} F_{r+1} &= (r+1)\,!\,(n-r)\,!\,\Delta\Delta'\sum_{i=1}^{n} X_i^2\,\Sigma^i\frac{1}{B_1B_2 \ . \ . \ . \ B_rA_{r+1} \ . \ . \ . \ A_n} \\ &= (r+1)\,!\,(n-r)\,!\,\Delta'\sum_{i=1}^{n} A_iX_i^2\,\Sigma^i\,C_1C_2 \ . \ . \ . \ C_r \end{aligned}\right\} . \quad (5)$$

where F_{r+1} is the value of f_{r+1} when the quadratics are taken in their canonical forms and $C_j = A_j/B_j$.

§ 21. *To find the non-symbolic expression for the covariant of the* n^{th} *order of the two quadratics* $(A \mid X^2)$ *and* $(B \mid X^2)$.

This covariant is proportional to the Jacobian J of the n quadratic covariants F_{r+1} given by (5), and so

$$J = \mu\begin{vmatrix} A_1X_1, & A_2X_2, \ . \ . \ . & A_nX_n \\ A_1X_1\Sigma^1C_1, & A_2X_2\Sigma^2C_1, \ . \ . \ . & A_nX_n\Sigma^nC_1 \\ . \ . \ . \ . \ . & . \ . \ . \ . \ . & . \ . \ . \ . \ . \\ . \ . \ . \ . \ . & . \ . \ . \ . \ . & . \ . \ . \ . \ . \\ A_1X_1\Sigma^1C_1C_2 \ . \ . \ . \ C_{n-2}, & A_2X_2\Sigma^2C_1C_2 \ . \ . \ . \ C_{n-2}, & . \ . \ . \ A_nX_n\Sigma^nC_1C_2 \ . \ . \ . \ C_{n-2} \\ A_1X_1C_2C_3 \ . \ . \ . \ C_n, & A_2X_2C_1C_3 \ . \ . \ . \ C_n, & . \ . \ . \ A_nX_nC_1C_2 \ . \ . \ . \ C_{n-1} \end{vmatrix}$$

where

$$\mu = 2^n\Delta'^n[n\,!\,(n-1)\,! \ . \ . \ . \ 3\,!\,2\,!\,1\,!]^2. \quad . \quad . \quad . \quad (6)$$

Hence

$$J = \mu\,\Delta\,X_1X_2 \ . \ . \ . \ X_n\begin{vmatrix} 1 & 1 & 1 \\ \Sigma^1C_1 & \Sigma^2C_1, \ . \ . \ . & \Sigma_nC_1 \\ \Sigma^1C_1C_2 & \Sigma^2C_1C_2, \ . \ . \ . & \Sigma^nC_1C_2 \\ . \ . \ . \ . \ . & . \ . \ . \ . \ . & . \ . \ . \ . \ . \\ . \ . \ . \ . \ . & . \ . \ . \ . \ . & . \ . \ . \ . \ . \\ \Sigma^1C_1C_2 \ . \ . \ . \ C_{n-2} & \Sigma^2C_1C_2 \ . \ . \ . \ C_{n-2}, \ . \ . \ . & \Sigma^nC_1C_2 \ . \ . \ . \ C_{n-2} \\ C_2C_3 \ . \ . \ . \ C_n & C_1C_3 \ . \ . \ . \ C_n, \ . \ . \ . & C_1C_2 \ . \ . \ . \ C_{n-1} \end{vmatrix}$$

$$= \mu\,\Delta\,X_1X_2 \ . \ . \ . \ X_nJ', \text{ say.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If $C_r = C_k$, $J' = 0$ because the r^{th} and the k^{th} columns become identical.

Accordingly $C_r - C_k$ is a factor for every value of $r, k = 1, 2, \ldots n, r \neq k$.

To prove that

$$J' = \prod_{r,\,k=1}^{r,\,k=n} (C_r - C_k), \quad [r \neq k]. \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

Proof.—There are $\frac{n(n-1)}{2}$ factors $(C_r - C_k)$: and as the degree of J′ in the C's is $\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$ our proposition must be true to a numerical factor ν, unless J′ vanishes identically.

Let the factors in the product on the right-hand side of (8) be written in the order $(C_r - C_k)$ where $r > k$ and consider the coefficient of $C_1^{n-1} C_2^{n-2} C_3^{n-3} \ldots C_{n-1}$. On the right-hand side the coefficient is ν and on the left-hand side it is $+1$ since the term considered occurs only once and then in the leading diagonal.

Consequently

$$J = \mu \Delta X_1 \ldots X_n \prod_{r,\,k=1}^{r,\,k=n} \left(\frac{A_r}{B_r} - \frac{A_k}{B_k} \right),$$

where $r > k$ and μ is non-zero by (6).

This shows clearly that the Jacobian of the n Quadratic Covariants represents the polyhedron of n vertices self-conjugate with respect to both quadratics.

(*Issued separately May* 14, 1925.)

XV.—The Law of Blackening of the Photographic Plate at Low Densities. By E. A. Baker, B.Sc. *Communicated by* Professor R. A. Sampson, F.R.S.

(MS. received January 12, 1925. Read January 12, 1925.)

I. Plan of the Investigation.

The following paper is ancillary to certain investigations in stellar photometry which are in progress at the Royal Observatory, Edinburgh, and require the establishment of data relating to the blackening of the plates employed. It describes the laboratory work carried out for this purpose, generally over the very restricted range of exposure times and densities used in the stellar work, but occasionally extended to cover a wider field. The conditions are chosen in order to give the maximum accuracy and range to the stellar work and not with a view to highly accurate laboratory work; on the other hand, the periodical change of emulsion which must occur in any lengthy investigation introduces an element which, though of considerable interest, is lacking in most of the published work on the subject. The plates used are Ilford Special Rapid Panchromatic, Imperial "B" Panchromatic, and Wellington Spectrum Panchromatic, all rapid panchromatic plates differing little in speed and having a sharp fall in sensitiveness at 660 $\mu\mu$. The stellar negatives cover a range of density from 0·05 to 1·0 and of wave-length from 380 $\mu\mu$ to 660 $\mu\mu$; and though their exposures range from a minute to an hour or more, yet in consequence of the small width of the image which is trailed over the plate the actual exposure time of any particular region of the plate is only one-twentieth of the total exposure time.

I would refer first to the method employed by Professor Sampson in the above investigations, as it determines the notation and to some extent the treatment that follows. Assuming a uniform emulsion and constant working conditions, the four quantities concerned are the illumination I of the plate, its wave-length λ, the duration t of the exposure and the resulting density D. In a preliminary paper * Professor Sampson made certain assumptions as to the relation between these four quantities; but in his later work † he makes no such assumptions, using

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* R. A. Sampson, *Monthly Notices R.A.S.*, **83**, 1923, p. 174.

† *Ibid.*, **85**, 1925, p. 212.

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the perfectly general expansion of log I in terms of two quantities Δ and T, thus:—

$$\log_{10} \mathrm{I}/\mathrm{I}_0 = a\Delta + b\mathrm{T} + \tfrac{1}{2}c\Delta^2 + d\Delta\mathrm{T} + \tfrac{1}{2}e\mathrm{T}^2 + \dots \qquad (1)$$

where Δ is a function of D only, $\mathrm{T} = \log_{10} t$, and I_0 and the coefficients a, b, c . . . are functions of the wave-length λ. As Professor Sampson points out, if Δ is replaced by D the first two terms of the expansion (1) give Hurter and Driffield's formula as modified by Schwarzschild. The form chosen for Δ in this paper is $\log_{10}(10^{\mathrm{D}} - 1)$ or $\log_{10}(\mathrm{O} - 1)$ where O is the opacity; this approaches D so rapidly that above unity Δ and D are practically interchangeable. As so defined, Δ is zero at about the middle of its range. A similar result is secured for T by choosing the unit of time as 30 seconds.

To arrive at definite values for the coefficients in (1), the quantity I_0 which is not required is eliminated by differentiation and two quantities p, q are introduced, defined by the equations

$$\left.\begin{aligned} p &= -\frac{\partial(\log \mathrm{I})}{\partial(\log t)} = -b - d\Delta - e\mathrm{T} - \dots \\ q &= +\frac{\partial(\log_{10} \mathrm{I})}{\partial\Delta} = a + c\Delta + d\mathrm{T} + \dots \end{aligned}\right\} \qquad (2)$$

It will be noticed that p is identical with Schwarzschild's index, while over a higher density range the gamma of the plate, taken as usual with reference to the exposure time, is given by $\frac{p}{q}$. By plotting p and q, first against Δ for $\mathrm{T} = 0$ (*i.e.* for 30 seconds' exposure time) and then against T for $\Delta = 0$, curves are obtained which, in so far as they can be regarded as straight lines, give the coefficients in (1) as far as those of the second degree terms, and indicate by their departures from straight lines the magnitudes of the higher terms of the expansion. When this process had been carried out for several batches of plates it became clear that the coefficient d is small, and generally that the terms of (1), involving both Δ and T, are negligible in the stellar work; but that for some emulsions and wave-lengths higher powers of Δ than the second are needed to express the results with accuracy. Accordingly Professor Sampson works with a formula equivalent to

$$\log \mathrm{I}/\mathrm{I}_0 = \theta(\lambda, \mathrm{D}) + \phi(\lambda, \mathrm{T}) \qquad (3)$$

and showed that by his method of reduction the term involving T is eliminated to a remarkable extent, completely, in fact, if that term can be put in the form

$$\phi(\lambda, \mathrm{T}) = \mathrm{A}(\lambda) + \mathrm{B}(\mathrm{T}) + \lambda\mathrm{C}(\mathrm{T}).$$

This is equivalent to the condition that, for any value of T within the range used, the relation between p and λ is linear. It is then possible to put the numerical results as far as they are required for application to the stellar spectra in the form of curves connecting log I with the deflection of the Koch photometer by which the spectra are recorded, so eliminating all reference to Δ or D. In what follows, however, no presupposition is made as to the special value of d or any other coefficient.

The measurement of p and q thus involves varying the time, the illumination, and the value of $(O-1)$ in known ratios. An accuracy of 1 per cent. is aimed at, hence the measurement of time-ratios involves no difficulties, at any rate for times over ten seconds. In this work it is more important to have a continuous and well-defined scale of opacities than one which is theoretically correct, and the scale used is to some extent arbitrary though closely approximating to the theoretical scale of "opacities in parallel light." It is obtained by throwing a beam of light of fixed relative aperture (f : 6) on to the plate and measuring, by a photo-electric cell, that portion of the light emerging within an equal relative aperture. Illumination ratios are measured by superposing two nearly equal illuminations, so giving an illumination double the mean of the separate illuminations. The practical difficulties involved in this fundamental method are very slight compared, for instance, with the elaborate screening arrangements required for the inverse square law method and the difficulty of realising a point source; while the illumination ratio 2:1 is specially appropriate to the present work, for it is sufficient to allow of accurate measures and yet so small that the variations of p and q over the corresponding differences of T and Δ are insensible.

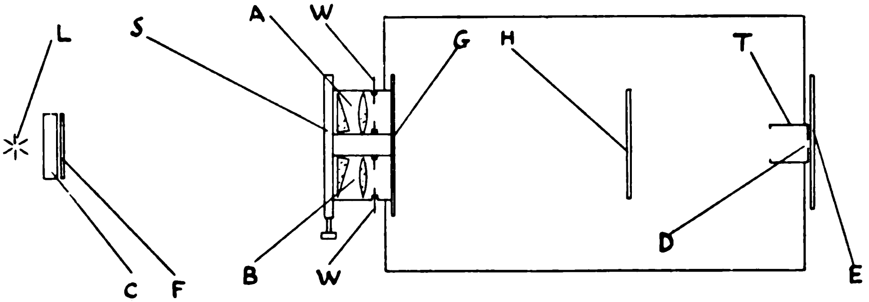

FIG. 1.—Section of camera.

The camera used is shown in fig. 1. It consists of a light-tight box about 20 inches long by 9 inches square, with two similar lens-prism systems A, B designed to throw two beams of light from the lamp L towards the plate E. A hand shutter S covers or exposes both lenses; but either beam may be occulted by substituting a piece of blackened brass for one of the Waterhouse stops W, W. Except for experiments involving high illuminations, the lamp is kept well out of focus; and to give uniform illumination over the diaphragm D, which limits the area of plate exposed, ground glass is inserted at G and H. The plate, of lantern size 3¼ inches square, is held in a double-slide plate-holder almost in contact with the aperture D, which is ⅛ inch in diameter, so that by moving the plate by ¼ inch after each exposure in one direction or the other 100 exposures are made on each plate. Later, in place of the single tube T, four such tubes were inserted with apertures spaced at the corners of a square of 1½ inches side, so that four regions of the plate, distinguished by making the apertures of different shapes, are exposed simultaneously, a full plate having 25 sets of exposures.

The source of light is a 100-cp. Pointolite lamp fed by a battery of large storage cells and controlled by a rheostat and a standard ammeter. Different regions of the spectrum are isolated by inserting at F one of the Wratten filters 71 to 76 (monochromatic series) or 18 (ultra-violet). In the later arrangement four filters can be used simultaneously by placing one in front of each of the four tubes T. A cell of N/5 copper sulphate solution 2 cm. thick is placed at C to make the effects of the filtered regions comparable: without this cell the red and orange filters give results greatly in excess of the blue and violet. So far as can be seen in a small direct-vision spectroscope this cell removes the whole of the deep red light otherwise transmitted by some of the green and blue filters. The wave-lengths used in Table VI are deduced on the assumption that the curves given by the makers to show the transmissions of the filters represent also the effective composition of the light passed by them with the copper sulphate cell in place, neglecting light at the red end beyond 670 $\mu\mu$. Stray light is eliminated by enclosing the lamp in a lantern (which must be well ventilated, for the lamp is unsteady when too hot), and by working in a dark room. The exposure times are taken from a deck watch with a luminous centre seconds hand, which moves with a sharp click at one-second intervals.

Values of q are obtained by exposing to lenses A, A + B, B in succession for equal times t, moving the plate after each exposure. Calling the corresponding illuminations I_1, $I_1 + I_2$, I_2 and the resulting blackenings Δ_1, Δ_3, Δ_2, the unavoidable difference between I_1 and I_2 may be allowed for by assuming p and q constant over small intervals of T and Δ. In such cases equations (2) give on integration

$$\log I/I' = q\Delta - pT \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Putting this equation in the equivalent form

$$c \cdot \exp\left(\frac{q\Delta}{\mu}\right) = \mathrm{I},$$

where μ is the modulus of common logarithms and $c = \mathrm{I}' \exp\left(-\frac{p\mathrm{T}}{\mu}\right)$, we have, if Δ' is the blackening which would result from an exposure to the illumination $\frac{1}{2}(\mathrm{I}_1 + \mathrm{I}_2)$ for time t,

$$\begin{aligned} c \exp\left(\frac{q\Delta'}{\mu}\right) &= \tfrac{1}{2}(\mathrm{I}_1 + \mathrm{I}_2) \\ &= \tfrac{1}{2}c\left\{\exp\left(\frac{q\Delta_1}{\mu}\right) + \exp\left(\frac{q\Delta_2}{\mu}\right)\right\} \\ &= c \exp\left(\frac{q(\Delta_1 + \Delta_2)}{2\mu}\right) \cosh \frac{q(\Delta_1 - \Delta_2)}{2\mu}, \end{aligned}$$

so that

$$\Delta' = \tfrac{1}{2}(\Delta_1 + \Delta_2) + \frac{1}{q} \log_{10} \cosh \frac{q(\Delta_1 - \Delta_2)}{2\mu} \quad . \quad . \quad . \quad (5)$$

The final term, which is given with sufficient accuracy by a very rough knowledge of the value of q, seldom exceeds 0·01. The value of q corresponding to the time t and to the mean blackening $\frac{1}{2}(\Delta_3 + \Delta')$ is then $\frac{\log_{10} 2}{\Delta_3 - \Delta'}$. Values for other densities and the same exposure time are found by repeating this process, either with another pair of Waterhouse stops, the areas of which increase roughly in the ratio 2 : 1 from 1 to 64, or for lower illuminations with a neutral absorbing screen (*e.g.* a fogged process plate developed with quinol) in front of the lamp. A 30-cp. Pointolite lamp was occasionally used.

The values of p are found by exposing successively to I_1 for time t, to $\mathrm{I}_1 + \mathrm{I}_2$ for time t_4, and to I_2 for time t, choosing the value of t_4 such that the blackenings Δ_1, Δ_4, Δ_2 obtained are approximately equal. Putting as before T for $\log_{10} t$ the value of p corresponding to the blackening Δ_4 and the mean of T and T_4 is given by

$$p(\mathrm{T} - \mathrm{T}_4) = \log_{10} 2 + q(\Delta' - \Delta_4)$$

where Δ' is given by (5). Here the value of q is supposed known with all the accuracy required from experiments on plates of the same batch if not on the same plate.

The blackening of the deposits is measured by an automatic densitometer designed in the first place for measuring the sizes of stellar images. Two photo-electric cells A, B (figs. 2–4) are connected together, and to one plate of an electrometer C, as in Koch's microphotometer.* The ball of

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* P. P. Koch, *Ann. d. Physik*, 39, 1912, p. 705.

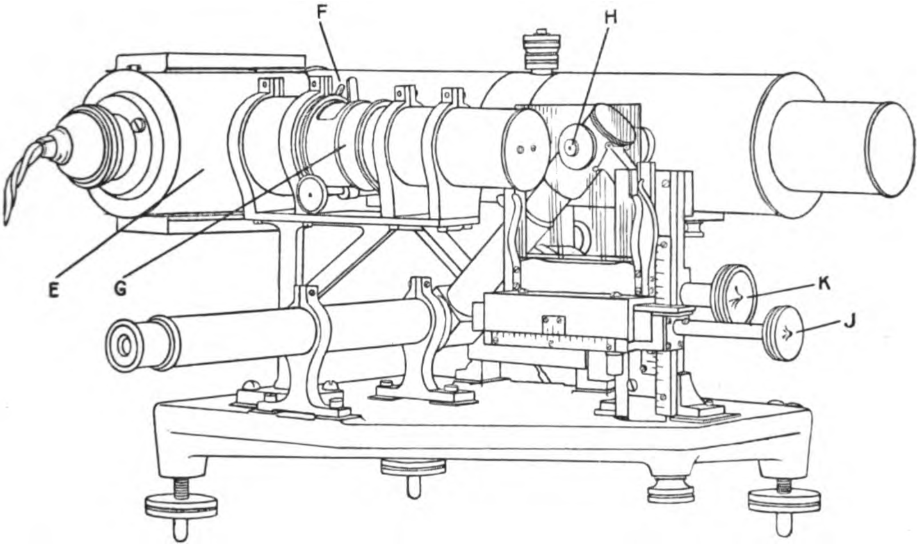

Fig. 2.—Densitometer, front view.

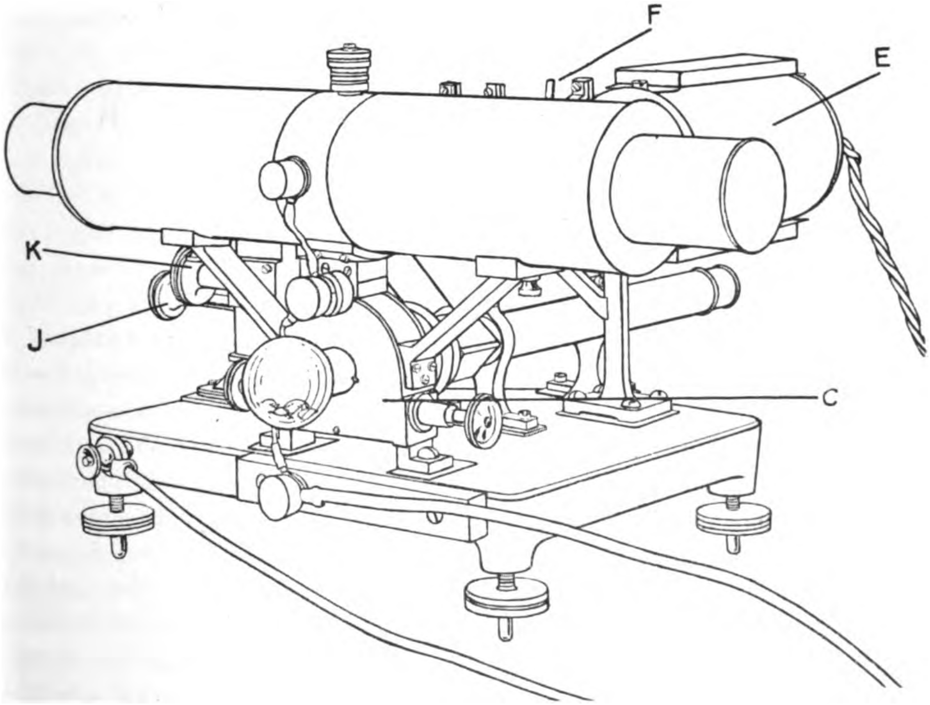

Fig. 3.—Densitometer from rear, showing electrometer.

a Pointolite lamp contained in the ventilated lantern E illuminates cell A directly through the adjustable aperture F, and is focussed by a Petzval lens G on a pinhole H immediately behind and almost in contact with the film of the photographic plate. The last-named is held in a double-slide plate-holder and can be moved either horizontally or vertically in its own plane by the milled heads J, K. Part of the emergent light is reflected from the face of a prism L of small angle to give an image of the pinhole magnified about 50 diameters in the eyepiece of the

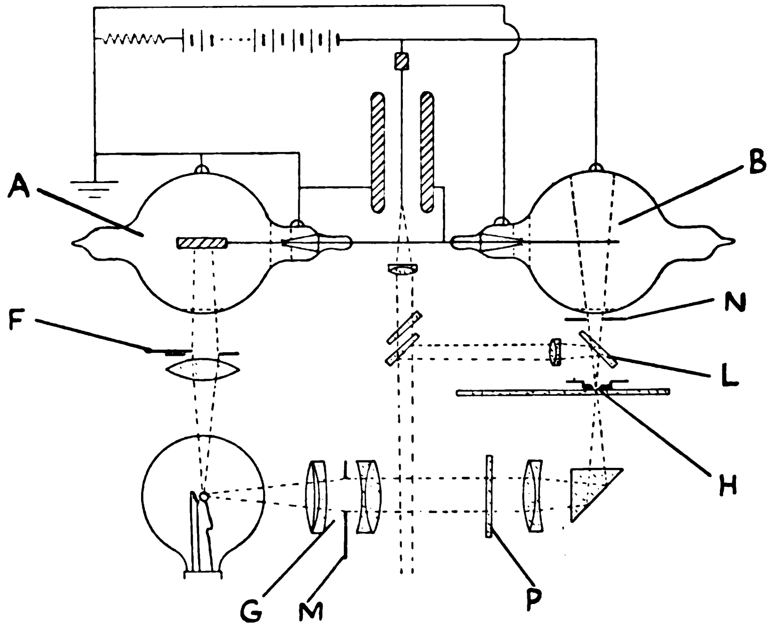

FIG. 4.—Densitometer, electrical and optical arrangements.

electrometer microscope. This arrangement, intended for centring a stellar image in the pinhole, was found useful in the present work as it allowed a part of the film free from slight defects such as dust particles to be selected for measurement The stops M, N limit the apertures of the incident and emergent beams to the ratio f : 6, as already mentioned. The electrometer is described elsewhere *; in this work it has been used as a deflection instrument, keeping the plates fixed and using a battery of fifty large Leclanche cells to charge both the moving wire and the cathode of cell B. The readings are independent of changes in the lamp current, while the wire is dead beat and comes to rest within one second. The deflection of the wire is very nearly proportional to the square of

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* *Journ. Sci. Instruments*, **1**, 1924, p. 345. It was made in the Observatory.

the transparency, giving a very open scale at the low density end. Higher densities than 0·5 are obtained by substituting for the neutral glass P (density 0·95) either of two similar but more transparent glasses (densities 0·39, 0·06).

The scale was calibrated by taking advantage of the fact that the cell B is known to give currents very closely proportional to the illumination. It was disconnected from cell A and connected to a Broca galvanometer with a standard universal shunt, when by using in place of the normal pinhole of 1 mm. diameter an aperture of about 5 sq. mm. area the deflections of the galvanometer were large enough for measurement with an accuracy of 0·5 per cent. The variations of the lamp were eliminated by taking a sufficiently large number of measures. In this way a series of fogged strips were measured and were subsequently used as standards for calibrating both this instrument and the Koch photometer used for recording the stellar spectra. No changes in the scale can be traced from the start of this work. The cells are of the sensitised helium-filled type and were made in 1918; no record has been kept of their absolute sensitiveness, but changes in the calibration of the scale have nowhere exceeded 2 per cent. since the first calibration some six months after the cells were made, and there has been no apparent decrease in the speed of working such as would have followed any considerable fall in sensitiveness.

The thread image is brought to the zero (opacity) end of the scale, first when the thread and plates are earthed, by moving the microscope, and again after charging the thread and cell, by moving the plates, with the result that the zero is independent of variations in the potential of the thread. The aperture F is adjusted to give the full deflection ($\Delta = -\infty$) for the clear plate in the immediate neighbourhood (*i.e.* within 2 mm.) of the deposit to be measured, and the deflection for the deposit is read; this is repeated, using as standard the clear plate on the opposite side of the deposit, and the mean of the deflections is noted. This method of dealing with variations of density over the clear parts of the plate is not ideal; it does not eliminate the effect of fog which is measurably greater towards the edges of even freshly obtained plates, especially those packed with paper separators, and it is unsuited to the measurement of high densities where the Eberhard development effect * first becomes prominent and for still higher densities is masked by fog due to scattered light (halation). For very low densities, however, where the variation in density of the clear plate over a few centimetres may be much greater than the effect of the exposure, it is necessary to adopt some such method. Readings of Δ are taken to 0·01 between $\bar{2}$ and $\bar{1}$ and to 0·002 from $\bar{1}$ upwards.

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* G. Eberhard, *Phys. Zeit.*, 13, 1912, p. 288.

The plates are developed in open dishes rocked by hand, first bringing dish and developer to 65° F. The time is fixed separately for each batch of plates, the aim being the maximum density consistent with freedom from fog. The development formula given by the makers of the plates has almost invariably been used—for the Ilford plates, pyro-soda; and for the Wellington and Imperial plates, metol-quinol. The makers' tables have been found useful guides in finding the most suitable times of development; more than this is not to be expected from these tables since different lots of developer chemicals, even though obtained from the same source, cannot be relied upon to give the same results.[*]

Many of the plates have been well washed in running tap water and dried before exposure. This procedure, recommended by Messrs Ilford for increasing the red and green sensitiveness,[†] produces very different results on different emulsions, especially in regard to their keeping properties, for while some batches of plates have kept for over a month after washing, retaining all the while a greatly increased sensitiveness, others have given little increase in sensitiveness or have fogged within a few days. As a rule the increased sensitiveness has been found to be greater than the 20 to 30 per cent. stated by Messrs Ilford. One batch of Imperial plates for which exact measures were made had their sensitiveness increased between 80 and 90 per cent. in the red and yellow, the sensitiveness in the blue and ultra-violet remaining unaffected. Tap water appears to be more effective than distilled water, possibly owing to its greater alkalinity (P_H 7·5); but the more strongly alkaline solutions tried—lime water of P_H 9·5 and N/50 sodium carbonate solution—led to fog and appeared to give no further increase of red sensitiveness, while a borax-boracic acid buffer solution (5 grms. boric acid, 2 grms. borax per litre) of P_H 8 gave the same relative increase of red sensitiveness but a slight all-over decrease of sensitiveness for the same development time consequent on the known restraining properties of borax on the developer.[‡] These experiments were made by soaking the plates with frequent rocking for 15 minutes, for it has been found that little advantage, if any, is obtained by prolonging the washing or by the use of running water.

The accuracy of the measures depends almost wholly on the evenness of the clear plate in the neighbourhood of the image, and it is seldom that the measures of Δ can be relied upon to 0·01 even for values above $\bar{1}$. Occasionally, however, plates are obtained which are exceptionally uniform over the unexposed parts and which give consistent results within the accuracy of 1 per cent. expected from the lamp and timing arrangement. The method of obtaining the values of p and q by a comparison of an image with its two neighbours goes far to eliminate irregularities of development, etc., such as have been investigated by Bloch * and others. None of the washed plates has given exceptionally uniform readings.

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[*] *Cf.* L. A. Jones and E. Huse, *Opt. Soc. Amer. Journ.*, 7, 1923, p. 1086.

[†] *Panchromatism* (3rd edition), 1922, p. 18.

[‡] *Brit. Journ. Phot.*, 1921, p. 639.

Although two plates from the same batch and subjected to the same treatment give consistent values for q, the value of p appears to depend to a considerable extent on the age of the plate or on some other factor which varies from month to month. As an example, the values of p for plates 44 and 45 given in Table VIII may be compared with those for the same batch and other circumstances given in Table VI. The former plates were taken in August 1924, the latter in April of the same year; and in the interval the value of p appears to have fallen by 0·06.† For this reason the values of p and q tabulated below are as far as possible from single plates; indeed it appears that the thirty or forty values of p or q given by one plate afford more reliable data on a particular point than a mass of less homogeneous material.

II. RESULTS FOR PANCHROMATIC PLATES AND FILTERED LIGHT.

The effects of varying the development conditions will first be considered. and to show clearly the contrast in this respect between high- and low-density work some figures given by Sheppard and Mees ‡ for a Barnet Photomechanical plate developed for 4 and for 8 minutes are shown in figs. 5 and 6, using first the given values of D, and then the corresponding values of Δ as ordinates. In spite of the close connection between γ the development factor and $\frac{1}{q}$ it is evident that for densities below 1·5 q is almost independent of the duration of development. Results obtained here from two Imperial plates, equally exposed but developed, one for 1½ minutes, the other for 4 minutes in pyro-soda, are plotted in fig. 7, separately for the

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* O. Bloch, *Phot. Journ.*, 1921, p. 425; H. T. Stetson and E. F. Carpenter, *Astrophys. Journ.*, **58**, 1923, p. 36; Mlle Clavier, *Sci. et Ind. Phot.*, **4** M, 1924, p. 9.

† Messrs Wellington and Ward, who were consulted on this point, state that they expect their panchromatic emulsions to give slightly lower values of p after, say, six months.

‡ Sheppard and Mees, *Theory of the Photographic Process*, London, 1907, p. 282.

red (filter 71) and the ultra-violet (filter 18). The main effect of increasing the development is to diminish I_0 of formula (1) in a ratio which depends on the wave-length; q is slightly increased at low densities and diminished at high densities—that is to say, the coefficient c is affected by development though the coefficient a is not. Temperature variations give exactly similar results. The result of omitting the potassium bromide from

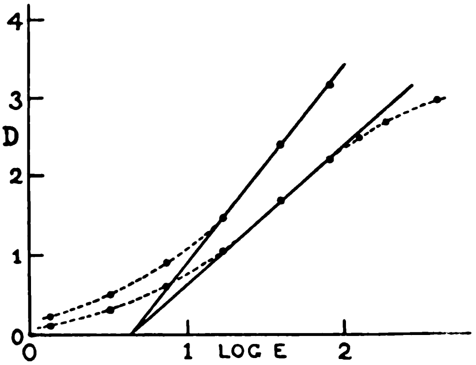

Fig. 5.—Barnet Photomechanical plate, developed for 4 and for 8 minutes (Sheppard and Mees).

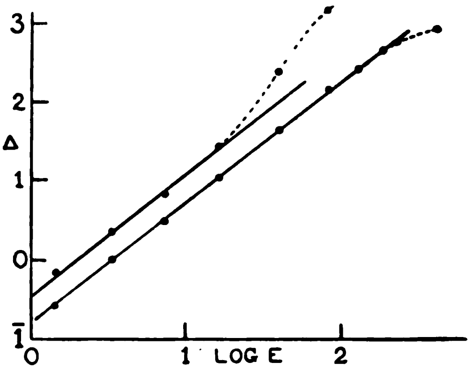

Fig. 6.—Result of replacing D of fig. 5 by Δ.

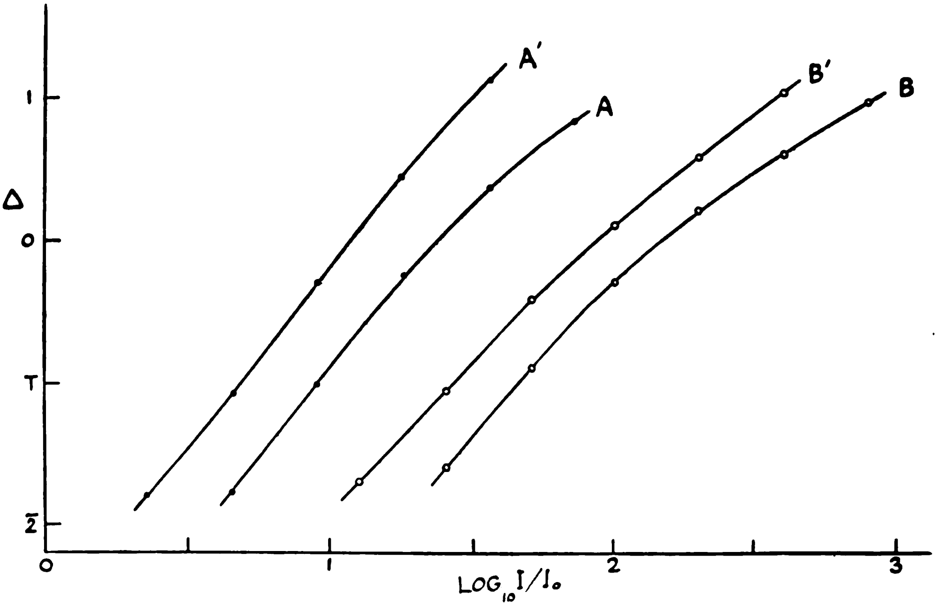

Fig. 7.—Result of increasing development time from $1\frac{1}{2}$ minutes (A, B) to 4 minutes (A′, B′). A, A′, red light (filter 71). B, B′, ultra-violet light (filter 18).

the developer was to confirm Sheppard and Mees's conclusion* that the effect of bromide in the developer is to move the characteristic curve as a whole parallel to the exposure axis; that is, to change I_0 only. The substitution of metol-quinol for pyro-soda left q entirely unaffected, both on Imperial and Wellington plates.

The effect of development variations on the values of p is easily measurable. An example is given in Table I of the effect of under-development.

TABLE I.

Plates 52, 53. Effect of development time on p or b.
Wellington 8896, pyro-soda developer.

Filter No.	71	73	75	18
Developed 1½ min.	0·98	0·97	0·98	0·96
Developed 4 min.	0·95	0·93	0·93	0·93

Additional bromide in the developer produces a similar effect to under-development.

TABLE II.

Plates 59, 60. Effects of bromide in developer on p or b.
Imperial 4154, metol-quinol developer.

Filter No.	71	73	75	18
Imperial M.Q. with 3 grms. KBr per litre .	0·92	0·94	0·93	0·91
Same developer without bromide . .	0·91	0·91	0·88	0·86

Washing and drying the plates before exposure, in so far as no fog occurs, leaves the value of q unaffected by more than 2 per cent. Fog, whatever its cause, appears to lower the value of q.

The values of p are distinctly lower on washed plates.

TABLE III.

Plates 66, 67, 70, 71. Effect of previous washing on p or b.
Imperial 4585, metol-quinol developer.

Filter No.	71	73	75	18
Washed in tap water	0·88	0·85	0·80	0·85
Not washed	0·89	0·91	0·84	0·85

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* *Loc. cit.*, pp. 112, 287.

Here, as in general, the values for the washed plates are somewhat erratic, and it is not certain that the effect on p resembles the increase of sensitiveness in being confined to the red end of the spectrum.

The determination of the variations of q with wave-length is on a less satisfactory basis than the remainder of the work. The convenience of working with the quantities p and q lies in their comparatively slow variations with the density and time, but in the variation of q with wave-length it is fairly certain that round about H_β (486 $\mu\mu$) a large change occurs over a wave-length-interval of the same order as that involved in changing from one filter to the next. In the orange about 600 $\mu\mu$ (compare, for instance, the values for q in Table IV, filters 73 and 71) and in the violet about 400 $\mu\mu$ q changes only very slowly if at all with wave-length, and its values are definitely measurable; but in the green about 500 $\mu\mu$ the effective wave-length of the filtered light becomes indefinite, depending not only on the sensitiveness of the particular batch of plates over this region but also, by reason of the variation of q, on the density of the deposit. Experiments are in progress on this point with monochromatic light, but in the meantime it is only possible to give results with filtered light in the orange where q appears to have a minimum value, and in the violet and ultra-violet where q rises slowly (corresponding to lower contrast) as the wave-length decreases. There is apparently little change in q at 660 $\mu\mu$ where the sensitiveness of the plates falls sharply with increase of wave-length; but here again filtered light is unsatisfactory.

The manner in which q varies with the value of Δ for constant exposure time can be deduced from curves such as are shown in figs. 5–7; but it appears more clearly when q is plotted against the corresponding mean value of Δ, as in figs. 8 and 9. The forms of the curves appear to be characteristic of the makes of plate though their actual positions vary with the batch. The comparative flatness of the curves for the Wellington plates in fig. 9 is noticeable: it is of course possible to choose Δ such that any one of the curves in figs. 8 and 9 becomes a straight line parallel to the axis of Δ, but it appears on further examination (though since q depends on the choice of Δ this is not immediately obvious from the figures) that on the whole no marked flattening of the curves would result from another choice of variable to represent the blackening. The different values for q shown in fig. 9 for two batches of Wellington plates are by no means exceptional or confined to this make, and may be compared with the different values of p for different emulsions given in Table VI. It appears that q does not fluctuate more than about 30 per cent. from its mean value 0·5.

The question as to the dependence of q on the value of T at constant density is of importance in fixing the value of d. Since q depends to a considerable extent on the value of Δ, and since particular values of Δ can be found only by a succession of trials on plates of the same batch, direct information on this point is not easy to obtain. Tables IV and V give some results in cases where the densities for high and low values of T correspond almost perfectly. For comparison the corresponding values of p, deduced from observations on the same plates, are also given.

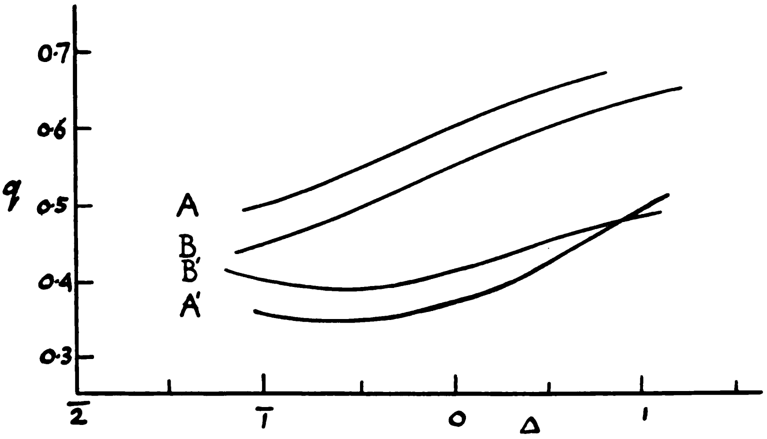

FIG. 8.—q as a function of Δ for constant exposure time 30 seconds.

A, Ilford plate, filter 18 (ultra-violet); A′, Ilford plate, filter 72 (orange); B, Imperial plate, filter 18 (ultra-violet); B′, Imperial plate, filter 72 (orange).

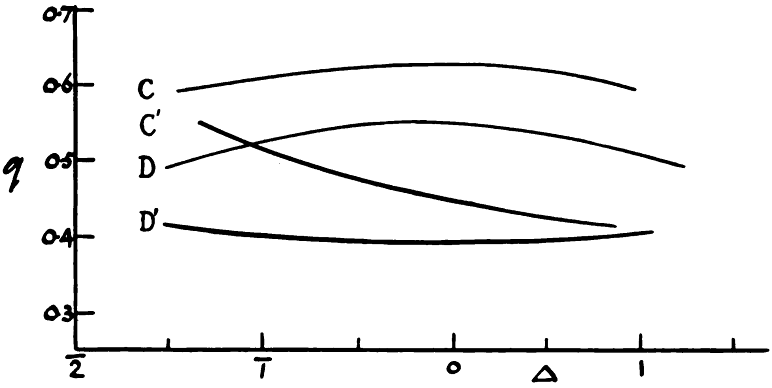

FIG. 9.—q as a function of Δ for constant exposure time 30 seconds.

C, D, filter 18 (ultra-violet); C′, D′, filter 72 (orange). Wellington plates.

TABLE IV.

Imperial B 4585.

Time (seconds).	Filter No. 18.			Filter No. 73.			Filter No. 71.		
	Mean Δ.	q.	p.	Mean Δ.	q.	p.	Mean Δ.	q.	p.
20	0·43	0·627	0·85	$\bar{1}$·75	0·368	0·91	$\bar{1}$·97	0·368	0·90
45	0·43	0·640	0·83	$\bar{1}$·81	0·384	0·92	0·02	0·380	0·89
90	0·38	0·618	0·82	$\bar{1}$·79	0·380	0·91	$\bar{1}$·99	0·379	0·88
200	0·28	0·624	0·79	$\bar{1}$·62	0·384	0·88	$\bar{1}$·83	0·382	0·85

TABLE V.

Wellington 8896.

Time (seconds).	Filter No. 18.			Time (seconds).	Filter Nos. 71, 73 (mean).		
	Mean Δ.	q.	p.		Mean Δ.	q.	p.
25	0·00	0·536	0·96	11	$\bar{1}$·95	0·431	0·96
120	0·00	0·549	0·90	45	0·01	0·414	0·95
360	0·27	0·526	0·84	180	$\bar{1}$·90	0·408	0·91

The values of p are found to depend greatly on the emulsion. Examples are given in Table VI, which refers to densities and exposure times corresponding closely to $\Delta = 0$, $T = 0$.

TABLE VI.

p as a function of the wave-length for constant blackening $\Delta = 0$ and constant exposure time 30 seconds.

Filter No.	18	76	75	74	73	72	71
Approximate wave-lengths, $\mu\mu$.	360	430	490	540	580	610	640
Ilford 5811 E	0·89	0·90	0·89	0·90	0·91	0·92	0·92
„ 5869 E	0·78	0·78	0·77	0·81	0·82	0·82	0·84
Imperial 4154	0·90	...	0·93	...	0·93	...	0·92
„ 4192	0·83	0·84	0·82	0·85	0·87	0·86	0·85
„ 4492	0·86	0·88	0·92	0·92	0·92	0·92	...
„ 4585	0·83	...	0·83	...	0·88	...	0·86
Wellington 8896	0·92	...	0·93	...	0·94	...	0·93

The extent of the variation with wave-length, which is small, appears to depend on the magnitude of p, being greater the lower the value of p. This is again evident in figs. 10 to 13. Figs. 10 and 11 show the relation between p and the exposure time for two regions of wave-length and for two batches of plates. It appears that while the coefficient b in (1) depends on the emulsion and varies little with wave-length, the coefficient e depends greatly on the wave-length and appears to have much the same values for different emulsions. Thus the results given in Table VI do not conflict with those of Hnatek * for Agfa Panchromatic plates and filtered light, for though he shows a considerable rise of p in the green and yellow his exposure times (2–5 minutes) are greater. A plate taken to test this point gave the following results:—

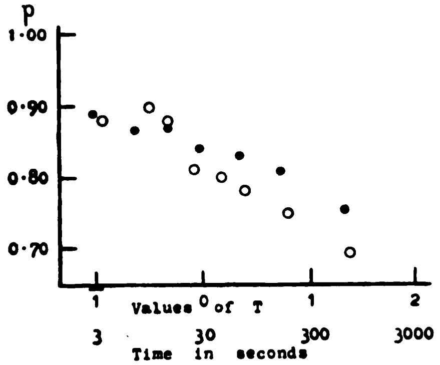

Fig. 10.—Imperial B 4192.

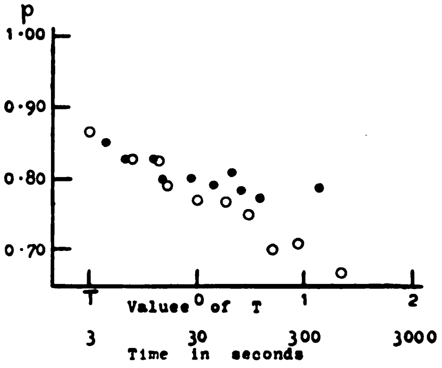

Fig. 11.—Ilford 5891 E.

p for constant density range, $\bar{1}\cdot5 - 0\cdot5$. Dots, filter 72 (orange); circles, filter 18 (ultra-violet).

Table VII.

Values of p for various times and wave-lengths.
Wellington 8896.

Time (seconds).	Filter No.			
	18. ($\Delta = 0\cdot0$).	75. ($\Delta = \bar{1}\cdot9$).	73. ($\Delta = \bar{1}\cdot6$).	71. ($\Delta = \bar{1}\cdot6$).
7½	0·980	0·972	0·972	0·952
30	0·945	0·949	0·958	0·944
120	0·887	0·935	0·941	0·923
450	0·824	0·879	0·878	0·832

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* A. Hnatek, *Zeits. wiss. Phot.*, **22**, 1923, p. 177.

It appears that the increase of p in the green is appreciable only for longer exposures than occur in the stellar work.

The variation of p with the density for constant-time range is shown in figs. 12 and 13 for the same plate-batches and wave-lengths as in figs. 10 and 11. The coefficient d in (1) is evidently very small; but higher order coefficients become appreciable for densities greater than unity.

The constancy of p for constant exposure time and of q for constant density lead at once to equation (3). This may be compared with the equation deduced by Dr J. Halm* from his own results and from those of Kron,† viz.

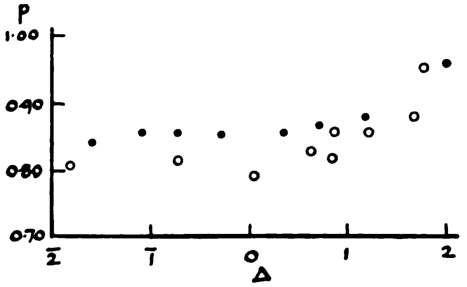

Fig. 12.—Imperial B 4192.

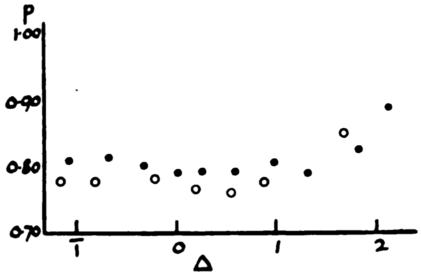

Fig. 13.—Ilford 5891 E.

p for constant time range 15–60 seconds. Dots, filter 72 (orange); circles, filter 18 (ultra-violet).

$$\log t = \phi(D) + \psi(\log I) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which asserts the constancy of p for constant illumination and the constancy of p/q for constant density. On the second point Tables IV–V give some information. Two plates taken, one with constant illumination, the other with constant time of exposure, gave the following results:—

Table VIII.

Imperial B 4192. Filter No. 18.

Illumination constant.				Time constant.		
Time (seconds).	T. (mean).	Δ.	p.	T. (mean).	Δ.	p.
2½–6	$\bar{1}$·15	$\bar{1}$·33	0·832	$\bar{1}$·82	$\bar{1}$·15	0·788
5–12	$\bar{1}$·40	$\bar{1}$·83	0·778	,,	$\bar{1}$·83	0·776
13–30	$\bar{1}$·82	0·33	0·772	,,	0·26	0·783
50–120	0·38	0·96	0·762	,,	1·20	0·831
100–240	1·18	1·44	0·736	,,	1·84	0·842

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* J. Halm, *Monthly Notices R.A.S.*, **75**, 1915, p. 159.

† E. Kron, *Publ. des astrophys. Observ. zu Potsdam*, **22**, 1913, p. 67.

Even if the shortest exposures are ruled out as subject to systematic errors of timing, there is a definite fall of p with increase of exposure time. On the other hand, the observations at higher densities, in particular the rise of p with density shown also in figs. 12 and 13, are in accordance with (6).

It is concluded that the blackening of the panchromatic plates used in this work for exposures between 5 seconds and 10 minutes, wave-lengths between 360 $\mu\mu$ and 650 $\mu\mu$, and densities below unity, can be represented by the formula

$$\log I/I_0 = \theta(D) + bT + \tfrac{1}{2}eT^2$$

where T is $\log_{10}\left(\frac{\text{exposure time}}{\text{30 seconds}}\right)$. The quantity I_0 is known to depend on the emulsion, wave-length and development, but is left undetermined. The function θ depends on the emulsion and wave-length, but not on the development within wide limits. The coefficient b depends on the emulsion and to some extent on the development, but is almost independent of the wave-length; while the coefficient e depends on the wave-length but not to any great extent on the emulsion. A simple expression giving a good approximation to $\theta(D)$ even for densities below 0·01 is $\frac{1}{2}\log_{10}(O-1)$ where O is the " opacity in parallel light," so that for constant time

$$O = 1 + kI^2,$$

the constant k depending on the unit chosen for I and on the emulsion and development.

III. Bibliography of Work on the Validity of the Reciprocity Law.

The appended bibliography is devoted to work on the validity of the reciprocity law for photographic plates, and for references to the closely allied subject of the intermittency effect the second paper by A. Odencrants should be consulted. I am indebted to that paper for a certain number of the items, in particular for the reference to the paper by Fizeau and Foucault, who found the reciprocity law invalid when working with daguerreotypes. The work by C. V. L. Charlier is noteworthy since in it he uses the law of blackening published some ten years later by Schwarzschild and known by his name. Several papers which discuss the results of other workers without introducing fresh material have been omitted from the list.

As might be expected from the variety of emulsions used, the different workers arrive at conclusions which are apparently contradictory.

Several have tried, and failed, to find a connection between the value of Schwarzschild's index and the speed of the plate, and have concluded from their results that the index had a value independent of the emulsion. In the present work no very exact speed determinations have been made, but there is no indication of a connection between the value of p and the plate speed. On the other hand, several workers mention a variation of p with the batch of plates used. Many find that p is independent of the range of exposure-time used, but where such a variation is found it is always in the direction found here—a decrease as the times increase. It appears that those who have examined the point agree that p is higher for under-developed than for fully developed plates.

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(*Issued separately May* 14, 1925.)

XVI.—The Theory of Compound Determinants from 1900 to 1920.

By **Sir Thomas Muir**, F.R.S.

(MS. received January 20, 1925. Read February 9, 1925.)

THE number of writings on compound determinants in the period 1900–1920 is almost identical with the number for the immediately preceding twenty-year period. The languages used by the writers are at least as numerous: but the contributions made by the various nationalities are relatively much altered in amount. The former strikingly high proportion of Magyar work is not at all maintained, and German has somewhat fallen off: on the other hand, English and Italian have increased. Although statements at variance with historical accuracy, and the publication of old results as new, are still more common than they ought to be, pleasing evidence is not awanting of improvement in these respects. There is nothing so glaring, for example, as the fact chronicled against the writers of the two preceding periods that they had unwittingly rediscovered ten times a theorem that had appeared in so well known a journal as Liouville's in 1851.

RAHUSEN, A. E. (1889).

(See under this heading in the chapter on Orthogonants.)

NANSON, E. J. (1900).

(See under this heading on p. 76 of *Hist.*, iv.)

CARLINI, L. (1900 $/_{12}$).

[Sul prodotto di due matrici rettangolari conjugate. *Periodico di Mat.*, (2) iii, pp. 193–198; iv, pp. 175–179.]

The type of compound determinant here considered is that in which the rows are in part taken from a compound of $|a_{1n}|$ and in part from the complementary compound, the points of inquiry being as to the power of $|a_{1n}|$ which is contained in it as a factor, and as to the form of the cofactor of this power. The author writes apparently without any knowledge of Hunyady's paper of 1880 on the subject (*Hist.*, iv, pp. 204–205).

VOGT, H. (1901 $^1/_5$).

[Théorème relatif aux mineurs d'un déterminant. *Nouv. Annales de Math.*, (4) i, pp. 211–214.]

The theorem in question, though here spoken of in connection with Netto, is in reality Sylvester's of March 1851 (*Hist.*, ii, pp. 193–194). It is most easily remembered as the extensional of the futile-looking identity

$$\begin{vmatrix} a_1 & a_2 & \dots \\ b_1 & b_2 & \dots \\ \dots & \dots & \dots \end{vmatrix} = |a_1 b_2 \dots| .$$

The fresh proof which Vogt submits is meant as an alternative to Netto's of 1893 (*Hist.*, iv, p. 216).

ZERR, G. B. M. (1902 $/_{11}$).

[Miscellaneous Question 116. *American Math. Monthly*, viii, p. 180; ix, pp. 268–269.]

The question is avowedly carried over by the proposer from Muir's text-book (p. 194). Zerr's proof is ample and suitable, but note should be taken that the equality sought to be established is shown elsewhere by Muir to be a case of Sylvester's of 1851 (*Hist.*, iv, pp. 206–207).

ORLANDO, L. (1902).

[Relazioni fra i minori d'ordine s d'un determinante di caratteristica p. 4 pp. Messina.]

[Relazioni fra i minori d'ordine p d'una matrice quadrata di caratteristica p. *Giornale di Mat.*, xl, pp. 275-277.]

The first of these publications is of a very unusual kind. The text, which extends to only twenty-four (24) lines, occupies the third of the four unnumbered pages: the first page is like the title-page of an ordinary book: the second page contains only two words intimating copyright, and a foot-line giving the name of the printing firm and the date "10th August 1902": and the fourth page is entirely blank. The communication to the *Giornale* appeared in September, but, from a comparison of the titles, it seems not unlikely that it was the first written of the two. In any case, it is the theorem of the leaflet, as being the more general, that we have to record. In effect it is: *If the non-zero rank of* $|\mathbf{a}_{1n}|$ *be* $\mathbf{k}$, *then the non-zero rank of the* m^{th} *compound of* $|\mathbf{a}_{1n}|$ *is* $(\mathbf{k})_{\mathbf{m}}$. The main part of the proof rests on the simple fact that any minor of a compound of $|a_{1n}|$ is a compound of a minor of $|a_{1n}|$, and therefore a power of this minor. In the case of the less general theorem a different proof is given.

STUYVAERT, M. (1903 $^1/_8$).

[Question 1429. *Mathesis*, (3) iii, p. 184; xxxvi, pp. 64–65.]

The equality here proposed for proof concerns the 4-by-6 array

$$\left\| \begin{matrix} a & b & c & d & & \\ 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \right\|,$$

being

$$\Big| \, |b_4c_5d_6| \quad |a_1c_5d_6| \quad |a_1b_2d_6| \quad |a_1b_2c_3| \, \Big| = |a_1b_4c_5d_6| \cdot |a_1b_2c_5d_6| \cdot |a_1b_2c_3d_6| .$$

It closely resembles Metzler's of 1897 (*Hist.*, iv, p. 221), but both are cases of Reiss' of 1867 (*Hist.*, iii, p. 189).

BAKER, H. F. (1904 $/_4$).

[Note on Sylvester's theorems on determinants. *Collected Math. Papers of J. J. Sylvester*, i, pp. 647–650.]

The theorems thus vaguely referred to are those brought forward in Sylvester's papers of March and August 1851 (*Hist.*, ii, pp. 58–61, 61–62, 193–197). The one which at present concerns us is the last of the three, being that now known as the extensional of the main property of the k^{th} compound of $|a_{1n}|$,—that is to say, the extensional of the equality

$$\Big| \, |a_{1n}| \, \Big|_k = |a_{1n}|^{(n-1)k-1}.$$

The exact character of the proof given by Baker will be better understood if we recall the fact that to establish the extensional of any equality E we may either simply use the Law of Extensible Minors and so attain our end at one stroke, or we may, for the special case of E, follow at full length the procedure by which the said Law itself in all its generality was established,—that is to say, employ on E in the appropriate manner the fundamental properties of the adjugate. Now Baker's procedure is neither of these, although necessarily akin to the second. It differs from the latter in the first place by beginning with the adjugate and not calling in E till a late stage: and, in the second place, in using the adjugate in the variant form of the reciprocal or "inverse." In illustration of it let us present in our own way the case where n is 4 (the initial determinant being $|a_1b_2c_3d_4|$). k is 2, and the extension is $e_5f_6g_7$: and where, therefore,

$$\Big|\, |a_1b_2c_3d_4| \,\Big|_2 = |a_1b_2c_3d_4|^3,$$

and the compound determinant to be considered is

$$\left|\begin{array}{cccc} |a_1b_2e_5f_6g_7| & |a_1b_3e_5f_6g_7| & \cdots & |a_3b_4e_5f_6g_7| \\ |a_1c_2e_5f_6g_7| & |a_1c_3e_5f_6g_7| & \cdots & |a_3c_4e_5f_6g_7| \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ |c_1d_2e_5f_6g_7| & |c_1d_3e_5f_6g_7| & \cdots & |c_3d_4e_5f_6g_7| \end{array}\right|, \text{ or X say.}$$

Denoting $|a_1b_2c_3d_4e_5f_6g_7|$ by Δ and its adjugate by $|A_1B_2 \ldots G_7|$ we have

$$\begin{aligned} X = & \left|\begin{array}{cccc} |C_3D_4|\Delta^{-1} & |C_2D_4|\Delta^{-1} & \cdots & |C_1D_2|\Delta^{-1} \\ |B_3D_4|\Delta^{-1} & |B_2D_4|\Delta^{-1} & \cdots & |B_1D_2|\Delta^{-1} \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ |A_3B_4|\Delta^{-1} & |A_2B_4|\Delta^{-1} & \cdots & |A_1B_2|\Delta^{-1} \end{array}\right| \\ = & \,\Delta^{-6} \cdot \text{second compound of } |A_1B_2C_3D_4| \\ = & \,\Delta^{-6}\, |A_1B_2C_3D_4|^3 = \Delta^{-6}\{\Delta^3 \cdot |e_5f_6g_7|\}^3 \\ = & \,|a_1b_2c_3d_4e_5f_6g_7|^3 \cdot |e_5f_6g_7|^3, \end{aligned}$$

as required, and as already given as an illustration on p. 196 of *Hist.*, ii.

Dickson, L. E. (1905 $^1/_1$).

[Expressions for the elements of a determinant in terms of the minors of a given order. Generalisation of a theorem of Studnička. *American Math. Monthly*, xii, pp. 217–221.]

This is the first direct attempt to find expressions of the kind in question: anything cognate of earlier date being of little consequence (*Hist.*, iv, p. 371). To begin with, a solution involving any unnecessary variety of radicals is deprecated: it being indeed affirmed before proof begins that the a's of $|a_{1n}|$ are expressible in terms of the k-line minors of $|a_{1n}|$ without using any irrationality save the k^{th} root of a rational function of the said minors. One particular case is first solved, not merely on its own account but in order to be used as a base for the solution of the others, namely, the case where $k=n-1$; in other words, where it is the primary minors that are given. The case where $k=2$ and the case where $k=3$ are then dealt with in detail in order to indicate the general procedure, the first stage of which when $k=2$ is to find 3-by-3 of the elements from knowing the adjugate elements, and when $k=3$ to find 4-by-4 of the unknown elements from similar data. With the help of these the remaining $n^2-(k+1)^2$ elements are obtained in instalments, the operations, roughly speaking, being substitutions for the newly-found elements and solutions of sets of linear equations so obtained.

An alternative mode of solution is also given for the case where $k=2$, the basis of which is Hermite's so-called condensation theorem of 1849 (*Hist.*, ii, p. 46), although not recognised as such. In connection therewith it is unfortunately also necessary to note that the theorem spoken of in the title as Studnička's is wrongly so called, and that even the wide generalisation of it effected by Dickson was already known in the early part of 1879 before Studnička's paper had appeared.

Maschke, H. (1905 $^{22}/_{4}$).

[Differential parameters of the first order. *Transac. American Math. Soc.*, vii, pp. 69–80, 571.]

In the introduction to this paper attention is drawn to four theorems in determinants, and the first section (pp. 69–72) is devoted to the formal enunciation and proof of them. As a result of this segregation, it is all the more disappointing to find that every one of the four is a rediscovery. Only one, it is true, belongs strictly to our present chapter—Bazin's important and oft-discovered* theorem of 1851 (*Hist.*, ii, pp. 206–208); but it is as well to point out once and for all that the first of the four is Sylvester's equally old expansion-theorem (*Hist.*, ii, pp. 61–62), and that the third is Casorati's theorem of 1874 on Jacobians (*Hist.*, iii, p. 269).

Muir, T. (1905 $^{6}/_{11}$).

[Elimination in the case of equality of fractions whose numerators and denominators are linear functions of the variables. *Transac. R. Soc. Edinburgh,* xlv, pp. 1–7.]

The procedure followed here will be readily guessed from observing the lemma established at the outset, namely: *If we have given*

$$\frac{a_1x_1+b_1x_2+\ .\ .\ .\ +l_1x_n}{\alpha_1x_1+\beta_1x_2+\ .\ .\ .\ +\lambda_1x_n}=\ .\ .\ .\ =\frac{a_nx_1+b_nx_2+\ .\ .\ .\ +l_nx_n}{\alpha_nx_1+\beta_nx_2+\ .\ .\ .\ +\lambda_nx_n}$$

$$=\frac{1}{r},$$

then

$$|\ a_1-r\alpha_1\quad b_2-r\beta_2\ .\ .\ .\ l_n-r\lambda_n\ |\ =0.$$

An additional equal fraction being given us we have at once the theorem; *the eliminant of the set of equations*

$$\frac{a_1x_1+b_1x_2+\ .\ .\ .\ +l_1x_n}{\alpha_1x_1+\beta_1x_2+\ .\ .\ .\ +\lambda_1x_n}=\ .\ .\ .\ =\frac{a_{n+1}x_1+b_{n+1}x_2+\ .\ .\ .\ +l_{n+1}x_n}{\alpha_{n+1}x_1+\beta_{n+1}x_2+\ .\ .\ .\ +\lambda_{n+1}x_n}$$

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* *E.g.* six times in the period 1860–1880.

is

$$\begin{vmatrix} \mathrm{D}_1 & \sum \mathrm{D}'_1 & \sum \mathrm{D}''_1 & \cdots \\ \mathrm{D}_2 & \sum \mathrm{D}'_2 & \sum \mathrm{D}''_2 & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ \mathrm{D}_{n+1} & \sum \mathrm{D}'_{n+1} & \sum \mathrm{D}''_{n+1} & \cdots \end{vmatrix}$$

where

$\mathrm{D}_1 = |a_1b_2 \ldots l_n|$, $\mathrm{D}_2 = |a_2b_3 \ldots l_{n+1}|$, $\ldots$, $\mathrm{D}_{n+1} = |a_{n+1}b_1 \ldots l_{n-1}|$,

and where D'_r *indicates that any one of the letters of* D_r *has been replaced by the corresponding letter of the other alphabet,* D''_r *that any two letters have been similarly treated, and so on.* By use of a similar process an alternative theorem is next got, namely: *The eliminant of the set of equations*

$$\frac{a_1x_1 + b_1x_2 + \ldots + l_1x_n}{\alpha_1x_1 + \beta_1x_2 + \ldots + \lambda_1x_n} = \ldots = \frac{a_{n+1}x_1 + b_{n+1}x_2 + \ldots + l_{n+1}x_n}{\alpha_{n+1}x_1 + \beta_{n+1}x_2 + \ldots + \lambda_{n+1}x_n}$$

is

$$\begin{vmatrix} \mathrm{D}_1 & \sum \mathrm{D}'_1 & \sum \mathrm{D}''_1 & \cdots \\ \mathrm{D}_2 & \sum \mathrm{D}'_2 & \sum \mathrm{D}''_2 & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ \mathrm{D}_n & \sum \mathrm{D}'_n & \sum \mathrm{D}''_n & \cdots \end{vmatrix} = 0,$$

where

$$\mathrm{D}_1 = |a_1b_2c_3 \ldots l_n\alpha_{n+1}|, \quad \mathrm{D}_2 = |a_1b_2c_3 \ldots l_n\beta_{n+1}|, \quad \ldots$$

and where D'_r *indicates that any one of the italic letters of* D_r *except the* r*th* *has been replaced by the corresponding Greek letter,* D''_r *that any two letters except the* r*th* *have been similarly treated, and so on.**

There is thus brought to light an equality between two compound determinants of different orders: for example,

$$\begin{vmatrix} |\alpha_1\beta_2\gamma_3| & \sum|a_1\beta_2\gamma_3| & \sum|a_1b_2c_3| & |a_1b_2c_3| \\ |\alpha_2\beta_3\gamma_4| & \sum|a_2\beta_3\gamma_4| & \sum|a_2b_3c_4| & |a_2b_3c_4| \\ |\alpha_3\beta_4\gamma_1| & \sum|a_3\beta_4\gamma_1| & \sum|a_3b_4c_1| & |a_3b_4c_1| \\ |\alpha_4\beta_1\gamma_2| & \sum|a_4\beta_1\gamma_2| & \sum|a_4b_1c_2| & |a_4b_1c_2| \end{vmatrix} = \begin{vmatrix} |a_1\alpha_2\beta_3\gamma_4| & |a_1\alpha_2b_3\gamma_4| + |a_1\alpha_2\beta_3c_4| & |a_1\alpha_2b_3c_4| \\ |b_1\alpha_2\beta_3\gamma_4| & |b_1\alpha_2\beta_3c_4| + |b_1\alpha_2\beta_3\gamma_4| & |b_1\alpha_2\beta_3c_4| \\ |c_1\alpha_2\beta_3\gamma_4| & |c_1\alpha_2\beta_3\gamma_4| + |c_1\alpha_2b_3\gamma_4| & |c_1\alpha_2b_3\gamma_4| \end{vmatrix}.$$

Finally, by finding the set of linear equations that are in so-called correspondence with the given set of homogeneous quadratics, the result obtained differs only from the first of the two preceding forms in being its conjugate.

* The problem here solved was set by Nanson at the beginning of 1905 in *Educ. Times*, lviii, p. 41, and a condensed solution was published early in 1906 in *Math. from Educ. Times*, (2) ix, pp. 109–110. See also (2) x, p. 106.

MUIR, T. (1905 $^{25}/_{11}$).

[Equality of two compound determinants of orders n and $n-1$. *Messenger of Math.*, xxxv, pp. 118–122.]

As this title is descriptive of the subject of the penultimate paragraph of the preceding paper, the theorem of the said paragraph is here first drawn attention to. Then follows the more important cognate theorem: *If there be* n *arrays, each of* n − 1 *rows and* n *columns, and two determinants be formed therefrom, namely, one whose* rth *column has for its elements the* n *determinants of the* (n − 1)th *order derivable from the* rth *array, and the other whose* rth *column has for elements the* n − 1 *determinants of the* nth *order formable by adding to the* rth *array one row of the* nth *array, the two determinants are equal:* for example, when n is 4,

$$\begin{vmatrix} |a_1b_2c_3| & |h_1k_2l_3| & |m_1n_2r_3| & |x_1y_2z_3| \\ |a_2b_3c_4| & |h_2k_3l_4| & |m_2n_3r_4| & |x_2y_3z_4| \\ |a_3b_4c_1| & |h_3k_4l_1| & |m_3n_4r_1| & |x_3y_4z_1| \\ |a_4b_1c_2| & |h_4k_1l_2| & |m_4n_1r_2| & |x_4y_1z_2| \end{vmatrix} = \begin{vmatrix} |a_1b_2c_3x_4| & |h_1k_2l_3x_4| & |m_1n_2r_3x_4| \\ |a_1b_2c_3y_4| & |h_1k_2l_3y_4| & |m_1n_2r_3y_4| \\ |a_1b_2c_3z_4| & |h_1k_2l_3z_4| & |m_1n_2r_3z_4| \end{vmatrix}.$$

To connect the theorem with results already known, there are noted (1) the case where

$$h,\ k,\ l,\ m,\ n,\ r,\ x,\ y,\ z \equiv b,\ c,\ d,\ c,\ d,\ e,\ d,\ e,\ f,$$

and (2) the case where farther $e, f = a, b$.

HUNYADY, E. (1905 $^{1}/_{12}$).

[Question 1416. *Nouv. Annales de Math.*, (4) v, pp. 568–570.]

A proof is at last here given of Hunyady's theorem of 1882 (*Hist.*, iv, p. 209). It consists in showing that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} A_1+A_1 & A_2+B_1 & A_3+C_1 \\ B_1+A_2 & B_2+B_2 & B_3+C_2 \\ C_1+A_3 & C_2+B_3 & C_3+C_3 \end{vmatrix} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \cdot \begin{vmatrix} a_1+a_1 & a_2+b_1 & a_3+c_1 \\ b_1+a_2 & b_2+b_2 & b_3+c_2 \\ c_1+a_3 & c_2+b_3 & c_3+c_3 \end{vmatrix}$$

and this is accomplished by performing the two multiplications indicated, the one rowwise and the other columnwise. The name of the ingenious author is not given.

NANSON, E. J. (1906 $^{28}/_{3}$).

[A theorem in compound determinants. *Messenger of Math.*, xxxvi, pp. 45–48.]

This follows immediately on Muir's second paper of the previous year, its main object being to show how the results of that paper may be deduced from a single general equality, namely, the equality now known as the unextended case of Bazin's theorem of 1851. Whether the new formulæ so founded are equally general with those sought to be reproduced it is a little difficult to say, because of uncertainty as to the meaning of the new notations employed; for example, whether

$$\begin{vmatrix} (a_4b_1c_1d_1) & (b_4b_1c_1d_1) & (c_4b_1c_1d_1) \\ (a_2a_4c_2d_2) & (a_2b_4c_2d_2) & (a_2c_4c_2d_2) \\ (a_3b_3a_4d_3) & (a_3b_3b_4d_3) & (a_3b_3c_4d_3) \end{vmatrix} = (A_1B_2C_3D_4)$$

is coextensive with

$$\left\| \begin{matrix} |a_1b_2c_3x_4| & |h_1k_2l_3x_4| & |m_1n_2r_3x_4| \\ |a_1b_2c_3y_4| & |h_1k_2l_3y_4| & |m_1n_2r_3y_4| \\ |a_1b_2c_3z_4| & |h_1k_2l_3z_4| & |m_1n_2r_3z_4| \end{matrix} \right\| = \left\| \begin{matrix} |a_1b_2c_3| & |h_1k_2l_3| & |m_1n_2r_3| & |x_1y_2z_3| \\ |a_2b_3c_4| & |h_2k_3l_4| & |m_2n_3r_4| & |x_2y_3z_4| \\ |a_3b_4c_1| & |h_3k_4l_1| & |m_3n_4r_1| & |x_3y_4z_1| \\ |a_4b_1c_2| & |h_4k_1l_2| & |m_4n_1r_2| & |x_4y_1z_2| \end{matrix} \right\| .$$

The interesting derivation of Sylvester's theorem regarding an aggregate of products of pairs of determinants we have already seen handled by Rubini in 1878 (*Hist.*, iii, p. 200).

PETR, K. (1906).

[Několik poznámek o determinantech. *Časopis pro pěstování math. a fys.*, xxxv, pp. 311–321: or, in Magyar, *Math. és phys. lapok*, xv, pp. 353–365: or, in German, *Math.-naturw. Berichte aus Ungarn*, xxv, pp. 95–105.]

The second section (pp. 97–103) of this paper concerns the latent roots of the compounds of a determinant. First, there is established Rados' theorem of 1891 (*Hist.*, iv, pp. 215, 217), the procedure being based on the theory of linear substitution. In the same manner and at almost equal length a wider theorem is then successfully dealt with, namely: *If* $| a_{hk}x + b_{hk}y |_n$ *be resolvable into* n *linear factors of the form* $\alpha x + \beta y$, *and* $a^{(m)}_{pq}$ *be any element of the* m*th* *compound of* $| a_{hk} |_n$, *then* $| a^{(m)}_{pq}x + b^{(m)}_{pq}y |_{(n)_m}$ *is similarly resolvable into* $(n)_m$ *factors.* A purely determinantal proof of the latter is a desideratum.

JAHNKE, E. (1906 $^8/_{11}$).

[Aufgaben und Lehrsätze. Nr. 125. *L'Intermédiaire des Math.*, xii, p. 172: *Archiv d. Math. u. Phys.*, (3) ix, p. 91; (3) xi, p. 140.]

The subject here is a quadric relation between six 5-line minors of the second compound of $| a_1b_2c_3d_4 |$. Grassmann's semi-geometrical method is used, but the relation is easily established otherwise. Calling the compound in question S, we have from Franke's theorem

$$\text{cof } |\, a_1 b_2 \,| \text{ in S} = |\, a_1 b_2 c_3 d_4 \,|^2 \cdot |\, c_3 d_4 \,| ,$$

and thus

$$\begin{aligned} \text{cof } |\, a_1 b_2 \,| \cdot \text{cof } |\, a_3 b_4 \,| - \text{cof } |\, a_1 b_3 \,| \cdot \text{cof } |\, a_2 b_4 \,| + \text{cof } |\, a_1 b_4 \,| \cdot \text{cof } |\, a_2 b_3 \,| \\ = |\, a_1 b_2 c_3 d_4 \,|^4 \cdot \Big\{ - |\, c_1 d_2 \,|\,|\, c_3 d_4 \,| + |\, c_1 d_3 \,|\,|\, c_2 d_4 \,| - |\, c_1 d_4 \,|\,|\, c_2 d_3 \,| \Big\} \\ = |\, a_1 b_2 c_3 d_4 \,|^4 \cdot 0 = 0. \end{aligned}$$

It will be noted that the six 5-line minors of S cannot be taken arbitrarily, but are those which are cofactors of elements of a single column of S.

MUIR, T. (1907 $^{20}/_{8}$).

[The theory of compound determinants in the historical order of its development up to 1860. *Proceed. R. Soc. Edinburgh*, xxviii, pp. 197–209.]

This, our first historical paper on the present subject, is introduced by a few paragraphs pointing out the early foreshadowings of the "adjugate" by Lagrange (1773), the first direct investigations of the same by Cauchy and by Jacobi, Cauchy's advance to other "systèmes dérivés" of $|\, a_{1n} \,|$ besides the adjugate, and Jacobi's carrying of the idea into the special field of "functional determinants." Then follow separate notices of three of Sylvester's papers and the text-books of Spottiswoode, Brioschi, and Bellavitis.

MUIR, T. (1907 $^{7}/_{10}$).

[The product of the primary minors of an m-by-$(m+1)$ array. *Proceed. R. Soc. Edinburgh*, xxviii, pp. 210–216.]

The second part of this paper concerns compound determinants of a special type, namely, those whose elements are products of pairs of determinants. The first to be considered is of order $\frac{1}{2}m(m-1)$ and whose component determinants are 2-line minors of an m-by-$(m+1)$ array; for example, when m is 4 and the array is

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 , \end{array}$$

the equality is

$$\left| \begin{array}{cccc} |\, a_1 b_2 \,| \cdot |\, a_1 b_3 \,| & |\, a_1 c_2 \,| \cdot |\, a_1 c_3 \,| & \cdots & |\, c_1 d_2 \,| \cdot |\, c_1 d_3 \,| \\ |\, a_1 b_2 \,| \cdot |\, a_1 b_4 \,| & |\, a_1 c_2 \,| \cdot |\, a_1 c_4 \,| & \cdots & |\, c_1 d_2 \,| \cdot |\, c_1 d_4 \,| \\ \cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot \\ |\, a_1 b_4 \,| \cdot |\, a_1 b_5 \,| & |\, a_1 c_4 \,| \cdot |\, a_1 c_5 \,| & \cdots & |\, c_1 d_4 \,| \cdot |\, c_1 d_5 \,| \end{array} \right|$$

$$= a_1{}^2 b_1{}^2 c_1{}^2 d_1{}^2 \cdot |\, a_1 b_2 c_3 d_4 \,| \cdot |\, a_1 b_2 c_3 d_5 \,| \cdot |\, a_1 b_2 c_4 d_5 \,| \cdot |\, a_1 b_3 c_4 d_5 \,| .$$

The prominence of a_1, b_1, c_1, d_1 on the right is due to the fact that the component 2-line minors on the left all have their first columns coincident with the first column of the array. No direct proof is given.

Next comes a series of results in which the component determinants are the primary minors of $|a_1b_2c_3d_4|$, or Δ say. Among the more interesting are

$$\begin{vmatrix} A_1A_2 & A_2A_3 & A_3A_4 & A_4A_1 \\ B_1B_2 & B_2B_3 & B_3B_4 & B_4B_1 \\ C_1C_2 & C_2C_3 & C_3C_4 & C_4C_1 \\ D_1D_2 & D_2D_3 & D_3D_4 & D_4D_1 \end{vmatrix} = \Delta^2 \begin{vmatrix} a_2A_2 & a_4A_2 & a_4A_4 & a_2A_4 \\ b_2B_2 & b_4B_2 & b_4B_4 & b_2B_4 \\ c_2C_2 & c_4C_2 & c_4C_4 & c_2C_4 \\ d_2D_2 & d_4D_2 & d_4D_4 & d_2D_4 \end{vmatrix}$$

$$= -\Delta^4 \,|\, a_1a_2 \quad b_2b_3 \quad c_3c_4 \quad d_4d_1 \,|;$$

and

$$|\, A_2A_3A_4 \quad B_3B_4B_1 \quad C_4C_1C_2 \quad D_1D_2D_3 \,| = \Delta^2 |\, a_2A_3A_4 \quad b_1B_3B_4 \quad C_1C_2c_4 \quad D_1D_2d_3 \,|.$$

MUIRHEAD, R. F. (1908 $^{13}/_{3}$).

[To express a determinant of the n^{th} order in terms of compound determinants of the 2^{nd} order. *Proceed. Edinburgh Math. Soc.*, xxvi, pp. 15-17.]

The series of 2-line determinants here,

$$T_1, T_2, T_3, \ldots,$$

is such that the elements of T_2 are quantities like T_1, the elements of T_3 are quantities like T_2, and so on: for example,

$$a_1, \quad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \quad \begin{Vmatrix} |a_1b_2| & |a_1b_3| \\ |a_1c_2| & |a_1c_3| \end{Vmatrix}, \quad \ldots$$

where T_1 is necessarily an alien, and $T_3, T_4, T_5, \ldots$ notwithstanding their increasing complexity, are quite simply shown to be equal to

$$a_1\,|a_1b_2c_3|, \qquad a_1^2\,|a_1b_2|\cdot|a_1b_2c_3d_4|, \qquad a_1^4\,|a_1b_2|^2\cdot|a_1b_2c_3|\cdot|a_1b_2c_3d_4e_5|, \qquad \ldots$$

The result thus obtained is

$$|a_1b_2c_3d_4e_5| = T_5{}^1T_4{}^0T_3{}^{-1}T_2{}^{-2}T_1{}^{-3},$$

and so generally.

FISCHER, E. (1909 $^{15}/_{2}$).

[Verallgemeinerung des Sylvesterschen Determinantensatzes. *Crelle's Journ.*, cxxxv, pp. 306–318.]

The title of this interesting paper is not helpful as to its contents, the actual subject being the problem of the factorisation of minors of the m^{th} compound of $|a_{1n}|$. Further, the problem is restricted in two directions:

the minors of the compound have to be coaxial and their factors have to be leading coaxial minors of $|a_{1n}|$. For illustrative examples it is perhaps most convenient to go to the exposition of 1853 by Spottiswoode (*Hist.*, ii, p. 203), whom, by the way, Fischer might more appropriately have referred to than to Sylvester. When n is 5 and m is 3, and to construct our minor we take all the triads formable from the first five integers except 245, 345, we obtain from Spottiswoode's first equality

$$\left|\begin{vmatrix}123\\123\end{vmatrix}\ \begin{vmatrix}124\\124\end{vmatrix}\ \begin{vmatrix}125\\125\end{vmatrix}\ \begin{vmatrix}134\\134\end{vmatrix}\ \begin{vmatrix}135\\135\end{vmatrix}\ \begin{vmatrix}145\\145\end{vmatrix}\ \begin{vmatrix}234\\234\end{vmatrix}\ \begin{vmatrix}235\\235\end{vmatrix}\right| = \begin{vmatrix}1\\1\end{vmatrix}\cdot\begin{vmatrix}123\\123\end{vmatrix}\cdot\begin{vmatrix}12345\\12345\end{vmatrix}^4,$$

and leaving out the two additional triads 145, 235, we obtain from his fourth equality

$$\left|\begin{vmatrix}123\\123\end{vmatrix}\ \begin{vmatrix}124\\124\end{vmatrix}\ \begin{vmatrix}125\\125\end{vmatrix}\ \begin{vmatrix}134\\134\end{vmatrix}\ \begin{vmatrix}135\\135\end{vmatrix}\ \begin{vmatrix}234\\234\end{vmatrix}\ \begin{vmatrix}235\\235\end{vmatrix}\right| = \begin{vmatrix}123\\123\end{vmatrix}^2\cdot\begin{vmatrix}12345\\12345\end{vmatrix}^3.$$

Sets of combinations, such as we have here, namely,

123, 124, 125, 134, 135, 145, 234, 235,

and

123, 124, 125, 134, 135, 234, 235,

are said to be "closed" or "compact" sets, because the replacing of any integer in any combination of the set by a lower integer does not produce a combination which is not already there.

With this explanation the reader will readily understand our formulation of the theorem to which the paper under consideration is devoted. *The necessary and sufficient condition that a given coaxial minor of the* m^th^ *compound of an* n*-line determinant be expressible as a product of powers of leading coaxial minors of the said determinant is that the set of* m*-ads of* 1, 2, 3, . . ., n *involved in the specification of the given minor be a "closed" set.* The proof (pp. 309–313) is based on work of Mertens', dating from 1880, on the conditions for the divisibility of any rational integral function of the elements of $|a_{1n}|$ by a power of $|a_{1n}|$ (*Hist.*, iv, p. 31).

Nothing is said about the finding of the exponents of the powers in the result of the factorisation, nor indeed is a single example vouchsafed of the application of the theorem. This is the more to be regretted because a dangerously plausible rule for obtaining them lies suggestively on the surface. For example, the given closed set of combinations being

123, 124, 125, 134, 135, 145, 234,

and it being noted that the integers of the set may be combined thus, $1^6 2^4 3^4 4^4 5^3$, and redistributed thus, $(12345)^3\,(1234)\,(1)^2$, we feel inclined to conclude that

$$\left\| \begin{vmatrix}123\\123\end{vmatrix} \begin{vmatrix}124\\124\end{vmatrix} \begin{vmatrix}125\\125\end{vmatrix} \begin{vmatrix}134\\134\end{vmatrix} \begin{vmatrix}135\\135\end{vmatrix} \begin{vmatrix}145\\145\end{vmatrix} \begin{vmatrix}234\\234\end{vmatrix} \right\| = \begin{vmatrix}1\\1\end{vmatrix}^2 \cdot \begin{vmatrix}1234\\1234\end{vmatrix} \cdot \begin{vmatrix}12345\\12345\end{vmatrix}^3 ;$$

and such indeed is the case. But the same procedure being followed in the case of the unclosed set

123, 124, 134, 135, 145, 234, 235, 245,

we should be wrong to conclude, as Spottiswoode actually does, that the result is

$$\begin{vmatrix}1234\\1234\end{vmatrix} \begin{vmatrix}12345\\12345\end{vmatrix}^4,$$

Franke's theorem of 1862 affording a check by giving the true cofactor of $\begin{vmatrix}12345\\12345\end{vmatrix}^4$.

The remaining pages of the paper are occupied in showing that a quite similar theorem holds in the case of the Reiss-Picquet "mixed" compound (*Hist.*, iii, pp. 198–199).

KOWALEWSKI, G. (1909 /$_4$).

[Einführung in die Determinantentheorie . . . vi + 550 pp. Leipzig.]

Half of the sixteen pages (pp. 83–88, 99–111) devoted to the consideration of the adjugate and other compound determinants is taken up, strangely enough, with three independent proofs of Sylvester's ungeneralised theorem of 1851 (*Hist.*, ii, pp. 59–61, 193–194). First, it is viewed as a condensation-theorem, the procedure being that with which we are already familiar (*Hist.*, iii, p. 81). In the second proof the fundamental step is the formation of a bordered determinant of which the compound concerned is both a minor and a factor: for example, to prove the simple extensional

$$\left\| \; |a_1b_2c_3d_4| \;\; |a_1b_2c_3e_5| \;\; |a_1b_2c_3f_6| \;\; |a_1b_2c_3g_7| \; \right\| = |a_1b_2c_3|^3 \cdot |a_1b_2c_3d_4e_5f_6g_7|,$$

a determinant equal to

$$|a_1b_2c_3| \cdot \left\| \; |a_1b_2c_3d_4| \;\; |a_1b_2c_3e_5| \;\; |a_1b_2c_3f_6| \;\; |a_1b_2c_3g_7| \; \right\|$$

would be formed, namely,

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\ \cdot & \cdot & \cdot & |a_1b_2c_3d_4| & |a_1b_2c_3d_5| & |a_1b_2c_3d_6| & |a_1b_2c_3d_7| \\ \cdot & \cdot & \cdot & |a_1b_2c_3e_4| & |a_1b_2c_3e_5| & |a_1b_2c_3e_6| & |a_1b_2c_3e_7| \\ \cdot & \cdot & \cdot & |a_1b_2c_3f_4| & |a_1b_2c_3f_5| & |a_1b_2c_3f_6| & |a_1b_2c_3f_7| \\ \cdot & \cdot & \cdot & |a_1b_2c_3g_4| & |a_1b_2c_3g_5| & |a_1b_2c_3g_6| & |a_1b_2c_3g_7| \end{vmatrix}.$$

Then row_4 of this determinant would be increased by

$$\text{row}_1 \cdot |b_1c_2d_3| - \text{row}_2 \cdot |a_1c_2d_3| + \text{row}_3 \cdot |a_1b_2d_3|,$$

and would thus take the form

$$d_1\,|a_1b_2c_3|,\quad d_2\,|a_1b_2c_3|,\quad d_3\,|a_1b_2c_3|,\quad \ldots,\quad d_7\,|a_1b_2c_3|.$$

Similarly row_5 would be treated, becoming

$$e_1\,|a_1b_2c_3|,\quad e_2\,|a_1b_2c_3|,\quad e_3\,|a_1b_2c_3|,\quad \ldots,\quad e_7\,|a_1b_2c_3|,$$

and so with the others. The determinant would then be seen to be equal to

$$|a_1b_2c_3|^4 \cdot |a_1b_2c_3d_4e_5f_6g_7|;$$

and nothing further would be necessary save division of both of its equivalents by $|a_1b_2c_3|$. The third proof is still less direct, arising from the equatement of two expressions obtainable for the adjugate of a minor of the adjugate of any determinant: for example, the basic determinant being, as in the preceding, $|a_1b_2c_3d_4e_5f_6g_7|$, or Δ say, two expressions can be obtained for the adjugate of $|D_4E_5F_6G_7|$, namely,

$$|D_4E_5F_6G_7|^3 \qquad \textit{i.e.} \qquad \{|a_1b_2c_3| \cdot \Delta^3\}^3 \qquad \textit{i.e.} \qquad |a_1b_2c_3|^3 \cdot \Delta^9,$$

and, by removing Δ^2 from each element,

$$(\Delta^2)^4 \begin{vmatrix} |a_1b_2c_3d_4| & |a_1b_2c_3e_5| & |a_1b_2c_3f_6| & |a_1b_2c_3g_7| \end{vmatrix},$$

the equating of which two expressions shows that the compound determinant is equal to

$$|a_1b_2c_3|^3 \cdot \Delta$$

as required. This third mode of proof is again used to establish the generalised theorem (*Hist.*, ii, pp. 194–197): in essentials it does not differ from Baker's mode of 1904 (see above, p. 189). As for the two other theorems dealt with, we only note that the names given to them deserve reconsideration.

MUIR, T. (1912 $^{20}/_{3}$).

[The resultant of a set of homogeneous lineo-linear equations. *Transac. R. Soc. S. Africa*, ii, pp. 373–380.]

This seeks, for one thing, to clear up the haze surrounding Sylvester's doings with "double determinants," entities first spoken of by him with heated enthusiasm and then quietly dropped (*Hist.*, iii, pp. 178–179). At the outset are obtained the two forms of resultant which he had spoken of as being unsatisfactory, and then a justifiable guess is made as to the unpublished form which more than met with his approval. A simple mode of obtaining this form is given, and the process is shown to be of perfectly general application, the set of equations considered with a view to the elimination of the $m+n$ unknowns being

$$0 = \left(\begin{array}{cccc} a_1 & a_2 & \ldots & a_{mn} \\ b_1 & b_2 & \ldots & b_{mn} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \multicolumn{4}{c}{(m+n-1 \text{ rows})} \end{array}\right)(x_1, x_2, \ldots, x_m)(y_1, y_2, \ldots, y_n)$$

where

$$(\alpha, \beta, \gamma, \ldots)(p, q, r, \ldots)$$

is used to stand for

$$\alpha p, \beta p, \gamma p, \ldots, \alpha q, \beta q, \gamma q, \ldots, \alpha r, \beta r, \gamma r, \ldots$$

Special attention is given to the case where $m, n = 3, 3$.

VIVANTI, G. (1912 $^{23}/_{5}$): SIBIRANI, F. (1913 $^{3}/_{7}$).

[Un teorema sui determinanti. *Rendic. . . . Ist. Lombardo, . . .* (2) xlv, pp. 556–559.]

[Un teorema sui determinanti. *Rendic. . . . Ist. Lombardo, . . .* (2) xlvi, pp. 822–830.]

The determinant here dealt with is of the order $\frac{1}{2}m(m+1)$, every row of it consisting of the squares and other binary products of m magnitudes. Simpler instances of it we have seen appearing in the work of Hunyady and others: for example,

$$\left|\begin{array}{cccccc} x_1^2 & y_1^2 & z_1^2 & y_1z_1 & z_1x_1 & x_1y_1 \\ x_2^2 & y_2^2 & z_2^2 & y_2z_2 & z_2x_2 & x_2y_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_6^2 & y_6^2 & z_6^2 & y_6z_6 & z_6x_6 & x_6y_6 \end{array}\right|$$

(*Hist.*, iii, pp. 202–204). The important difference now is that instead of the simple variables x_1, y_1, . . . Vivanti has determinants, namely, the $(m-1)$-line minors of an m-by-$(m+1)$ array, his theorem being a generalisation of the equality

$$\begin{vmatrix} a_1^2 & b_1^2 & a_1b_1 \\ a_2^2 & b_2^2 & a_2b_2 \\ a_3^2 & b_3^2 & a_3b_3 \end{vmatrix} = |a_1b_2| \cdot |a_2b_3| \cdot |a_3b_1|$$

(*Hist.*, iii, p. 132), and expressible as follows:—*The determinant of order* $\frac{1}{2}m(m+1)$ *whose every row is of the form*

$$a^2 \quad b^2 \quad c^2 \ . \ . \ . \ y^2 \quad z^2 \quad ab \quad ac \ . \ . \ . \ yz,$$

and whose first m *columns have for elements the squares of the* (m − 1)*-line minors of an* m*-by-*(m + 1) *array is equal to the* (m − 1)*th power of the product of the* m*-line minors of the array*; for example, when m is 3 and the array is

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{matrix}$$

we have

$$\begin{Vmatrix} |a_1b_2|^2 & |a_1c_2|^2 & |b_1c_2|^2 & |a_1b_2|\cdot|a_1c_2| & |a_1b_2|\cdot|b_1c_2| & |a_1c_2|\cdot|b_1c_2| \\ |a_1b_3|^2 & |a_1c_3|^2 & |b_1c_3|^2 & |a_1b_3|\cdot|a_1c_3| & |a_1b_3|\cdot|b_1c_3| & |a_1c_3|\cdot|b_1c_3| \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ |a_3b_4|^2 & |a_3c_4|^2 & |b_3c_4|^2 & |a_3b_4|\cdot|a_3c_4| & |a_3b_4|\cdot|b_3c_4| & |a_3c_4|\cdot|b_3c_4| \end{Vmatrix}$$
$$= |a_1b_2c_3|^2 \cdot |a_1b_2c_4|^2 \cdot |a_1b_3c_4|^2 \cdot |a_2b_3c_4|^2.$$

The proof consists in showing, from an examination of the 1st, 2nd, 4th rows, that $|a_1b_2c_3|^2$ is a factor, then similarly that $|a_1b_2c_4|^2$, $|a_1b_3c_4|^2$, $|a_2b_3c_4|^2$ are factors, and finally that the only other possible factor is a power of -1.

As bringing additional interest to an already interesting theorem we draw attention to the fact that the product of the m-line minors of an m-by-$(m+1)$ array was shown in 1907 to be itself expressible as a determinant of order $\frac{1}{2}m(m+1)$, and that consequently we have the curious result that one special determinant of that order equals the $(m-1)$th power of another: for example, when m is 3 the 6-line compound determinant above

$$= \begin{vmatrix} a_1a_2 & b_1b_2 & c_1c_2 & a_1b_2+a_2b_1 & a_1c_2+a_2c_1 & b_1c_2+b_2c_1 \\ a_1a_3 & b_1b_3 & c_1c_3 & a_1b_3+a_3b_1 & a_1c_3+a_3c_1 & b_1c_3+b_3c_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_3a_4 & b_3b_4 & c_3c_4 & a_3b_4+a_4b_3 & a_3c_4+a_4c_3 & b_3c_4+b_4c_3 \end{vmatrix}^2 .$$

An independent proof of this would be of value.

Sibirani's paper is avowedly a close following-up of Vivanti's, the key to the generalisation made lying in the fact that the originating array is no longer of $m+1$ columns but of $m+p$. The order of the compound determinant is now $(m+p)_{p+1}$: the minors that go to the construction of it are as before of the order $m-1$, but the number of them involved in an element is no longer 2 but $p+1$. The right-hand side of the equality is in keeping with this; the minors involved are as before of the order m and the index of the power to which they are raised is $m-1$, but the number of them is now $(m+p)_m$.

METZLER, W. H. (1913 $^{17}/_4$).

[Some factorable minors of a compound determinant. *Proceed. R. Soc. Edinburgh*, xxxiv, pp. 27–31.]

The avowed object of this paper is to include in one generalisation Sylvester's theorem of 1851 (*Hist.*, ii, pp. 193–197) and a theorem of Muir's. Unfortunately, the width of the generalisation makes the statement of it distinctly troublesome: it and the explanations of its attendant symbolisms occupy a full page and a half (pp. 29–31) of the paper. We give instead an example where $m=5$ and $k=3$, the minor proposed for factorisation being

$$\Big|\, |a_x b_y c_3| \; |a_x b_y d_4| \; |a_x b_y e_5| \; |a_x c_3 d_4| \; |a_x c_3 e_5| \; |a_x d_4 e_5| \; |b_2 c_3 d_4| \; |b_2 c_3 e_5| \; |b_2 d_4 e_5| \; |c_3 d_4 e_5| \,\Big|,$$

or M say.

Multiplying M by $|a_1 b_2 c_3 d_4 e_5|^4$ in the form of the 2nd compound of $|a_1 b_2 c_3 d_4 e_5|$, namely,

$$\Big|\; |d_4 e_5| \quad |c_3 e_5| \quad |c_3 d_4| \quad |b_2 e_5| \quad |b_2 d_4| \quad |b_2 c_3| \quad |a_1 e_5| \quad |a_1 d_4| \quad |a_1 c_3| \quad |a_1 b_2| \;\Big|$$

with its 2nd, 5th, 7th, 9th rows altered in sign, we obtain a product determinant having nothing but zeros on one side of the diagonal, and its diagonal term equal to

$$|a_x b_y c_z d_4 e_5| \; |a_x b_y c_4 d_3 e_5| \; |a_x b_y c_5 d_3 e_4| \; |a_x b_3 c_4 d_2 e_5| \; |a_x b_3 c_5 d_2 e_4| \; |a_x b_4 c_5 d_2 e_3| \cdot |a_1 b_2 c_3 d_4 e_5|^4.$$

The result thus reached is *

$$\mathrm{M} = |\, a_x b_y c_z d_4 e_5 \,| \cdot |\, a_x b_y c_3 d_4 e_5 \,|^2 \cdot |\, a_x b_2 c_3 d_4 e_5 \,|^3.$$

---

* To be strictly accurate it should be added that Metzler's equality is the complementary of this, and therefore has an additional factor on the right-hand side. His notation for minors is unusual, $\left|\begin{smallmatrix} 1 & 2 & 3 \\ m & n & p \end{smallmatrix}\right|$ standing for the complementary of what is ordinarily denoted by that symbol—standing, in fact, for our $\left|\begin{smallmatrix} 1 & 2 & 3 \\ m & n & p \end{smallmatrix}\right|$ (*Transac. R. Soc. Edinburgh*, xl, pp. 511–533).

METZLER, W. H. (1914 $^1/_3$).

[On the expansion of certain minors of the 7th compound of a determinant as a function of the elements of a single line of the m^{th} compound. *Annals of Math.*; xv, pp. 166–169.]

This paper follows up Muir's paper of 1905 on the equality of two compound determinants. The nature of the generalisation effected will be understood from the fact that it depends on the multiplication of the k^{th} compound by a minor of the $(m-k)^{\text{th}}$ compound without the resulting determinant being a minor of the m^{th} compound. Specialisations leading to factorisation of the product-determinant afford results of varying interest. The example given is

$$\Big|\ |a_1b_2|\quad |a_3c_4|\quad |a_5d_6|\quad |b_7c_8|\quad |b_9d_\tau|\quad |c_\zeta d_\sigma|\ \Big| \cdot |a_9b_\tau c_\sigma d_\zeta|$$
$$= \Big|\ |a_1b_2c_\tau d_\zeta|\quad |a_3b_4c_\tau d_\sigma|\quad |a_5b_6c_9d_\zeta|\quad |a_7b_8c_9d_\sigma|\ \Big|.$$

It may be readily verified by multiplying the 6-line determinant on the left by the second compound of $|a_9b_\zeta c_\sigma d_\tau|$ and striking out $|a_9b_\zeta c_\sigma d_\tau|^2$ from both sides.

The other theorem of 1905 is also generalised.*

THAER, F. (1914): THAER, C. (1914).

[Ein Determinantensatz. *Mittheil. d. math. Ges. zu Hamburg*, v, pp. 141–142.]

[Ein Determinantensatz. *Archiv d. Math. u. Phys.*, (3) xxiii, pp. 92–93.]

The theorem in question asserts the divisibility of $|A_{11}^pA_{22}^p \ldots A_{nn}^p|$ by $|a_{11}a_{22} \ldots a_{nn}|^{n-1}$. Probably a better proof than those here given is readily got by using Chio's condensation-theorem (*Hist.*, ii, pp. 79–81).

MUIR, T. (1914 $^{16}/_9$).

[Note on Hesse's generalisation of Pascal's theorem. *Transac. R. Soc. S. Africa*, (2) v, pp. 39–43.]

This concerns Hesse's equalities of 1872 (*Hist.*. iii, pp. 437–438) regarding the determinants $[\alpha, \beta]$, $[\alpha\gamma, \beta\delta]$ got by the bordering of $|u_{1n}|$. It is first shown that Hesse's result

$$|u_{1n}| \cdot [\alpha\gamma, \beta\delta] = [\alpha\beta, \gamma\delta] - [\alpha\delta, \gamma\beta]$$

---

* These wide generalisations of Metzler's might well have been referred to at the end of my recent paper in the *Proceedings* (xlv, pp. 51–55): the effect of the footnote there given would thus have been much enhanced.

can be followed up by

$$|u_{1n}|\cdot[\alpha\gamma\epsilon,\ \beta\delta\zeta] = [\alpha,\beta]\cdot[\gamma\epsilon,\ \delta\zeta] + [\alpha,\delta]\cdot[\gamma\epsilon,\ \beta\zeta] + [\alpha,\zeta]\cdot[\gamma\epsilon,\ \beta\delta],$$

and others like it: then that the same result written in the form

$$|u_{1n}|\cdot[\alpha\gamma,\ \beta\delta] = \begin{vmatrix} [\alpha,\beta] & [\alpha,\delta] \\ [\gamma,\beta] & [\gamma,\delta] \end{vmatrix}$$

is also the opening member of another series, of which the next member is

$$|u_{1n}|^2\cdot[\alpha\gamma\epsilon,\ \beta\delta\zeta] = \begin{vmatrix} [\alpha,\beta] & [\alpha,\delta] & [\alpha,\zeta] \\ [\gamma,\beta] & [\gamma,\delta] & [\gamma,\zeta] \\ [\epsilon,\beta] & [\epsilon,\delta] & [\epsilon,\zeta] \end{vmatrix}:$$

and finally that this second series can by despecialisation be led up to an already well-known "extensional" (*Hist.*, iv, p. 8).

Pascal, M. (1914 $/_{11}$).

[Un teorema relativo ai determinanti composti. *Giornale di Mat.*, lii, pp. 372–376.]

The first three pages of this paper are occupied with the two series of so-called "condensation-theorems" already more than once rediscovered (see above, p. 191). In the remaining two pages, however, an interesting deduction, not hitherto brought to light, is drawn attention to and discussed. It is arrived at quite simply by suitably permuting the columns of the basic determinant. For example, by Hermite's initial theorem of one of the said series, we have

$$|a_1b_2c_3d_4|\cdot a_2a_3 = \begin{vmatrix} |a_1b_2| & |a_2b_3| & |a_3b_4| \\ |a_1c_2| & |a_2c_3| & |a_3c_4| \\ |a_1d_2| & |a_2d_3| & |a_3d_4| \end{vmatrix},$$

and from this and the companion equality got by mere interchange of the suffixes 1 and 4 we deduce

$$\Big|\ |a_1b_2|\quad |a_2c_3|\quad |a_3d_4|\ \Big| = \Big|\ |a_2b_4|\quad |a_2c_3|\quad |a_3d_1|\ \Big|.$$

The question of proving such an equality otherwise is not entered on.

Muir, T. (1916 $^1/_5$, 1917 $^1/_6$, 1917 $^1/_7$).

[Question 18215. *Math. Quest. and Sol.*, ii, pp. 35–36.]
[Question 18448. *Math. Quest. and Sol.*, iii, part 6.]
[Question 18467. *Math. Quest. and Sol.*, iv, pp. 75–76.]

Of the three results here the first and second are the more interesting, the third being merely an alternative proof of a known equality. The first effects the resolution of

$$\begin{vmatrix} |x_1y_\alpha| & |x_2y_\alpha| & |x_1y_\beta| & |x_2y_\beta| & |x_1y_\gamma| & |x_2y_\gamma| \\ |x_3y_\alpha| & |x_4y_\alpha| & |x_3y_\beta| & |x_4y_\beta| & |x_3y_\gamma| & |x_4y_\gamma| \\ |x_5y_\alpha| & |x_6y_\alpha| & |x_5y_\beta| & |x_6y_\beta| & |x_5y_\gamma| & |x_6y_\gamma| \end{vmatrix}$$

into

$$|x_\alpha y_\beta|\cdot|x_\beta y_\gamma|\cdot|x_\gamma y_\alpha|\cdot\begin{vmatrix} x_1x_2 & y_1y_2 & x_1y_2+x_2y_1 \\ x_3x_4 & y_3y_4 & x_3y_4+x_4y_3 \\ x_5x_6 & y_5y_6 & x_5y_6+x_6y_5 \end{vmatrix};$$

and the second is the theorem that in the second compound of any 4-line determinant we may have one 3-line minor equal to another although differing from it in form and situation, namely:

$$\begin{vmatrix} p & 7-p & r \\ p & 7-p & r \end{vmatrix} = \begin{vmatrix} q & 7-q & r \\ q & 7-q & r \end{vmatrix}$$

where p, $7-p$, q, $7-q$, r are the ordinal numbers of rows and columns which the minors occupy in the compound.

WHITTAKER, E. T. (1916 $^1/_{11}$).

(See under this heading in Chapter on Latent Roots.)

MUIR, T. (1917 $^{11}/_4$).

[Note on the resolvability of the minors of a compound determinant. *Transac. R. Soc. S. Africa*, vii, pp. 97–102.]

The subject of detailed investigation here is the 3-line minors of the 2nd compound of $|a_1b_2c_3d_4|$, and the essence of the paper is to be found in the following classification and census of the said minors, 400 in number:—

Irresolvable, like	$\Big\vert\ \vert a_1b_2\vert\ \ \vert a_1c_3\vert\ \ \vert b_2d_4\vert\ \Big\vert$		144
Resolvable, like	$\Big\vert\ \vert a_1b_2\vert\ \ \vert a_1d_3\vert\ \ \vert b_2c_3\vert\ \Big\vert$	$= \vert a_1b_2c_3\vert\cdot\vert a_1b_2d_3\vert$	96
,, ,,	$\Big\vert\ \vert a_1b_2\vert\ \ \vert a_1c_3\vert\ \ \vert b_1d_4\vert\ \Big\vert$	$= \vert a_1b_2\vert\cdot\vert a_1b_2c_3d_4\vert$	96
,, ,,	$\Big\vert\ \vert a_1b_2\vert\ \ \vert a_1c_3\vert\ \ \vert b_2c_3\vert\ \Big\vert$	$= \vert a_1b_2c_3\vert^2$	16
,, ,,	$\Big\vert\ \vert a_1b_2\vert\ \ \vert a_1c_3\vert\ \ \vert a_1d_4\vert\ \Big\vert$	$= a_1{}^2\ \vert a_1b_2c_3d_4\vert$	16
Vanishing like	$\Big\vert\ \vert a_1b_2\vert\ \ \vert a_1c_3\vert\ \ \vert b_1c_4\vert\ \Big\vert$		32

MUIR, T. (1918 $^{10}/_{2}$).

[Theorems connected with minors of an m-by-m^2 array. *Messenger of Math.*, xlvii, pp. 184–190.]

The fundamental theorem here is: *If the* m^2 *columns of an* m-*by*-m^2 *array be separated into* $\mathrm{m}+1$ *sets, namely,* m *sets of* $\mathrm{m}-1$ *columns each and a single set of* m *columns, then the determinant whose* $(\mathrm{r}, \mathrm{s})^{th}$ *element is the determinant formed from the* r^{th} *set and the* s^{th} *column of the last set has the determinant of the last set for a factor, the cofactor being the compound determinant formed from the first* m *sets as the former compound determinant was formed from all the sets.* For example, when n is 4, and $|\, p, q, t, u \,|$ is used to stand for the determinant whose columns are the p^{th}, q^{th}, t^{th}, u^{th} columns of the array

$$\begin{array}{cccccccccccccccc} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & g_1 & h_1 & i_1 & j_1 & k_1 & l_1 & x_1 & y_1 & z_1 & w_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_4 & b_4 & c_4 & d_4 & e_4 & f_4 & g_4 & h_4 & i_4 & j_4 & k_4 & l_4 & x_4 & y_4 & z_4 & w_4 \end{array}$$

we have

$$\left|\begin{array}{cccc} |\,1, 2, 3, 13\,| & |\,1, 2, 3, 14\,| & |\,1, 2, 3, 15\,| & |\,1, 2, 3, 16\,| \\ |\,4, 5, 6, 13\,| & |\,4, 5, 6, 14\,| & |\,4, 5, 6, 15\,| & |\,4, 5, 6, 16\,| \\ |\,7, 8, 9, 13\,| & |\,7, 8, 9, 14\,| & |\,7, 8, 9, 15\,| & |\,7, 8, 9, 16\,| \\ |\,10, 11, 12, 13\,| & |\,10, 11, 12, 14\,| & |\,10, 11, 12, 15\,| & |\,10, 11, 12, 16\,| \end{array}\right|$$

$$= |\,13, 14, 15, 16\,| \left|\begin{array}{ccc} |\,1, 2, 3, 10\,| & |\,1, 2, 3, 11\,| & |\,1, 2, 3, 12\,| \\ |\,4, 5, 6, 10\,| & |\,4, 5, 6, 11\,| & |\,4, 5, 6, 12\,| \\ |\,7, 8, 9, 10\,| & |\,7, 8, 9, 11\,| & |\,7, 8, 9, 12\,| \end{array}\right| .$$

The proof rests on the condensation-theorem of 1905, which in turn itself receives development: for example, its simplest case

$$\left|\begin{array}{ccc} |\,a_1b_2\,| & |\,c_1d_3\,| & |\,e_2f_3\,| \end{array}\right| = \left|\begin{array}{cc} |\,a_1b_2e_3\,| & |\,a_1b_2f_3\,| \\ |\,c_1d_2e_3\,| & |\,c_1d_2f_3\,| \end{array}\right|$$

is shown to lead to

$$\left|\begin{array}{ccc} |\,a_1b_2\,| & |\,c_1d_3\,| & |\,e_2f_4\,| \end{array}\right| + \left|\begin{array}{ccc} |\,a_1b_2\,| & |\,c_1d_4\,| & |\,e_2f_3\,| \end{array}\right|$$

$$= \left|\begin{array}{cc} |\,a_1b_2e_3\,| & |\,a_1b_2f_4\,| \\ |\,c_1d_2e_3\,| & |\,c_1d_2f_4\,| \end{array}\right| + \left|\begin{array}{cc} |\,a_1b_2e_4\,| & |\,a_1b_2f_3\,| \\ |\,c_1d_2e_4\,| & |\,c_1d_2f_3\,| \end{array}\right| ,$$

and thence to another with *three* compound determinants on either side, and finally to another with *four*, the complete set forming *a series of* *relations connecting three-line minors of the second compound of* $|\,a_1b_2c_3d_4e_5f_6\,|$ *with two-line minors of the third compound.* In the concluding paragraph the minors of another compound are also brought into the circle of relationships.

WHITTAKER, E. T. (1918 $^{10}/_{9}$).

[On determinants whose elements are determinants. *Proceed. Edinburgh Math. Soc.*, xxxvi, pp. 107–115.]

The fresh idea here brought forward is interesting and suggestive, namely, the possibility of expressing certain compound determinants as determinants that are not compound, but have elements of the ordinary monomial kind; for example,

$$\begin{vmatrix} |\,a_1b_2e_3\,| & |\,a_1b_2f_3\,| \\ |\,c_1d_2e_3\,| & |\,c_1d_2f_3\,| \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} |\,a_1b_2\,| & |\,a_1b_3\,| & |\,a_2b_3\,| \\ |\,c_1d_2\,| & |\,c_1d_3\,| & |\,c_2d_3\,| \\ |\,e_1f_2\,| & |\,e_1f_3\,| & |\,e_2f_3\,| \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 & \cdot & \cdot & \cdot \\ b_1 & b_2 & b_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & c_1 & c_2 & c_3 \\ \cdot & \cdot & \cdot & d_1 & d_2 & d_3 \\ e_1 & e_2 & e_3 & e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 & f_1 & f_2 & f_3 \end{vmatrix}.$$

The types found to be so expressible are those of Muir's second paper of 1905, and the non-compound now shown to give rise to them is of the form

$$\begin{vmatrix} \{P_1\} & \{0\} & \{0\} & \cdots & \{0\} & \{0\} \\ \{0\} & \{P_2\} & \{0\} & \cdots & \{0\} & \{0\} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \{0\} & \{0\} & \{0\} & \cdots & \{0\} & \{P_m\} \\ \{P_{m+1}\} & \{P_{m+1}\} & \{P_{m+1}\} & \cdots & \{P_{m+1}\} & \{P_{m+1}\} \end{vmatrix},$$

where the $\{P\}$'s are m-by-$(m+1)$ arrays of any elements and the $\{0\}$'s are m-by-$(m+1)$ arrays of zero elements, so that the order of the determinant is the $(m^2+m)^{\text{th}}$, the number of variables involved is $m^2(m+1)^2$, and the orders of the two compound determinants are the m^{th} and $(m+1)^{\text{th}}$ respectively. The transformation from the non-compound determinant to either of the others is effected by direct application of Laplace's expansion-theorem.

As a deduction there is given a theorem which may be looked on as a generalisation (from 1 to p) of the "unextended" case of Bazin's of 1851, namely: *The determinant whose rows are the first* p *rows of the adjugate of* A *and the last* n − p *rows of the adjugate of* B *is equal to*

$$B^{n-p-1}\begin{vmatrix} (A_1, B_1) & (A_1, B_2) & \dots & (A_1, B_p) \\ (A_2, B_1) & (A_2, B_2) & \dots & (A_2, B_p) \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ (A_p, B_1) & (A_p, B_2) & \dots & (A_p, B_p) \end{vmatrix}$$

where n *is the order of* A *and* B, *and* (A_r, B_s) *denotes the determinant got from* A *by replacing its* r^th^ *row by the* s^th^ *row of* B.

Muir, T. (1918 $^1/_{12}$).

[Note on the determinant of the primary minors of a special set of $(n-1)$-by-n arrays. *Proceed. R. Soc. Edinburgh,* xxxix, pp. 35–40.]

When n is 4 the arrays in question are

$$\left\|\begin{matrix} 1 & a+b+c & ab+ac+bc & abc \\ 1 & e+f+g & ef+eg+fg & efg \\ 1 & i+j+k & ij+ik+jk & ijk \end{matrix}\right\|, \quad \left\|\begin{matrix} 1 & b+c+d & bc+bd+cd & bcd \\ 1 & f+g+h & fg+fh+gh & fgh \\ 1 & j+k+l & jk+jl+kl & jkl \end{matrix}\right\|$$

$$\left\|\begin{matrix} 1 & c+d+e & \dots \\ 1 & g+h+i & \dots \\ 1 & k+l+a & \dots \end{matrix}\right\|, \quad \left\|\begin{matrix} 1 & d+e+f & \dots \\ 1 & h+i+j & \dots \\ 1 & l+a+b & \dots \end{matrix}\right\|$$

and the associated determinant referred to, Θ say, has for its $(r, s)^{\text{th}}$ element the minor got from the r^{th} array by deleting the s^{th} column. On using the condensation-theorem of 1905 (see above, p. 193) Θ is changed into a 3-line compound whose $(1, 1)^{\text{th}}$ element is

$$\begin{vmatrix} 1 & a+b+c & \dots \\ 1 & b+c+d & \dots \\ 1 & f+g+h & \dots \\ 1 & i+j+k & \dots \end{vmatrix}.$$

The fundamental property of determinants of this type is next given, so as to aid in the evaluation of Θ, and especially to ascertain under what circumstances Θ vanishes. Of the properties of Θ thus reached one of the most interesting is that when $g, l, i \equiv d, c, e$, its value is expressible as a product of eighteen differences, namely,

$$\begin{aligned} &(k-h)(d-j)(f-j)(a-e)(b-e)(c-e) \\ \cdot\ &(k-b)(d-a)(e-a)(d-c)(h-c)(e-c) \\ \cdot\ &(h-e)(c-f)(b-f)(j-d)(k-d)(e-d). \end{aligned}$$

(Muir, T. (1919 $^{16}/_{1}$).

[Note on compound determinants expressible as simple determinants. *Quart. Journ. of . . . Math.*, xlviii, pp. 379–384.]

This note follows up Whittaker's of 1918. The two results arrived at in it may for comparison's sake be stated thus: *If* $\{\mathrm{m}, \mathrm{m}+1\}$ *denote an* m-*by*-$(\mathrm{m}+1)$ *array of any elements, and* $\{\mathrm{o}\}$ *an* m-*by*-$(\mathrm{m}+1)$ *array of zeros, the non-compound determinant of the* $(\mathrm{pm}+\mathrm{p})^{th}$ *order*

$$\begin{vmatrix} \{m, m+1\}_1 & \{o\} & \{o\} & \cdots & \{o\} & \{o\} \\ \{o\} & \{m, m+1\}_2 & \{o\} & \cdots & \{o\} & \{o\} \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ \{o\} & \{o\} & \{o\} & \cdots & \{o\} & \{m, m+1\}_p \\ \{p, m+1\}_{p+1} & \{p, m+1\}_{p+1} & \{p, m+1\}_{p+1} & \cdots & \{p, m+1\}_{p+1} & \{p, m+1\}_{p+1} \end{vmatrix}$$

is equal to a p-*line compound determinant with* $(\mathrm{m}+1)$-*line elements; and the similar determinant of the* $(\mathrm{pm})^{th}$ *order*

$$\begin{vmatrix} \{m, m+1\}_1 & \{o\} & \{o\} & \cdots & \{o\} & \{o\} \\ \{o\} & \{m, m+1\}_2 & \{o\} & \cdots & \{o\} & \{o\} \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ \{o\} & \{o\} & \{o\} & \cdots & \{m, m+1\}_{p-1} & \{o\} \\ \{m, m+1\}_p & \{m, m+1\}_p & \{m, m+1\}_p & \cdots & \{m, m+1\}_p & \{m, -p+m+1\}_p \end{vmatrix}$$

is equal to a p-*line compound determinant with* m-*line elements.* In the paper, however, a slightly different notation for the arrays makes greater definiteness possible: for example, when p is 3 we have

$$\begin{vmatrix} \mathrm{A}(m, m+1) & \{o\} & \{o\} \\ \{o\} & \mathrm{B}(m, m+1) & \{o\} \\ \{o\} & \{o\} & \mathrm{C}(m, m+1) \\ \mathrm{Z}(3, m+1) & \mathrm{Z}(3, m+1) & \mathrm{Z}(3, m+1) \end{vmatrix} = (-1)^m \begin{vmatrix} \mathrm{A}_{+1} & \mathrm{A}_{+2} & \mathrm{A}_{+3} \\ \mathrm{B}_{+1} & \mathrm{B}_{+2} & \mathrm{B}_{+3} \\ \mathrm{C}_{+1} & \mathrm{C}_{+2} & \mathrm{C}_{+3} \end{vmatrix}$$

where A_{+r} stands for the determinant got by affixing to the array A the r^{th} row of the last array Z; and

$$\begin{vmatrix} \mathrm{A}(m, m+1) & \{o\} & \{o\} \\ \{o\} & \mathrm{B}(m, m+1) & \{o\} \\ \mathrm{C}(m, m+1) & \mathrm{C}(m, m+1) & \mathrm{C}(m, -3+m+1) \end{vmatrix} = (-1)^{m-1} \,|\, \mathrm{A}_{-1}\mathrm{B}_{-2}\mathrm{C}_{-3} \,| ,$$

where A_{-r} stands for the determinant got from the array A by deleting the r^{th} column, and $\mathrm{C}(m, -r+m+1)$ stands for the array got from C by deleting the first r columns.

The original suggesting theorem is the case of the first above when p is taken equal to m, and the case of the second when p is taken equal to

$m+1$, the compound determinant got from the first being of the m^{th} order with $(m+1)$-line elements, and that got from the second being of the $(m+1)^{\text{th}}$ order with m-line elements.

MUIR, T. (1919 $^{15}/_{7}$).

[The eliminant of two binary quantics with determinantal coefficients. *Messenger of Math.*, xlix, pp. 37–41.]

The curious result here lit upon is that the eliminant of

$$\left.\begin{array}{l} |\,a_1b_2c_3\,|x^3 - |\,a_1b_2c_4\,|x^2 + |\,a_1b_3c_4\,|x - |\,a_2b_3c_4\,| = 0 \\ |\,a_2b_3c_4\,|x^3 - |\,a_2b_3c_5\,|x^2 + |\,a_2b_4c_6\,|x - |\,a_3b_4c_5\,| = 0 \end{array}\right\}$$

mimics Bezout's eliminant of

$$\left.\begin{array}{l} a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 = 0 \\ b_1x^4 + b_2x^3 + b_3x^2 + b_4x + b_5 = 0 \end{array}\right\},$$

being

$$\left|\begin{array}{cccc} |\,123\,| & |\,124\,| & |\,134\,| & |\,234\,| \\ |\,124\,| & |\,134\,| + |\,125\,| & |\,234\,| + |\,135\,| & |\,235\,| \\ |\,134\,| & |\,234\,| + |\,135\,| & |\,235\,| + |\,145\,| & |\,245\,| \\ |\,234\,| & |\,235\,| & |\,245\,| & |\,345\,| \end{array}\right|.$$

In seeking for an explanation of this, the additional result is obtained that the eliminant of

$$\left\|\begin{array}{cccc} a_1y - l_1x & a_2y - l_2x & a_3y - l_3x & a_4y - l_4x \\ b_1y - m_1x & b_2y - m_2x & b_3y - m_3x & b_4y - m_4x \\ c_1y - n_1x & c_2y - n_2x & c_3y - n_3x & c_4y - n_4x \end{array}\right\| = 0$$

is

$$\left|\begin{array}{ccc} |\,a_1b_2c_3l_4\,| & |\,a_1b_2n_3l_4\,| + |\,a_1m_2c_3l_4\,| & |\,a_1m_2n_3l_4\,| \\ |\,a_1b_2c_3m_4\,| & |\,a_1b_2n_3m_4\,| + |\,l_1b_2c_3m_4\,| & |\,l_1b_2n_3m_4\,| \\ |\,a_1b_2c_3n_4\,| & |\,a_1m_2c_3n_4\,| + |\,l_1b_2c_3n_4\,| & |\,l_1m_2c_3n_4\,| \end{array}\right|.$$

MUIR, T. (1919 $^{24}/_{9}$).

[Additional note on the resolvability of the minors of a compound determinant. *Transac. R. Soc. S. Africa*, viii, 229–233.]

Elimination by the "method of instalments" here furnishes a suggestion for factorisation. If, for example, the set of six homogeneous linear equations

$$\left.\begin{array}{l} a_1u + a_2v + a_3w + a_4x + a_5y + a_6z = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ f_1u + f_2v + f_3w + f_4x + f_5y + f_6z = 0 \end{array}\right\}$$

be taken, and u, v, w be eliminated from every subset of four, there are obtained 15 equations in x, y, z, and then from the $C_{15,3}$ triads of these are got 455 eliminants like

$$\begin{vmatrix} |a_1b_2c_3d_4| & |a_1b_2c_3d_5| & |a_1b_2c_3d_6| \\ |a_1b_2c_3e_4| & |a_1b_2c_3e_5| & |a_1b_2c_3e_6| \\ |a_1b_2c_3f_4| & |a_1b_2c_3f_5| & |a_1b_2c_3f_6| \end{vmatrix},$$

of which the true eliminant $|a_1b_2c_3d_4e_5f_6|$, or Δ_6 say, must be a factor. Full factorisation of a large number of 3-line minors of the 4th compound of Δ_6 is thus seen to be rendered easy; and, the requisite work having been performed, the following census of results is given: *Of* 455 *three-line minors of the fourth compound of* Δ_6

15 *minors are of the type* $\begin{vmatrix} |a_1b_2d_3| & |a_1b_2c_3| \\ |d_1e_2f_3| & |c_1e_2f_3| \end{vmatrix} . \Delta_6,$

180 *minors are of the type* $|a_1b_2c_3| \,.\, |a_1d_2e_3| \,.\, \Delta_6,$

180 *minors are of the type* $|a_1b_2c_3| \,.\, |a_1b_2d_3| \,.\, \Delta_6,$

20 *minors are of the type* $|a_1b_2c_3|^2 \,.\, \Delta_6,$

and 60 *minors are equal to* 0.

Muir, T. (1919 $^5/_{12}$).

[Note on the m^{th} compound of a determinant of the $(2m)^{\text{th}}$ order. *Messenger of Math.*, xlix, pp. 62–64.]

The result here reached is really a generalisation connected with the compound mentioned. When m is 2 it concerns a 4-by-9 array of elements, and is

$$\begin{vmatrix} |a_1b_2| & |a_1b_3| & |a_1b_4| & |a_2b_3| & |a_2b_4| & |a_3b_4| \\ |a_1c_2| & |a_1c_3| & |a_1c_4| & |a_2c_3| & |a_2c_4| & |a_3c_4| \\ |b_1c_2| & |b_1c_3| & |b_1c_4| & |b_2c_3| & |b_2c_4| & |b_3c_4| \\ |d_1e_2| & |d_1e_3| & |d_1e_4| & |d_2e_3| & |d_2e_4| & |d_3e_4| \\ |f_1g_2| & |f_1g_3| & |f_1g_4| & |f_2g_3| & |f_2g_4| & |f_3g_4| \\ |h_1k_2| & |h_1k_3| & |h_1k_4| & |h_2k_3| & |h_2k_4| & |h_3k_4| \end{vmatrix} = \begin{vmatrix} |a_1b_2d_3e_4| & |a_1b_2f_3g_4| & |a_1b_2h_3k_4| \\ |a_1c_2d_3e_4| & |a_1c_2f_3g_4| & |a_1c_2h_3k_4| \\ |b_1c_2d_3e_4| & |b_1c_2f_3g_4| & |b_1c_2h_3k_4| \end{vmatrix}.$$

An interesting stage in the course of degeneration to the compound is arrived at by putting the e's equal to the c's, and the g's equal to the b's.

Muir, T. (1915 $^{12}/_{11}$).

(See under this heading in the chapter on General Determinants.)

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(*Issued separately May* 18, 1925.)

XVII.—Discontinuities in the Atmosphere. By A. H. R. Goldie, M.A. (With Four Plates and Three Text Figures.)

(MS. received February 21, 1925. Read March 9, 1925.)

THIS paper contains the results of an investigation, on empirical and dynamical lines, into the following matters:—

(1) The origin of surfaces of discontinuity in the atmosphere;
(2) The precise processes occurring at such surfaces in well-defined cases;
(3) The physical processes which lead to the obliteration or movement of the discontinuities;
(4) The part played by the discontinuities in the development of the cyclones of temperate latitudes.

An "inversion" of the normal lapse rate of temperature in the vertical, and the associated changes of wind direction and velocity, are regarded as normally marking a surface of discontinuity or bounding surface between two currents differing in their mean motions and potential temperatures.

1. The frequency of occurrence of "inversions" at various heights has recently been determined by Peppler (1) to be somewhat as follows. The highest frequency is in the lowest stratum, below a height of 200 metres. Above that the most important maxima occur at heights of about 1500 and 2500 metres respectively; the frequency falls rapidly above 3000 metres and becomes very small above 4000 metres. The surface inversions, *i.e.* those below 200 metres, are probably to be accounted for in many cases by chilling of the air in contact with the surface, and by slow eddy diffusion of the cooling effect upward in the way described by Taylor (2).

During 1922 the present writer made an examination of all conspicuous cases of inversions—apart from surface inversions—noted in upper-air observations made from aeroplanes in that year, the examination being conducted with reference to the origin or past history of the air in the upper and lower currents respectively. There were 25 marked cases. In 16 of these the air above was of "equatorial" and the air below of "polar" origin, in the Bjerknes sense of these terms; in 8 cases the entire air column appeared to be of polar origin, the evidence indicating as a rule that the lower colder current had arrived from northerly regions by a more direct route than had the upper current; in the remaining case the questions of origin could not be definitely decided. This result, that difference of geographical origin and recent past history are by far the most important causes giving rise to inversions in the upper air and in all probability also to a considerable percentage of surface inversions, has been fully supported by individual cases noted and studied since that time. In view of the frequency of occurrence of the simple case of equatorial air above polar air, tables are given on pp. 227, 228 of the mean temperatures in each corresponding to various pressures and also of the mean difference of temperature. These mean values were derived by classifying all upper-air observations made from aeroplanes in this country in the sixteen months September 1921 to December 1922, the origin of the air in each case being determined by a study of the synoptic charts (4 daily) and other information contained in the Daily Weather Report of the Meteorological Office. This is not to be taken as implying that discontinuities observed are of quite the same magnitude as the differences of temperature shown in these tables; most of the attendant circumstances, including mixing and dynamical warming of the lower current (as, in the process of undercutting the warmer current (3), it is subjected to increasing pressure), operate in the direction of reducing the magnitude of discontinuity.

2. The conditions for the persistence of a stationary or quasi-stationary surface of separation between two currents of different densities and velocities have been set out by Margules (4), Helmholz (5), Bjerknes (6), and others. The possible occurrence of waves at the bounding surface has been studied mathematically by Helmholz (7), Kelvin (8), Lamb (9), and Schmidt (10).

In a paper (11) which will be referred to hereafter as "the previous paper," the present writer recently discussed the problem of two overlapping currents of mean velocities U in the lower and U′ in the upper, making angles a, a', with the direction of propagation of the waves, on the assumption of constant densities ρ and ρ' in the two regions. The x-axis is taken as the direction of propagation of waves of the shape $z=a\sin k(x-\mathrm{V}t)$, corresponding to the propagation with velocity V of waves of wave-length $\lambda=\frac{2\pi}{k}$ and of period $\mathrm{T}=\frac{2\pi}{k\mathrm{V}}$. The height of the bounding surface above ground-level is denoted by h. Formulæ are derived connecting the variables which are measured in upper-air observations, or can be obtained from the records of barographs, microbarographs, and anemographs on the ground. The actual observational data, in so far as they are available for certain particular cases, are shown to be in good agreement with the hydrodynamical theory given. The records of the ground-level instruments, taken in conjunction with the reasoning there given, appear to indicate that waves with amplitudes at the bounding surface of from 100 to 750 metres or perhaps more, with lengths of the order of 5 to 20 kilometres, and with periods of from a few minutes up to forty minutes (and in exceptional cases, to be referred to later, of one to two hours), are of comparatively frequent occurrence.

In the present paper the interest is concentrated mainly on the thermodynamical processes occurring in the upper air, and it is necessary therefore to extend the formulæ of the previous paper to apply to the case where the wave movement takes place not under conditions of constant air density but under conditions very closely approximating to adiabatic change. At the ground, by reason of the fact that the motion of a given air-mass is constrained to be horizontal, and that its changes of pressure are therefore limited in magnitude to the small changes (of, as a rule, less than ½ millibar) recorded on microbarographs, the adiabatic change of temperature is so slight as not to be detected by the ordinary means of observation. But at the bounding surface, or anywhere near it, the changes of pressure experienced by a given air-mass may be very considerable indeed, and it is therefore desirable to examine the question fully.

The pressure equation, in the case of adiabatic conditions, becomes

$$\frac{\delta\phi}{\delta t}+\tfrac{1}{2}q^2-\Phi+\frac{\gamma}{\gamma-1}\cdot\frac{p}{\rho}=\text{const}$$

where Φ is the potential of the extraneous forces. It can be shown that this involves no change in the form of the equation given in the previous paper, for the wave-length in the case where it is long compared with the height of the bounding surface above ground-level, *i.e.*

$$\lambda=\frac{2\pi(\mathrm{U}'\cos\alpha'-\mathrm{V})^2}{g\left(1-\frac{\rho'}{\rho}\right)-\frac{(\mathrm{U}\cos\alpha-\mathrm{V})^2}{h}} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

As to the effect which the waves produce on autographic records of pressure at ground-level, we find

$$\delta p=\frac{\rho_0\cdot v^2\cdot \mathrm{C}}{1-\frac{\rho_0 v^2\mathrm{C}}{\gamma p_0}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in lieu of the former equation

$$\delta p=\rho_0\cdot v^2\cdot \mathrm{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where δp is the amplitude of the pressure wave as recorded, ρ_0, p_0 are the (mean) ground-level density and pressure, v is written for $(\mathrm{U}\cos\alpha - \mathrm{V})$ and C for $\frac{a}{h}\cdot\frac{\frac{2\pi h}{\lambda}}{\sinh\frac{2\pi h}{\lambda}}$; in the case where λ is six or more times h, C becomes for all practical purposes $\frac{a}{h}$. The differences resulting from using equation (3) in lieu of equation (2) are quite negligible—of the order of 1 in 1000 or less.

The other important working equation, relating to the amplitude of the changes of wind at ground-level, *i.e.*

$$\delta\mathrm{U} = v\cdot\mathrm{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

remains unaltered. Any numerical results given in the previous paper therefore remain equally true for the adiabatic case.

Turning now to the question of the pressure at the bounding surface, we have in the lower current near this bounding surface

$$\frac{\gamma}{\gamma-1}\cdot\frac{p}{\rho} = \text{const} - \{ga - ka\coth kh(\mathrm{U}\cos\alpha - \mathrm{V})^2\}\sin k(x - \mathrm{V}t) - \tfrac{1}{2}\mathrm{U}^2$$
$$+ \text{terms involving } a^2,$$

and taking P_0, R_0 to represent the mean pressure and density at the bounding surface and $\delta\mathrm{P}$ the amplitude of the pressure variation occurring, we arrive at the result

$$\delta\mathrm{P} = \mathrm{R}_0(v^2\cdot\mathrm{C} - ga).$$

The term $\mathrm{R}_0v^2\mathrm{C}$ is obviously of slightly less account at height h than at the surface, and therefore may be neglected in the consideration of possible effects in the way of condensation. The other term, however, $-ga\mathrm{R}_0$, may be quite important. An air-mass near the boundary when raised to the crest of a wave suffers diminution of pressure to this extent as compared with its mean position or position before the wave-motion set in. Taking a by no means extreme case where h is 4000 metres and a is 500 metres, the reduction of pressure is about 40 millibars, and this at the level mentioned involves an adiabatic change of temperature of about 8° F. It is evident, therefore, that waves in the atmosphere cannot ordinarily develop very far without leading to cloud formation. If actual condensation takes place the whole conditions of the movement are altered. It is interesting to note that one possible effect of this condensation may be to dry the layer just above the discontinuity. At each wave crest a certain amount of condensed moisture will be undergoing a process of slow precipitation, and any that falls through the discontinuity will not again be able to return to the upper layer so long as the discontinuity persists and no mixing of the two layers takes place. It seems likely that some process of this sort may be at least a contributory cause, not necessarily the main one, of the dryness so very frequently found in the layer just above an inversion of temperature. The process would also tend to add to the vapour content of the layer just below the inversion, though the mere fact that an inversion forms a " lid " to ordinary convectional processes would probably largely account for the high relative humidity in that layer.

3. Apart from the question of cause, let us consider the case, which we know to be of very frequent occurrence, where the top of the lower current is practically saturated and the bottom of the upper current has a vapour content so low that condensation need not be considered. In the crests of any waves formed we shall then have the condition that on the upper side of the boundary temperature is being reduced at the dry adiabatic rate, whilst on the lower side temperature is being reduced at the saturated adiabatic rate. The difference between these two rates (as deduced from a Hertz (12) diagram), at a height centred at 2 km., is 4°·3 F. per 500 metres, and at a height of 5 km. it is 3°·4 F. Inversions of these amounts would therefore be wiped out at the crests of waves of 500 metres amplitude, whilst if the waves were of any greater amplitude actual instability would arise and the two layers at such points would mix. Looked at from this aspect, the addition of water-vapour to a lower cold layer by evaporation from the ground- or sea-surface is a contributory cause by means of which discontinuities can be wiped out and replaced by patches of small lapse rate of temperature, or possibly by means of which they may be transferred to different levels. It is to be noted that in cases where waves originate by reason of the existence of a discontinuity, streaks of cloud may be formed at levels some distance from the discontinuity and either above or below it, for all layers are affected by the wave-motion, the amplitude of the motion, of course, decreasing as a function of the distance from the bounding surface; the only condition is that the water content of the layer shall be sufficient to result in condensation when an air-mass is raised to the crest of its oscillation. Similarly, mixing of layers and the more even diffusion of heat and water-vapour may be brought about.

It will be seen, then, that in order that there should be perfect stability in the case of small inversions of temperature, it is necessary and also sufficient that one of the following conditions should be satisfied :—

(1) That there should be no water-vapour at the levels chiefly affected, or so little that no actual condensation can happen as a result of waves of the greatest amplitude likely to be developed; or

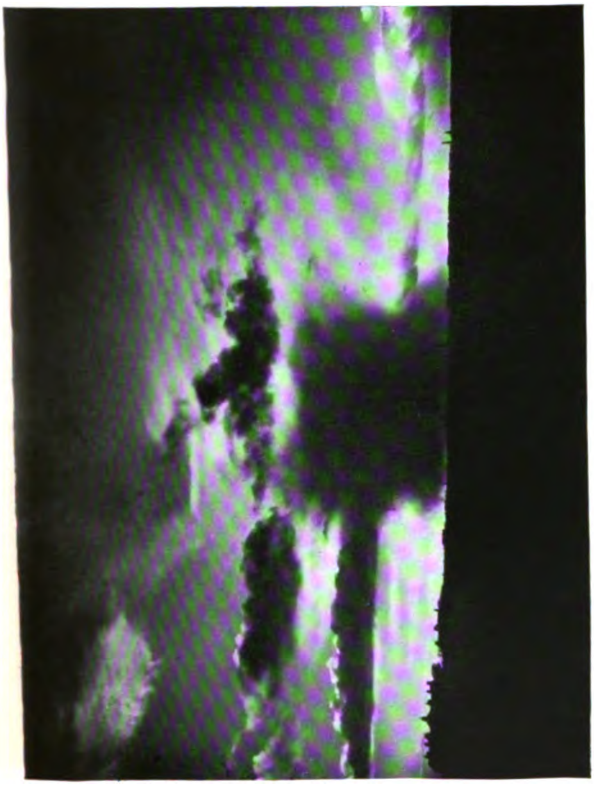
(2) that the saturation temperatures in the adjacent layers should differ by equal amounts from the actual temperatures; or

(3) that at the heights considered the dry adiabatics and the saturated adiabatics may for practical purposes be regarded as parallel.

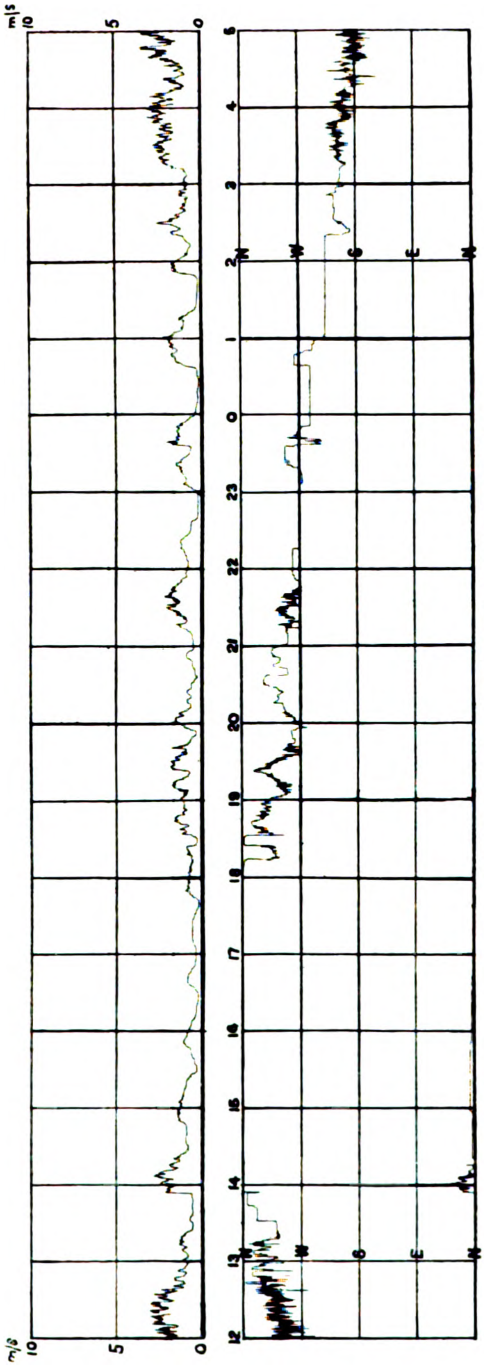
The application of these considerations to the upper regions of the atmosphere suggests an explanation of the sharpness and of the persistence of the tropopanse. If the upper limit of the highest clouds be regarded as marking the upper limit of water-vapour in the atmosphere, the first condition above would be satisfied, and any inversion once formed at a slightly higher elevation would tend to persist indefinitely, there being no mixing process but only the intensely slow process of conduction by means of which downward communication of heat could be brought about. If, on the other hand, following the view of Gold (13), the lower stratosphere be regarded as saturated, the second condition would be satisfied.

From his rich collection of cloud photographs Mr G. A. Clarke of Aberdeen has been able to find me illustrations of some of the points dealt with above, and to supply in addition descriptive notes of observations made on the days when the photographs were taken. A study of these photographs in relation to all other available information, as, for instance, records of autographic instruments, upper-air observations, synoptic charts, etc., has revealed many points of agreement of the hydrodynamical and thermodynamical theory with the facts of observation. A brief description of these will now be given.

(*a*) The first case (Plate I) is one of waves leading to the formation of bands of strato-cumulus cloud, with the development of a cumulo-nimbus cloud at or near the crest of an oscillation. The upper-air structure in the region of Aberdeen on that afternoon appears to have been complex, consisting first of a shallow N. to N.N.W. current, above that a westerly current, and finally (above 10,000 feet or so) a southerly wind, probably of the order of 20-25 m.p.h. There is no microbarograph at Aberdeen, but the anemograms, as will be seen, indicate very distinct oscillations of wind velocity with a period of 50 minutes, which continued so long as the northerly surface-wind persisted. It may be remarked that the trace of the anemograph here is typical of the case where, but for the waves in the upper air, the surface-wind would have fallen to a night calm. At about 18 hours the northerly current died out and the westerly one extended down to ground-level. Other waves less regular, with a less well-marked period of about 1 hour 45 minutes, then appeared, and continued till the

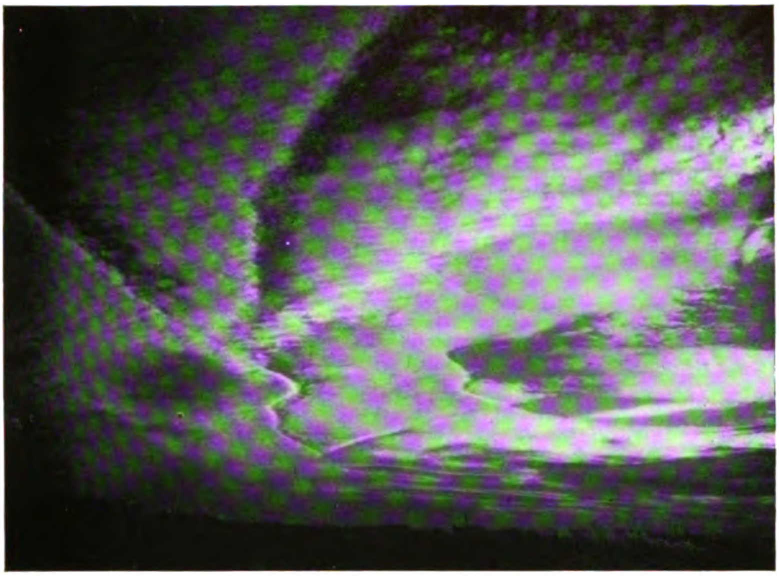


Cloud at Aberdeen on 28th October 1924, at 15 hours 30 minutes.

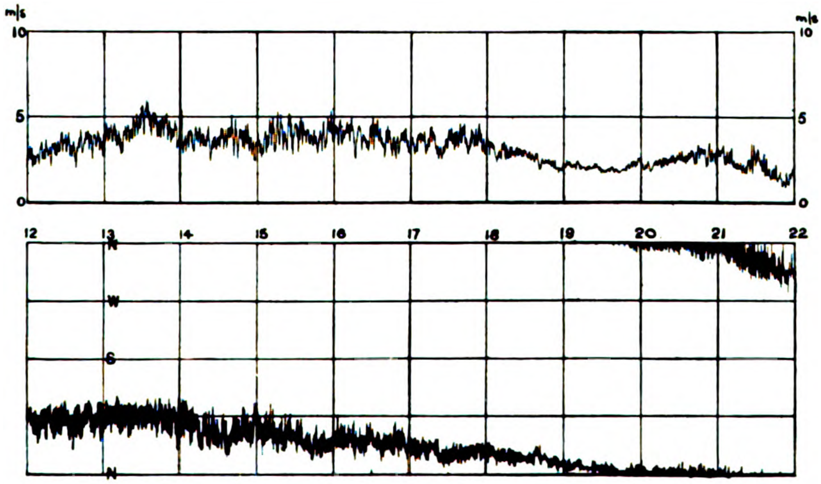


Pressure-tube Anemogram, Aberdeen, 28th–29th October 1924.

A. H. R. Goldie, M.A. [Plate I.



Cloud at Aberdeen on 3rd April 1924, at 17 hours 10 minutes.



Pressure-tube Anemogram, Aberdeen, 3rd April 1924.

southerly current in turn came down to surface-level. These processes were, in fact, associated with the advance from the Atlantic of the "warm-front" of a depression.

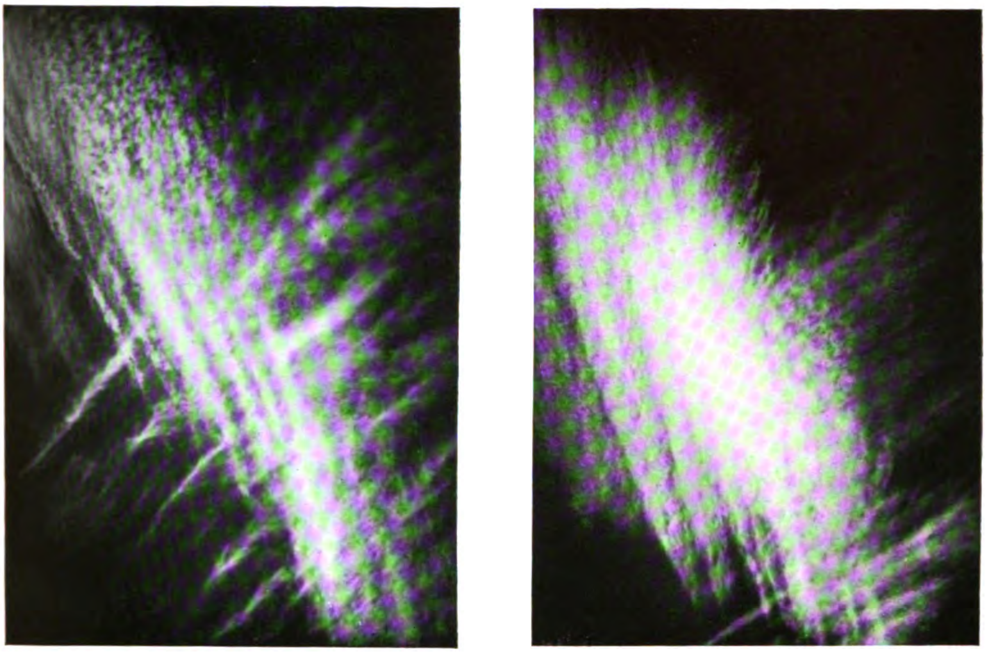
Mr Clarke happened to note that the cumulo-nimbus cloud was due west at 15·00 G.M.T., an observation which indicates that the pressure oscillation was in phase with the surface-wind oscillation, and therefore (see previous paper) that the direction of the surface-wind had a negative component along the direction of propagation of the waves. The photograph was taken at 15·30 G.M.T., when the anvil top of the cumulo-nimbus cloud, presumably entering the high southerly current, had become distended towards the rear into a sheet of "false cirrus." In this case the formation of the "false cirrus" need not have arisen from any electrical effect in the way suggested by Mr C. T. R. Wilson (14). The shearing effect at the upper discontinuity appears to afford quite sufficient reason for the distension.

(*b*) The photograph in the next case (Plate II), if looked at upside down, is exactly what one might see on a small scale on the surface of the sea when looking from a height of a few hundred feet. The case was associated with the gradual replacement of an easterly surface-wind by a W.N.W. wind, the wind at some height above having probably been westerly to north-westerly throughout the whole of the few hours considered; conditions were anticyclonic. The photograph was taken looking towards the south-west. Several lenticular clouds can be seen at the same level in the distance, suggesting the idea that such clouds are really wave-crest condensation effects—possibly only a slight mist being formed within them—which become conspicuous when looked at endwise. The anemogram on this occasion indicates a number of wave effects, none well developed, possibly owing to the continuous change in progress. The variations in direction shown on the direction record indicate that the direction of propagation of the waves was considerably inclined to the direction of the surface-wind, for, having regard to the smallness of the amplitudes and periods in the velocity trace, no other cause is likely to have produced them. So far as can be seen, there are evidences of periods of 2½ to 3 minutes, 5 to 5½ minutes, 10 to 11 minutes, and about 15 minutes, which are possibly harmonics of a common period; but the determinations are subject to much uncertainty. Indeed, for the study of cases of this sort it would be necessary to have a type of anemograph which would damp out the effects of gusts—that is, of the rapid variations in velocity and direction each lasting over only a few seconds. The pressure-tube anemograph is designed to record every such gust, whilst the other kind of anemograph in ordinary use, the cup anemograph, is of the integrating type, its records being of use chiefly where measurement of the total run over 1 hour or some similar period is desired.

(*c*) The third case (Plates III and IV) is a very beautiful one indeed. Neither microbarograms nor anemograms are available for Aberdeen for the period in question, but very complete observations were made of the clouds and their apparent movements. The synoptic situation at the time (fig. 1) was that Aberdeen lay right under the "occluded warm sector" of a depression centred to the east of Iceland; in other words, the cold air had completely surrounded the cyclone, so that in the surface layers it had become thermally symmetrical, the warm sector having been undercut and the last traces of it being just in process of being lifted from the ground, between the cold south-easterly wind in front and the cold invading westerly current in its rear. The long lines of cloud running north to south are probably the result in the upper air of the slight raising produced by the undercutting by the leading edge of the westerly current below; that the westerly current was doing some work of this sort from 13·00 onward we know from the slow rise of the barometer which then commenced. From observations of cloud movement made at 18 hours we also know that it was then at least a mile deep, its mean velocity, confirmed also by computation from pressure gradient, being W. 24 m.p.h.

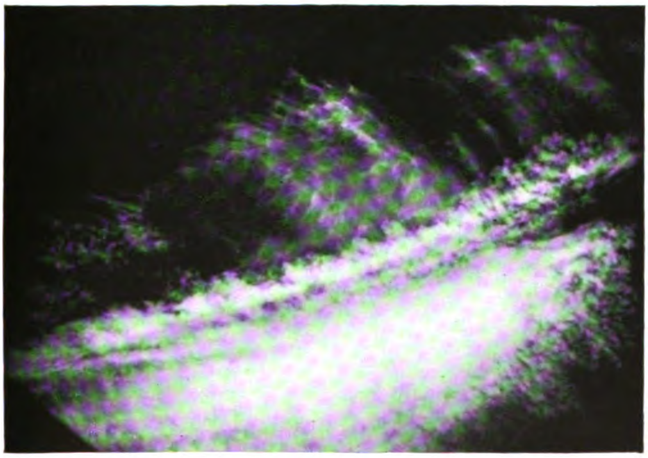
As to the drift of the warmer air above, a nephoscope observation of A-stratus cloud at 13·00 indicated a velocity (assuming the cloud at a height of 3 miles) of about 54 m.p.h. from S.W. by W., whilst observations of the rippled cirrus and cirro-cumulus clouds in the photographs between 15·00 and 18·00 indicated a velocity (assuming the height as being 3½ miles) of about 77 m.p.h. from S.S.W. Between the A-stratus level and the lower cold current there was therefore a relative velocity from about S.S.W., and the other set of waves indicated on the photographs are approximately normal to this direction. As to the velocity and direction indicated by the rippled clouds themselves, it would appear in theory that these should relate to the propagation of the waves and not to the movement of the actual air masses, for it is not really the same mass of air and condensed vapour that is under observation, but different masses of vapour which are being condensed and evaporated successively as they are subjected to the rise and fall of the passing waves; it is probably more than coincidence that in the above case acceptance of this idea brings the observed facts into agreement.

To prove or disprove these suggestions is a matter of much importance to meteorologists, for increased attention in recent years to nephoscope



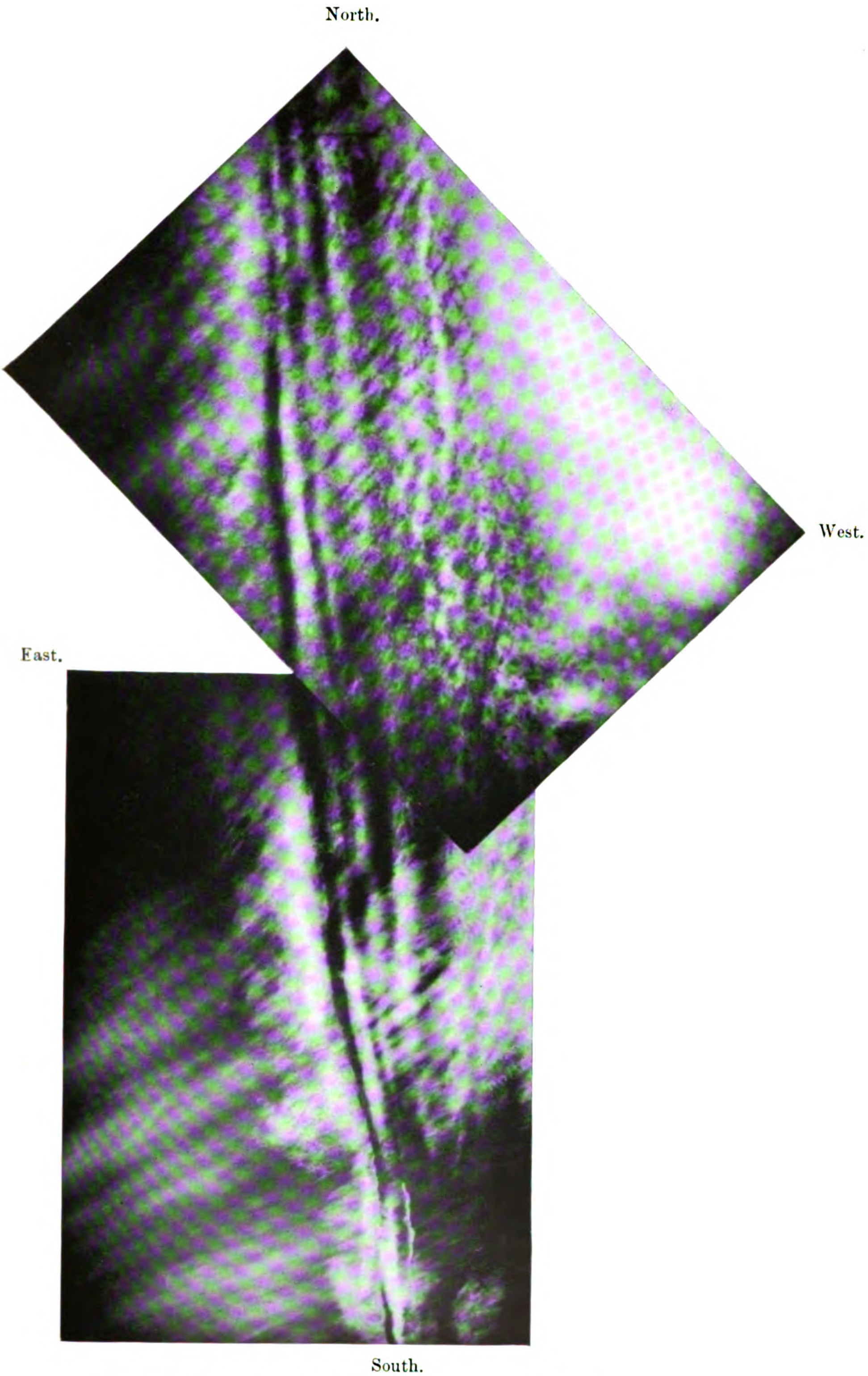
15 hours.

51 hours 5 minutes.



15 hours 8 minutes.

Cloud at Aberdeen, 20th June 1922.



Cloud at Aberdeen on 20th June 1922, at 15 hours 15 minutes.

observations has revealed numerous cases of apparently very high velocities in the cirrus level, and if many of these cases are merely the effects of

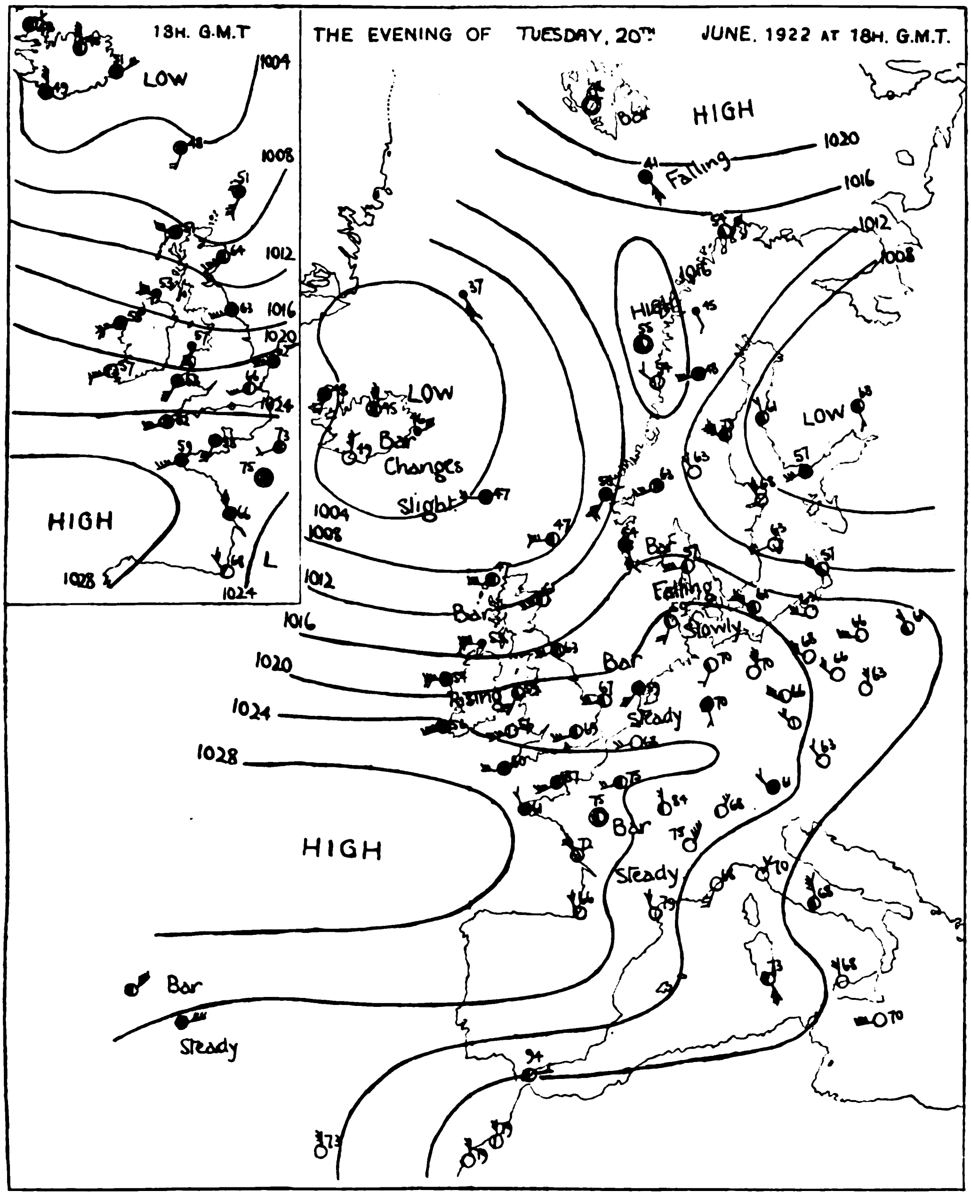


Fig. 1.

travelling waves and not of actual drift of air, the significance of the observations is entirely altered. That the cloud-forms in the particular case under discussion should be regarded as actually the result of wave-motion, is emphasised by Mr Clarke's observation that "the most striking features were the very clear-cut lanes of blue sky within the belt."

4. There are other ways in which the wave-motion can be broken up and through which greater developments may conceivably result. The first is in the case of a discontinuity at a small height, by the amplitude becoming too great in relation to the height of the bounding surface above the ground. The ratio of the amplitude to the height is given, in terms of the phenomena recorded at ground-level, by

$$\frac{a}{h}=\frac{\rho\delta U^2}{\delta p}.$$

If this is to approximate to unity, we may expect to find the anemogram indicating considerable squalls and the microbarogram indicating pressure waves of relatively small amplitude. Thus at Lerwick on 20th January 1924 there were squalls which had a period of 1 hour, and for which

$$2\delta U = 11 \text{ f.s.}$$

and

$$2\delta p = 0{\cdot}1 \text{ mb.},$$

giving

$$\frac{a}{h}=\frac{3}{4}.$$

It was estimated that the wave-length was of the order of 16 kilometres. Cases of squalls, which appear to be almost certainly wave effects, can be noted with periods up to about 2 hours, though as the period lengthens it becomes more and more difficult to detect the pressure wave on the barogram or microbarogram. Also in the case of these long periods an appreciable oscillation in surface-wind direction can be seen. This brings us to the next point, the effects of rotation of the earth when waves are set up. The first conceivable effect is that arising by reason of the transverse pressure gradient superposed (by the wave of pressure) on the previously existing gradient. Take the case where a steady motion of the surface layer of air exists originally under the influence of a pressure gradient of amount $\frac{\delta p}{\delta y}$ directed along the y-axis, and denote $\frac{1}{\rho}\frac{\delta p}{\delta y}$ by Γ. Take the frictional resistance to motion as proportional to the velocity—say μv—and assume that pressure waves perpendicular to the x-axis are set up so that the gradient in this direction is given by $\rho C \sin nt$. The equations of motion then become

$$\ddot{x} - \Omega\dot{y} = -\mu\dot{x} + C\sin nt$$

$$\ddot{y} + \Omega\dot{x} = \Gamma - \mu\dot{y}$$

where Ω is written for $2\omega \sin \phi$, ω being the angular velocity of rotation of the earth and ϕ the latitude of the place. Multiplying the second equation by $i(\sqrt{-1})$ and adding, we have, if v be written for $\dot{x}+i\dot{y}$,

$$\dot{v}+(\mu+i\Omega)v = C \sin nt + i\Gamma,$$

of which the solution is

$$v=(A+iB)e^{-(\mu+i\Omega)t}+\frac{i\Gamma}{\mu+i\Omega}+\frac{C\sin(nt-\beta)}{\sqrt{n^2+(\mu+i\Omega)^2}}$$

where $\tan\beta=\frac{n}{\mu+i\Omega}$.

Now, if t be measured from the time when v passes through its mean (gradient) value, we have

$$v=\frac{i\Gamma}{\mu+i\Omega}, \qquad \text{when } t=0.$$

So that the equation for v may now be written,

$$v=\frac{i\Gamma}{\mu+i\Omega}\{1-e^{-(\mu+i\Omega)t}\}+\frac{C}{\sqrt{n^2+(\mu+i\Omega)^2}}\{\sin(nt-\beta)+\sin\beta e^{-(\mu+i\Omega)t}\}.$$

As t increases the first term tends to the steady mean value of v, whilst the second part of the second term tends to vanish. The important part for our present purpose is the wind oscillation expressed by

$$\frac{C\sin(nt-\beta)}{\sqrt{n^2+(\mu+i\Omega)^2}}.$$

This is equivalent to

$$\frac{C\{\mu\sin nt-n\cos nt+i\Omega\sin nt\}\{k-2i\mu\Omega\}}{k^2+4\mu^2\Omega^2}$$

where $k=n^2+\mu^2-\Omega^2$.

Now the modulus of this expression, which is the maximum value the oscillation can give, is

$$C\sqrt{\frac{(\mu\sin nt-n\cos nt)^2+\Omega^2\sin^2 nt}{(n^2+\mu^2-\Omega^2)^2+4\mu^2\Omega^2}}.$$

Put $\mu=\cdot 4\Omega$, which is an approximate empirical value for the ground friction, and $n=m\Omega$.

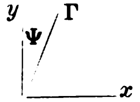
The expression then becomes

$$\frac{C}{\Omega}\sqrt{\frac{(\cdot 4\sin nt-m\cos nt)^2+\sin^2 nt}{(m^2-\cdot 84)^2+\cdot 64}}.$$

The maximum value of this is roughly $\frac{C}{m\Omega}$, *i.e.* $\frac{1}{m}$ of stationary gradient value for a gradient C which is itself very small. Now $\frac{2\pi}{\omega}=24$ hours, and $\frac{2\pi}{n}$ at most may reach 2 hours. Hence for latitude 60°, m is of the order of 7 or more. Thus the vector change of wind introduced by the transverse pressure gradient set up by the waves is quite negligible.

The next point to examine is the possible variation in $\dot{y}$ introduced by the waves in the component parallel to the x-axis. Let the wind velocity parallel to the x-axis become

$$\dot{x} = \frac{\Gamma}{\sqrt{\mu^2+\Omega^2}}(1 + C' \sin nt)$$



the mean value $\frac{\Gamma}{\sqrt{\mu^2+\Omega^2}}$ having been produced by a gradient as shown in the diagram, where $\cos \Psi = \frac{\Omega}{\sqrt{\mu^2+\Omega^2}}$.

Then we have, as the equation of motion parallel to the y-axis,

$$\ddot{y} + \Omega\dot{x} = \frac{\Gamma\Omega}{\sqrt{\mu^2+\Omega^2}} - \mu\dot{y},$$

i.e.

$$\ddot{y} + \mu\dot{y} = -\frac{C'\Gamma\Omega}{\sqrt{\mu^2+\Omega^2}}\sin nt.$$

$$\therefore \ \dot{y} = Ae^{-\mu t} - \frac{C'\Gamma\Omega}{\sqrt{(\mu^2+\Omega^2)(n^2+\mu^2)}} \cdot \sin(nt - \beta')$$

where $\tan \beta' = \frac{n}{\mu}$.

Determining the constants by the condition that when $t = 0$, $\dot{y} = 0$, the equation becomes

$$\dot{y} = -\frac{C'\Gamma\Omega}{\sqrt{(\mu^2+\Omega^2)(n^2+\mu^2)}}\{\sin(nt - \beta') + \sin\beta' e^{-\mu t}\}.$$

The oscillatory part of this after a time is a maximum in the direction of the y-axis, when

$$\sin(nt - \beta') = -1,$$

i.e.

$$nt = 270° + \beta'.$$

Now $\tan \beta' = \frac{n}{\mu}$. An average value for μ is $\cdot 4\Omega$, whilst for a period equal to 2 hours or less the value of n would be seven or more times Ω.

$$\therefore \ \beta' = 87° \text{ to } 90°,$$

i.e. there is a maximum backing practically at the times when $nt = 360°$, $0°$, etc., *i.e.* at the moment when $\dot{x}$ passes through its mean value. In this case

$$\dot{y} = \frac{C'\Gamma\Omega}{\sqrt{(\mu^2+\Omega^2)(n^2+\mu^2)}} = \frac{C'U}{m} \text{ approximately,}$$

where m is put for $n\Omega$, *i.e.* $\dot{y}$ is of the order of $\frac{\delta U}{m}$.

∴ Angle of maximum backing is given by $\tan\phi = \frac{\delta U}{mU}$.

This may be appreciable, for δU is frequently of the order of U with waves of 2-hour period. On the other hand, for waves of only ¼-hour period we should have this angle $= \frac{\frac{1}{16}\delta U}{U}$, which would not show up on an anemogram. For example, at Lerwick on 31st January 1924, a cold westerly wind commenced to undercut a warmer south-westerly current at about 17 hours G.M.T., resulting in the next few hours in one or two enormous wind-waves of period equal to 1½ hours (see fig. 2). From the records of the anemograph we obtain the following data:—

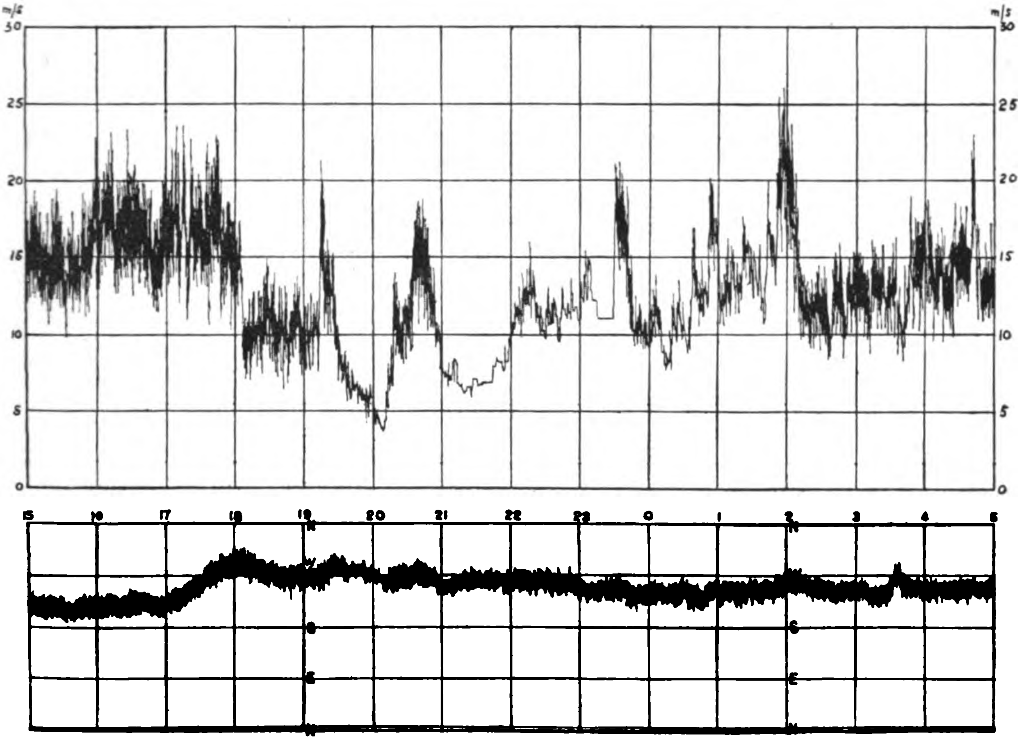


Fig. 2.

$$U = 26 \text{ m.p.h.}$$
$$2\delta U = 21 \text{ m.p.h.}$$
$$\text{Variation in direction} = 40°.$$

By the above theory, the variation in direction for the above period, wind velocities, and variation of velocity, should be through an angle (on each side of the mean) given by

$$\tan\phi = \tfrac{4}{9}, \quad i.e.\ \phi = 24°,$$

giving a total range of 48°, which is rather more than is indicated by the actual wind record.

Again at Lerwick on 30th January 1924, at about 3 a.m. to 5 a.m. (see fig. 3), there were waves of period 1 hour, for which the data were

$$U = 34 \text{ m.p.h.}; \quad 2\delta U = 12 \text{ m.p.h.}$$

By calculation,

$$\tan\phi = \cdot 012, \quad \therefore \ \phi = 0^{\circ}\cdot 7,$$

i.e. almost imperceptible, as was in fact actually the case.

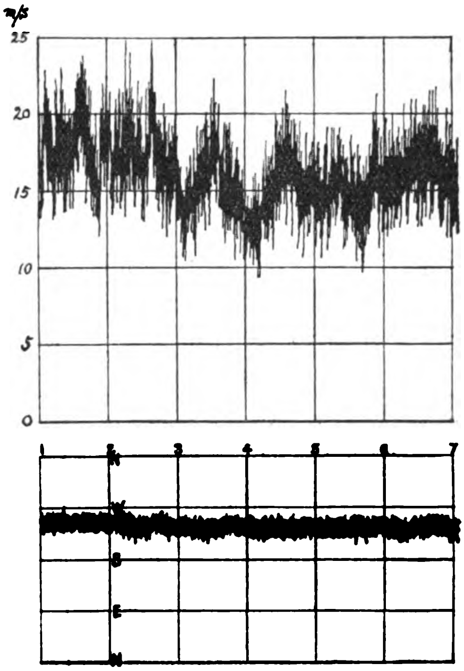


FIG. 3.

It may be added that the record in the former of these two cases (fig. 2) is typical of the case where the amplitude of the wave in the bounding surface is practically as great as the mean height of the bounding surface above ground-level. In the second case the amplitude is probably about one-third of the height of the discontinuity above ground-level.

It becomes evident, then, that where and when the two following conditions are satisfied, the wave movements, which with short periods can be confined to a vertical plane and remain of small magnitude, may be transformed as a result of the rotation of the earth into circulations with transverse horizontal components of such magnitude as to suggest that the next stage of the development must be towards a complete horizontal cyclonic circulation. The two conditions are:—

(1) The surface of discontinuity must be sufficiently close to ground level, *i.e.* to give a high value of δU; and

(2) the period must be lengthened, an effect which may result as mixing proceeds and the width of the mixing zone increases.

Further detailed investigation of the subject of discontinuities in their relation to the development of the cyclones of temperature latitudes is in hand. According to the theory of Professor V. Bjerknes and the "Bergen School," these cyclones originate as waves in the sloping boundary surface between air of polar and air of equatorial origin, and in the past few years much work has been done on the subject without, however, the details, and in particular the initial stages and immediate causes of the development of the cyclones, becoming clear. It is for this reason that in the above paper and in the previous one every aspect, empirical and mathematical, of the situation in regard to actually occurring discontinuities has been examined with a view to determining as far as may be possible the precise physical processes which occur at such discontinuities.

TABLE I.—MEAN TEMPERATURES (IN °F.) AT SURFACE-LEVEL AND AT 1000 FEET, AND THEREAFTER AT VARIOUS PRESSURES IN AIR OF "EQUATORIAL" ORIGIN.

Deduced from Observations made during Aeroplane Ascents in the British Isles.

Month.	Sur-face.	1000 ft.	Pressure in Millibars.									
			950.	900.	850.	800.	750.	700.	650.	600.	550.	500.
1921 Sept. .	61	59	57	53	50	47	42	37	31	24	19	...
Oct. .	59	59	58	54	51	47	42	36	29	23	15	...
Nov. .	50	49	48	46	42	36	32	27	23	19	14	...
Dec. .	51	49	47	45	43	40	36	30	24	16	9	1
1922 Jan. .	(49	47	46	39	33	30)	24	18	11	5	...	...
Feb. .	...	47	46	44	42	37	33	28	21	13	5	...
Mar. .	50	45	44	41	38	34	28	22	17	11	4	...
Apr. .	...	...	...	...	...	...	...	...	...	...	...	...
May .	66	63	60	55	50	45	40	34	27	20	13	5
June .	65	63	61	57	50	45	39	34	28	21	14	...
July .	58·5	55	54	50	48	46	41	36·5	27	24	20	...
Aug. .	(58	55	47	46	47)	44	40	34	31	27	21	...
Sept. .	59	56	52	49	45	42	40	38	...	...	...	...
Oct. .	58	54	51	47	43·5	41	35·5	30	24·5	18	11	3
Nov. .	49	47	45	41	38	34	31	26	21	14	9	...
Dec. .	46	45	43	40	38	35	29	26	23	...	...	...

TABLE II.—MEAN TEMPERATURES (IN °F.) AT SURFACE-LEVEL AND AT 1000 FEET, AND THEREAFTER AT VARIOUS PRESSURES IN AIR OF "POLAR" ORIGIN.

Deduced from Observations made during Aeroplane Ascents in the British Isles.

Month.	Surface.	1000 ft.	Pressure in Millibars.									
			950.	900.	850.	800.	750.	700.	650.	600.	550.	500.
1921 Sept. .	54	51	48	42	38	36	32	27	22	16	11	...
Oct. .	53	50	48	43	39	35	32	27	21	14	...	...
Nov. .	40	39	37	32	28	23	19	15	10	4	− 2	− 8
Dec. .	42	39	37	33	27	24	19	14	8	1	− 9	...
1922 Jan. .	40	37	35	31	25	20	15	8	1	− 6	−11	−21
Feb. .	43	40	38	33	27	23	18	11	2	− 7	...	...
Mar. .	42	38	35	29	23	19	15	9	4	− 4	...	...
Apr. .	50	41	39	34	30	23	18	14	6	− 1	−10	...
May .	47	38	37	31	26	24	20	16	9	4	− 3	...
June .	48	48	45	38	33	29	24	20	15	8	...	...
July .	57	53	51	45	39	33	28·5	25·5	20	14	...	...
Aug. .	55	52	48	44	39	34	30·5	26·5	21·5	13·5	6	− 6
Sept. .	55	52	49	45	39	33	28	24	18	12	8	...
Oct. .	48	44	39	33·5	28	24	21	17	11·5	5·5	0·5	− 4·5
Nov. .	39	38	34	30	24	19	14	9	4	...	...	...
Dec. .	39	37	36	31	27	24	18	13	...	...	...	...

TABLE III.—MEAN DIFFERENCE OF TEMPERATURE BETWEEN POLAR AND EQUATORIAL AIR AT SURFACE-LEVEL AND AT 1000 FEET, AND THEREAFTER AT VARIOUS PRESSURES.

Month.	Surface.	1000 ft.	Pressure in Millibars.										
			950.	900.	850.	800.	750.	700.	650.	600.	550.	500.	Mean from 600 to 950.
1921 Sept.	7	8	8	11	12	11	11	10	9	9	8	...	10
Oct.	6	7	10	11	12	11	10	9	9	8	...	...	10
Nov.	9	10	12	14	14	13	12	12	13	15	16	...	13
Dec.	10	10	11	13	16	16	17	16	15	15	18	...	15
1922 Jan.	(9	10	11	8	8	10)	9	10	10	11	...	...	10
Feb.	...	7	8	11	15	14	15	17	19	20	...	...	15
Mar.	8	8	11	14	16	16	14	13	13	15	...	...	14
Apr.	...	...	...	...	...	...	...	...	...	...	...	...	...
May	19	25	24	25	24	22	21	19	19	16	16	...	21
June	17	15	16	19	17	16	15	15	13	6	6	...	15
July	2	2	3	5	9	13	13	10	6	10	12	...	9
Aug.	(3	3	−1	2	8)	10	10	8	10	14	15	...	8
Sept.	4	4	3	4	6	9	12	14	...	...	...	...	8
Oct.	10	10	12	14	16	17	15	13	13	13	11	7	14
Nov.	10	9	11	11	14	15	17	17	17	...	...	...	15
Dec.	7	8	7	9	11	15	11	11	13	...	...	...	11

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(13) *Geophysical Memoirs*, No. 5.

(14) *Phil. Trans.*, Series A, **221**, pp. 73–115.

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XVIII.—The Equation of Conduction of Heat. By **Marion C. Gray**, M.A., J. E. Baxter Scholar of the University of Edinburgh, and Research Student of Bryn Mawr College, Pa. *Communicated by* Professor E. T. WHITTAKER, F.R.S.

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§ 1. INTRODUCTION.

THE differential equation of the conduction of heat in ordinary three-dimensional space is generally written in the form

$$\frac{\partial v}{\partial t}=\kappa\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right),$$

where v denotes the temperature of the medium at time t. For a medium in which the temperature varies only in one direction, *e.g.* an infinite cylinder with the temperature varying along the axis, the equation is

$$\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2}.$$

By taking κt as a new variable in place of t this may be written

$$\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{I}$$

Similarly, when the temperature depends on two co-ordinates, *e.g.* an infinite cylinder with all points on any line parallel to the axis at the same temperature, the equation will be

$$\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{II}$$

And lastly, the three-dimensional equation can be written

$$\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{III}$$

The purpose of this paper is to show how the various *general* solutions of these three equations can be derived from one another. For the present I do not consider the particular solutions for bodies of special forms, or for bodies with given boundary conditions.

§ 2. GENERAL SOLUTION OF EQUATION I.

The equation I has evidently an elementary solution of the form

$$e^{ix\alpha-t\alpha^2},$$

where a is an arbitrary constant. By building up from this solution we may write our general solution as

$$v=\int_{-\infty}^{\infty} e^{ixa-ta^2}f(a)da \quad . \quad . \quad . \quad . \quad . \quad 1.1$$

§ 3. Forsyth's Solution.

Forsyth * has shown how to derive a theoretical solution, which is

$$v=\int_0^{2\pi} e^{x\cos u+t\cos 2u}\cos(u+x\sin u+t\sin 2u)\mathrm{G}(u)du \quad . \quad . \quad 1.2$$

And this is easily shown to be equivalent to 1.1, for it may be written

$$v=\tfrac{1}{2}\int_0^{2\pi} e^{x\cos u+t\cos 2u}\{e^{i(u+x\sin u+t\sin 2u)}+e^{-i(u+x\sin u+t\sin 2u)}\}\mathrm{G}(u)du$$

$$=\tfrac{1}{2}\int_0^{2\pi} e^{x(\cos u+i\sin u)+t(\cos 2u+i\sin 2u)}e^{iu}\mathrm{G}(u)du$$

$$+\tfrac{1}{2}\int_0^{2\pi} e^{x(\cos u-i\sin u)+t(\cos 2u-i\sin 2u)}e^{-iu}\mathrm{G}(u)du.$$

In the first integral put $e^{iu}=ia$, and in the second $e^{-iu}=ia$.
Then

$$v=\tfrac{1}{2}\int_c e^{ixa-ta^2}\mathrm{G}_1(a)da+\tfrac{1}{2}\int_c e^{ixa-ta^2}\mathrm{G}_2(a)da,$$

where the contour c is a circle of unit radius surrounding the origin. It is easily seen that the contour can be deformed to become two straight lines from $-\infty$ to $+\infty$, in which case the solution may be written

$$v=\int_{-\infty}^{\infty} e^{ixa-ta^2}f(a)da.$$

Thus Forsyth's solution is equivalent to the solution 1.1.

§ 4. Fourier's Solutions.

The other general solutions we shall consider have been given by Fourier.† Of these one of the best known is

$$v=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}\phi(x')e^{-\frac{(x-x')^2}{4t}}dx', \quad . \quad . \quad . \quad . \quad . \quad 1.3$$

and we shall now show that this is equivalent to our general solution. First make the change of variable $x'-x=2\sqrt{t}u$.
Then v becomes

$$v=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\phi(x+2\sqrt{t}u)e^{-u^2}du \quad . \quad . \quad . \quad . \quad 1.4$$

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* Forsyth, *Theory of Differential Equations*, part iv, vol. vi, pp. 107–109.

† Fourier, *Théorie Analytique de la Chaleur.*

Now if $\phi(x)$ is analytic near the point x, $\phi(x+2\sqrt{tu})$ can be expanded in the form of a Taylor's series in the neighbourhood of x. On this assumption, therefore, we may write

$$v=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\left\{\sum\frac{(2\sqrt{tu})^n}{n!}\phi^n(x)\right\}e^{-u^2}du.$$

Now it is well known that

$$\int_{-\infty}^{\infty}e^{-u^2}du=\sqrt{\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 1.41$$

$$\left.\begin{aligned}\int_{-\infty}^{\infty}u^n e^{-u^2}du&=\frac{2m!}{2^{2m}m!}\sqrt{\pi}, \text{ when } n \text{ is even}=2m\\&=0 \text{ when } n \text{ is odd}\end{aligned}\right\} \quad . \quad . \quad 1.42$$

Hence, since the integral is uniformly convergent, we may integrate term by term, giving

$$v=\frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{t^m 2^{2m}}{2m!}\,\frac{2m!}{2^{2m}m!}\sqrt{\pi}\phi^{2m}(x)$$

$$=\sum_{n=0}^{\infty}\frac{t^m}{m!}\phi^{2m}(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 1.5$$

But if we expand the solution 1.1 as a power series in t, we obtain the same value for v. For consider

$$v=\int_{-\infty}^{\infty}e^{ix\alpha-t\alpha^2}f(\alpha)d\alpha$$

$$=\int_{-\infty}^{\infty}e^{ix\alpha}\left\{1-t\alpha^2+\frac{t^2\alpha^4}{2!}\ .\ .\ .+(-1)^n\frac{t^n\alpha^{2n}}{n!}+\ .\ .\ .\right\}f(\alpha)d\alpha \quad . \quad 1.51$$

When

$$t=0,\ v=\phi(x).$$

$$\therefore\ \phi(x)=\int_{-\infty}^{\infty}e^{ix\alpha}f(\alpha)d\alpha.$$

And if we may differentiate under the integral sign

$$\phi''(x)=\int_{-\infty}^{\infty}(-\alpha^2)e^{ix\alpha}f(\alpha)d\alpha,$$

which is the coefficient of t in the expansion above.

Similarly

$$\phi^{2n}(x)=\int_{-\infty}^{\infty}(i\alpha)^{2n}e^{ix\alpha}f(\alpha)d\alpha,$$

which is the coefficient of $\frac{t^n}{n!}$.

$$\therefore\quad v=\phi(x)+t\phi''(x)+\frac{t^2}{2!}\phi^{\text{iv}}(x)+\ .\ .\ .\ +\frac{t^n}{n!}\phi^{2n}(x)+\ .\ .\ .$$

$$=\sum\frac{t^m}{m!}\phi^{2m}(x).$$

Thus we have shown that the solutions 1.3, 1.4, and 1.5 are all equivalent forms of the solution 1.1.

§ 5. SERIES SOLUTION.

When we consider a series solution proceeding in ascending powers of x we must have the given values

$$(v)_{x=0}=g(t)\,;\ \left(\frac{\partial v}{\partial x}\right)_{x=0}=h(t).$$

Then we find easily

$$\begin{aligned} v&=g(t)+\frac{x^2}{2\,!}g'(t)+\frac{x^4}{4\,!}g''(t)+\ .\ .\ . \\ &\quad+xh(t)+\frac{x^3}{3\,!}h'(t)+\ .\ .\ . \\ &=\sum_{n=0}^{\infty}\left\{\frac{x^{2n}}{2n\,!}g^{n}(t)+\frac{x^{2n+1}}{(2n+1)!}h^{n}(t)\right\}\quad .\quad .\quad . \end{aligned} \qquad 1.6$$

This solution can also be found from 1.1.

For

$$v=\int_{-\infty}^{\infty}e^{-ta^2}\left\{1+ixa+\frac{(ixa)^2}{2\,!}+\ .\ .\ .\right\}daf(a).$$

$\therefore$ when $x=0$

$$g\,(t)=\int_{-\infty}^{\infty}e^{-ta^2}f(a)da.$$

$$h(t)=\int_{-\infty}^{\infty}iae^{-ta^2}f(a)da.$$

Then

$$g'(t)=\int_{-\infty}^{\infty}(-a^2)e^{-ta^2}f(a)da,$$

$$h'(t)=\int_{-\infty}^{\infty}ia(-a^2)e^{-ta^2}f(a)da,$$

and we are thus evidently led to the same series 1.6.

§ 6. RELATION BETWEEN THE TWO SERIES SOLUTIONS.

The two solutions 1.5 and 1.6 can be shown to be consistent by a further expansion in series. Thus from 1.6

$$\begin{aligned} v&=g(o)+tg'(o)+\frac{t^2}{2\,!}g''(o)+\ .\ .\ . \\ &\quad+x\left\{h(o)+th'(o)+\frac{t^2}{2\,!}h''(o)+\ .\ .\ .\ \right\} \\ &\quad+\frac{x^2}{2\,!}\left\{g'(o)+tg''(o)+\frac{t^2}{2\,!}g'''(o)+\ .\ .\ .\right\} \end{aligned}$$

$$+\frac{x^3}{3!}\left\{h'(o)+th''(o)+\frac{t^2}{2!}h'''(o)+\ .\ .\ .\right\}$$

$$+\ .\ .\ .$$

$$=g(o)+xh(o)+\frac{x^2}{2!}g'(o)+\frac{x^3}{3!}h'(o)+\ .\ .\ .$$

$$+t\left\{g'(o)+xh'(o)+\frac{x^2}{2!}g''(o)+\frac{x^3}{3!}h''(o)+\ .\ .\ .\right\}$$

$$+\frac{t^2}{2!}\left\{g''(o)+xh''(o)+\frac{x^2}{2!}g'''(o)+\ .\ .\ .\right\}+\ .\ .\ .$$

And if we now write

$$\phi(x)=g(o)+xh(o)+\frac{x^2}{2!}g'(o)+\ .\ .\ .,$$

this becomes

$$v=\phi(x)+t\phi''(x)+\frac{t^2}{2!}\phi^{\text{iv}}(x)+\ .\ .\ .,$$

which is precisely the solution 1.5.

In the same way, if we expand 1.5 as a power series in x, and if we write

$$g(t)=\phi(o)+t\phi''(o)+\frac{t^2}{2!}\phi^{\text{iv}}(o)+\ .\ .\ .$$

$$h(t)=\phi'(o)+t\phi'''(o)+\frac{t^2}{2!}\phi^{\text{v}}(o)+\ .\ .\ .$$

the solution becomes 1.6.

§ 7. INTEGRAL EQUATION SOLUTION.

The general solution 1.1 may be used to find the solution satisfying an arbitrary initial distribution $v=\phi(x)$.

For when $t=0$ we have

$$\phi(x)=\int_{-\infty}^{\infty}e^{ixa}f(a)da.$$

And the solution of this integral equation is well known,* namely,

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ixu}\phi(u)du.$$

Our general solution becomes therefore

$$v=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ixa-ta^2}\left[\int_{-\infty}^{\infty}e^{-iau}\phi(u)du\right]da$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}da\int_{-\infty}^{\infty}e^{-ta^2}\cos\left\{(x-u)a\right\}\phi(u)du\quad .\quad .\quad .\quad 1.7$$

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* Poincaré, *Théorie Analytique de la Chaleur*, chap. vii.

If we may change the order of integration, we can effect the integration with respect to a, since

$$\int_{-\infty}^{\infty} e^{-a^2x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}}{a}e^{-b^2/a^2} \quad . \quad . \quad . \quad . \quad 1.71$$

$$\therefore\ v = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sqrt{\pi}}{\sqrt{t}}e^{-\frac{(x-u)^2}{4t}}\phi(u)du$$

$$= \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-u)^2}{4t}}\phi(u)du,$$

and the solution 1.7 is simply a different form of the solution 1.3.

§ 8. SPECIAL EXAMPLE.

The particular case in which the initial disturbance is expressed as a Fourier Series is of importance. We have then, that when $t=0$,

$$v = \phi(x) = \sum_{n=0}^{\infty}(a_n\cos nx + b_n\sin nx).$$

Then, starting from the solution 1.4,

$$v = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}\phi(x+2\sqrt{t}u)du$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}\left\{\sum_{n=0}^{\infty}a_n\cos n(x+2\sqrt{t}u) + b_n\sin n(x+2\sqrt{t}u)\right\}du.$$

The integral is uniformly convergent, so we may integrate term by term. The coefficient of a_n is

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}(\cos nx\cos 2n\sqrt{t}u - \sin nx\sin 2n\sqrt{t}u)du$$

$$= \frac{1}{\sqrt{\pi}}\cos nx(\sqrt{\pi}e^{-n^2t}) + \sin nx \times 0. \quad \text{From } 1.71.$$

$$= \cos nxe^{-n^2t}.$$

Similarly the coefficient of b_n is $\sin nxe^{-n^2t}$.

$\therefore$ Finally

$$v = \sum_{n=0}^{\infty}(a_n\cos nx + b_n\sin nx)e^{-n^2t}. \quad . \quad . \quad . \quad 1.8$$

This is the form in which the solution is usually given. We see therefore that all these solutions are equivalent forms of the solution 1.1; and, in fact, by means of 1.4 or 1.7 we can write down immediately the solution corresponding to any assigned initial values of v.

§ 9. General Solution for Equation of Two Dimensions.

We pass on now to consider the equation of the conduction of heat in two dimensions. This equation II may be treated in the same way as I. The solution corresponding to the general solution 1.1 will be

$$v=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{ix\alpha+iy\beta-t(\alpha^2+\beta^2)}f(\alpha,\ \beta)d\alpha d\beta\ . \qquad . \quad . \quad . \quad 2.1$$

And we again take this as our general solution.

By generalising 1.2 we have another solution,

$$v=\int_0^{2\pi}ds\int_0^{2\pi}du e^{x\cos s+t\cos 2s}e^{y\cos u+t\cos 2u}\cos(s+x\sin s+t\sin 2s)$$

$$\cos(u+y\sin u+t\sin 2u)G(s,\ u) \qquad . \quad . \quad 2.2$$

which it would be difficult to justify analytically, but by direct differentiation it is easily shown to satisfy the differential equation II.

As before, we may write

$$v=\int_0^{2\pi}\int_0^{2\pi}e^{x\cos s+t\cos 2s+y\cos u+t\cos 2u}e^{i(s+x\sin s+t\sin 2s)}e^{i(u+y\sin u+t\sin 2u)}G(s,\ u)dsdu$$

$$=\int_0^{2\pi}\int_0^{2\pi}e^{x(\cos s+i\sin s)+y(\cos u+i\sin u)+t(\cos 2s+i\sin 2s+\cos 2u+i\sin 2u)}G_1(s,\ u)dsdu.$$

And again introducing new variables,

$$i\alpha=e^{is},\ i\beta=e^{iu},$$

we have

$$v=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{ix\alpha+iy\beta-t(\alpha^2+\beta^2)}f(\alpha,\ \beta)d\alpha d\beta.$$

Thus these two solutions are equivalent.

§ 10. Fourier's Solutions.

The solution corresponding to 1.3 may be written in the form

$$v=\frac{1}{4\pi t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\frac{(x-x')^2+(y-y')^2}{4t}}\phi(x',\ y')dx'dy' \qquad . \quad . \quad . \quad 2.3$$

And by the change of variable

$$\left.\begin{aligned}x'-x&=2\sqrt{t}s\\y'-y&=2\sqrt{t}u\end{aligned}\right\}$$

this becomes

$$v=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-s^2-u^2}\phi(x+2\sqrt{t}s,\ y+2\sqrt{t}u)dsdu \quad . \quad . \quad . \quad 2.4$$

Expand this in a double Taylor series, so

$$v=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-s^2-u^2}\left\{\sum_{n=0}^{\infty}\frac{1}{n!}\left[2\sqrt{t}s\frac{\partial}{\partial x}+2\sqrt{t}u\frac{\partial}{\partial y}\right]^n\phi(x,\, y)\right\}dsdu$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}ds\int_{-\infty}^{\infty}e^{-s^2-u^2}du\left\{\sum_{n=0}^{\infty}\frac{(2\sqrt{t})^n}{n!}\left[s^nD^n+ns^{n-1}D^{n-1}uD'+\ldots\right.\right.$$

$$\left.\left.+\frac{n(n-1)\ldots(n-r+1)}{r!}(sD)^{n-r}(uD')^r+\ldots\right]\phi(x,\, y)\right\},$$

where $D=\frac{\partial}{\partial x}$, and $D'=\frac{\partial}{\partial y}$. Interchanging the signs of summation and integration this may be written

$$v=\sum_{n=0}^{\infty}\frac{(2\sqrt{t})^n}{\pi n!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-s^2-u^2}\left\{\sum_{r=0}^{n}\frac{n!}{(n-r)!\,r!}s^{n-r}u^r\frac{\partial^n\phi\lambda}{\partial x^{n-r}\partial y^r}\right\}dsdu.$$

If n is odd the double integral must vanish, since either r or $n-r$ will be odd. Take n even, therefore, and equal to $2m$, and r must also be even.

Then

$$v=\sum_{n=0}^{\infty}\frac{2^{2m}t^m}{\pi 2m!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-s^2-u^2}\left\{\sum_{r=0}^{m}\frac{2m!}{(2m-2r)!\,2r!}s^{2m-2r}u^{2r}\frac{\partial^{2m}\phi}{\partial x^{2m-2r}\partial y^{2r}}\right\}dsdu$$

$$=\sum_{m=0}^{\infty}\sum_{r=0}^{m}\frac{2^{2m}t^m}{\pi(2m-2r)!\,2r!}\frac{\partial^{2m}\phi}{\partial x^{2m-2r}\partial y^{2r}}\int_{-\infty}^{\infty}e^{-s^2}s^{2m-2r}ds\int_{-\infty}^{\infty}e^{-u^2}u^{2r}du$$

$$=\sum_{m=0}^{\infty}\sum_{r=0}^{m}\frac{2^{2m}t^m}{\pi(2m-2r)!\,2r!}\frac{\partial^{2m}\phi}{\partial x^{2m-2r}\partial y^{2r}}\frac{(2m-2r)!}{2^{2m-2r}(m-r)!}\sqrt{\pi}\cdot\frac{2r!}{2^{2r}r!}\sqrt{\pi}$$

$$=\sum_{m=0}^{\infty}\sum_{r=0}^{m}\frac{t^m}{(m-r)!\,r!}\frac{\partial^{2m}\phi}{\partial x^{2m-2r}\partial y^{2r}}\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad 2.51$$

The same series solution may be obtained from the solution 2.1. For if we write it

$$v=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{ix\alpha+iy\beta}\left\{1-t(\alpha^2+\beta^2)+\frac{t^2}{2!}(\alpha^2+\beta^2)^2+\ldots\right\}f(\alpha,\, \beta)d\alpha d\beta,$$

we see that

$$\phi(x,\, y)=(v)_{t=0}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{ix\alpha+iy\beta}f(\alpha,\, \beta)d\alpha d\beta.$$

Also

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}-(\alpha^2+\beta^2)e^{ix\alpha+iy\beta}f(\alpha,\, \beta)d\alpha d\beta,$$

which is the coefficient of t in the expansion. In general the coefficient of t^n is

$$\frac{1}{n!}(\mathrm{D}^2+\mathrm{D}'^2)^n\phi(x,\,y)=\frac{1}{n!}\sum\frac{n(n-1)\ .\ .\ .\ (n-r+1)}{r!}\mathrm{D}^{2n-2r}\mathrm{D}'^{2r}\phi$$

$$=\sum\frac{1}{(n-r)!\,r!}\,\frac{\partial^{2n}\phi}{\partial x^{2n-2r}\partial y^{2r}}\,.$$

$$\therefore\ v=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{t^n}{(n-r)!\,r!}\,\frac{\partial^{2n}\phi}{\partial x^{2n-2r}\partial y^{2r}}\,.$$

Thus the two solutions are equivalent.

If we may break up the function $\phi(x,\,y)$ into the product of two functions, one a function of x only, and the other of y only,

$$\phi(x,\,y)=\phi_1(x)\phi_2(y),$$

the series solution 2.51 may be written alternatively in the form

$$v=\sum_{n=0}^{\infty}\frac{t^n}{n!}\phi_1^{\,2n}(x)\cdot\sum_{r=0}^{\infty}\frac{t^r}{r!}\phi_2^{\,2r}(y)\quad .\quad .\quad .\quad .\quad 2.52$$

i.e.

$$v=\left\{\phi_1(x)+t\phi_1''(x)+\frac{t^2}{2!}\phi_1^{\mathrm{iv}}(x)+\ .\ .\ .\right\}\left\{\phi_2(y)+t\phi_2''(y)+\frac{t^2}{2!}\phi_2^{\mathrm{iv}}(y)+\ .\ .\ .\right\}$$

$$=\phi_1(x)\phi_2(y)+t\{\phi_1''(x)\phi_2(y)+\phi_2''(y)\phi_1(x)\}+\frac{t^2}{2!}\{\phi_1(x)\phi_2^{\mathrm{iv}}(y)+2\phi_1''(x)\phi_2''(y)$$
$$+\phi_2(y)\phi_1^{\mathrm{iv}}(x)\}+\ .\ .\ .$$

$$=\phi(x,\,y)+t(\mathrm{D}^2+\mathrm{D}'^2)\phi(x,\,y)+\frac{t^2}{2!}(\mathrm{D}^2+\mathrm{D}'^2)^2\phi(x,\,y)+\ .\ .\ .$$

$$=\sum_{n=0}^{\infty}\frac{t^n}{n!}(\mathrm{D}^2+\mathrm{D}'^2)^n\phi(x,\,y).$$

$\therefore$ The two forms 2.51 and 2.52 are identical.

§ 11. SERIES SOLUTION.

The series solution corresponding to 1.6 will be

$$v=\left\{\sum_{n=0}^{\infty}\frac{x^{2n}}{2n!}f^n(t)+\frac{x^{2n+1}}{(2n+1)!}g^n(t)\right\}\left\{\sum_{r=0}^{\infty}\frac{y^{2r}}{2r!}h^r(t)+\frac{y^{2r+1}}{(2r+1)!}k^r(t)\right\}\quad .\quad 2.6$$

or, writing in full,

$$v=\left\{f(t)+xg(t)+\frac{x^2}{2!}f'(t)+\frac{x^3}{3!}g'(t)+\ .\ .\ .\right\}\left\{h(t)+yk(t)+\frac{y^2}{2!}h'(t)+\frac{y^3}{3!}k'(t)+\ .\ .\ .\right\}$$

$$=\Bigg[f(o)+tf'(o)+\frac{t^2}{2!}f''(o)+\ .\ .\ .\ +x\left\{g(o)+tg'(o)+\frac{t^2}{2!}g''(o)+\ .\ .\ .\right\}$$
$$+\frac{x^2}{2!}\left\{g'(o)+tg''(o)+\frac{t^2}{2!}g'''(o)+\ .\ .\ .\right\}+\ .\ .\ .\Bigg]$$

$$\times\left[h(o)+th'(o)+\frac{t^2}{2!}h''(o)+\ .\ .\ .\ +y\left\{k(o)+tk'(o)+\frac{t^2}{2!}k''(o)+\ .\ .\ .\right\}\right.$$

$$\left.+\frac{y^2}{2!}\left\{h'(o)+th''(o)+\frac{t^2}{2!}h'''(o)+\ .\ .\ .\right\}+\ .\ .\ .\right]$$

$$=\left[f(o)+xg(o)+\frac{x^2}{2!}f'(o)+\ .\ .\ .\ +t\left\{f'(o)+xg'(o)+\frac{x^2}{2!}f''(o)+\ .\ .\ .\right.\right.$$

$$\left.\left.+\frac{t^2}{2!}\left\{f''(o)+xg''(o)+\ .\ .\ .\right\}+\ .\ .\ .\right]$$

$$\times\left[h(o)+yk(o)+\frac{y^2}{2!}h'(o)+\ .\ .\ .\ +t\left\{h'(o)+yk'(o)+\frac{y^2}{2!}h''(o)+\ .\ .\ .\right\}+\ .\ .\ .\right]$$

$$=\left[F(x)+tF''(x)+\frac{t^2}{2!}F^{iv}(x)+\ .\ .\ .\right]\left[H(y)+tH''(y)+\frac{t^2}{2!}H^{iv}(y)+\ .\ .\ .\right]$$

$$=\sum_{n=0}^{\infty}\frac{t^n}{n!}F^{2n}(x)\times\sum_{r=0}^{\infty}\frac{t^r}{r!}H^{2r}(y),$$

where

$$\left.\begin{aligned}F(x)&=f(o)+xg(o)+\frac{x^2}{2!}f'(o)+\frac{x^3}{3!}g'(o)+\ .\ .\ .\\H(y)&=h(o)+yg(o)+\frac{y^2}{2!}h'(o)+\frac{y^3}{3!}g'(o)+\ .\ .\ .\end{aligned}\right\}$$

And again we see that the series solutions in powers of t, and in powers of x and y, are consistent, provided the arbitrary functions appearing in the series are consistent.

§ 12. Integral Equation Solution.

The general solution 2.1 again leads to an integral equation method of solving the heat equation with known initial conditions. For from 2.1, when $t=0$, we have

$$(v)_{t=0}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{ix\alpha+iy\beta}f(\alpha,\ \beta)d\alpha d\beta.$$

Now suppose the arbitrary function $f(\alpha,\ \beta)$ may be written $f_1(\alpha)f_2(\beta)$, and further that the initial value of $v=\phi(x,\ y)$ may be written $\phi_1(x)\phi_2(y)$.

Then

$$\phi_1(x)\phi_2(y)=\int_{-\infty}^{+\infty}e^{ix\alpha}f_1(\alpha)d\alpha\int_{-\infty}^{\infty}e^{iy\beta}f_2(\beta)d\beta.$$

$$\therefore\ \phi_1(x)=\int_{-\infty}^{\infty}e^{ix\alpha}f_1(\alpha)d\alpha,$$

$$\phi_2(y)=\int_{-\infty}^{\infty}e^{iy\beta}f_2(\beta)d\beta.$$

And the solutions of these integral equations are

$$f_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixs}\phi_1(s)ds,$$

$$f_2(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iyu}\phi_2(u)du.$$

∴ Our general solution may be written

$$v = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} da e^{ix\alpha - t\alpha^2}\int_{-\infty}^{\infty} e^{iy\beta - t\beta^2}d\beta\int_{-\infty}^{\infty} e^{-i\alpha s}\phi_1(s)ds\int_{-\infty}^{\infty} e^{-i\beta u}\phi_2(u)du$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} e^{-t\alpha^2}d\alpha\int_{-\infty}^{\infty} e^{-t\beta^2}d\beta\int_0^{\infty} \cos\alpha(x-s)\phi_1(s)ds\int_0^{\infty} \cos\beta(y-u)\phi_2(u)du \quad . \; . \; . \quad 2.7$$

Again, if we may change the order of integration, this may be reduced to a double integral form, for

$$v = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\phi_1(s)ds\int_{-\infty}^{\infty}\phi_2(u)du\int_{-\infty}^{\infty} e^{-t\alpha^2}\cos(x-s)\alpha d\alpha\int_{-\infty}^{\infty} e^{-t\beta^2}\cos(y-u)\beta d\beta$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\phi_1(s)ds\int_{-\infty}^{\infty}\phi_2(u)du \cdot \frac{\sqrt{\pi}}{\sqrt{t}}e^{-\frac{(x-s)^2}{4t}}\frac{\sqrt{\pi}}{\sqrt{t}}e^{-\frac{(y-u)^2}{4t}},$$

using 1.71. That is

$$v = \frac{1}{4\pi t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{(x-s)^2+(y-u)^2}{4t}}\phi(s, u)dsdu,$$

which is identical with the solution 2.3.

§ 13. GENERAL SOLUTION FOR THE EQUATION IN THREE DIMENSIONS.

Very similar results are obtained for the equation of conduction of heat in three dimensions. Our general solution will be

$$v = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ix\alpha+iy\beta+iz\gamma-t(\alpha^2+\beta^2+\gamma^2)}f(\alpha, \beta, \gamma)d\alpha d\beta d\gamma \quad . \; . \; . \quad . \quad . \quad 3.1$$

A further generalisation of Forsyth's solution enables us to write

$$v = \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} e^{ix\cos s+t\cos 2s}e^{iy\cos u+t\cos 2u}e^{iz\cos w+t\cos 2w}$$
$$\times \cos(s+x\sin s+t\sin 2s)\cos(u+y\sin u+t\sin 2u)\cos(w+z\sin w+t\sin 2w)$$
$$G(s, u, w)dsdudw,$$

and the proof of the equivalence of these two solutions is obvious.

§ 14. FOURIER'S SOLUTIONS.

The solution corresponding to the solution 1.3 will be

$$v=\frac{1}{8\pi^{3/2}t^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x', y', z')e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{t}}dx'dy'dz' \quad . \quad . \quad 3.3$$

or, with the usual change of variable,

$$x'-x=2\sqrt{t}a,$$
$$y'-y=2\sqrt{t}\beta,$$
$$z'-z=2\sqrt{t}\gamma,$$

this becomes

$$v=\pi^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x+2\sqrt{t}a,\ y+2\sqrt{t}\beta,\ z+2\sqrt{t}\gamma)e^{-a^2-\beta^2-\gamma^2}da\,d\beta\,d\gamma \quad . \quad 3.4$$

Expanding the function $\phi(x+2\sqrt{t}a,\ y+2\sqrt{t}\beta,\ z+2\sqrt{t}\gamma)$ as a Taylor's series in the neighbourhood of the point (x, y, z), we have

$$v=\pi^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-a^2-\beta^2-\gamma^2}\left[\sum\frac{(2\sqrt{t})^n}{n!}\left(a\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}+\gamma\frac{\partial}{\partial z}\right)^n\phi(x, y, z)\right]da\,d\beta\,d\gamma$$

$$=\pi^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\sum\frac{(2\sqrt{t})^n}{n!}\sum_{r=0}^{n}\sum_{s=0}^{r}\frac{n!}{(n-r)!\,(r-s)!\,s!}(aD_1)^{n-r}\right.$$
$$\left.(\beta D_2)^{r-s}(\gamma D_3)^s\phi\right]e^{-a^2-\beta^2-\gamma^2}da\,d\beta\,d\gamma,$$

where

$$D_1=\frac{\partial}{\partial x},\ D_2=\frac{\partial}{\partial y},\ D_3=\frac{\partial}{\partial z}.$$

Now if n is odd one of the numbers $n-r$, $r-s$, and s must be odd, and therefore one of the three repeated integrals involved in the triple integral must vanish. We retain therefore only those terms in the expansion for which n, r, and s are all even.

$$\therefore\ v=\pi^{-3/2}\sum_{m=0}^{\infty}\frac{2^{2m}t^m}{2m!}\sum_{r=0}^{m}\sum_{s=0}^{r}\frac{2m!}{(2m-2r)!\,(2r-2s)!\,2s!}\ \frac{\partial^{2m}\phi}{\partial x^{2m-2r}\partial y^{2r-2s}\partial z^{2s}}$$
$$\times\int_{-\infty}^{\infty}e^{-a^2}a^{2m-2r}da\int_{-\infty}^{\infty}e^{-\beta^2}\beta^{2r-2s}d\beta\int_{-\infty}^{\infty}e^{-\gamma^2}\gamma^{2s}d\gamma$$

i.e.

$$v=\pi^{-3/2}\sum_{m=0}^{\infty}\sum_{r=0}^{m}\sum_{s=0}^{r}\frac{2^{2m}t^m}{(2m-2r)!\,(2r-2s)!\,2s!}\cdot\frac{(2m-2r)!\,\sqrt{\pi}}{2^{2m-2r}(m-r)!}\cdot\frac{(2r-2s)!\,\sqrt{\pi}}{2^{2r-2s}(r-s)!}\cdot$$
$$\frac{2s!\,\sqrt{\pi}\partial^{2m}\phi}{2^{2s}s!\,\partial x^{2m-2r}\partial y^{2r-2s}\partial z^{2s}}$$

$$=\sum_{m=0}^{\infty}\sum_{r=0}^{m}\sum_{s=0}^{r}\frac{t^m}{(m-r)!\,(r-s)!\,s!}\frac{\partial^{2m}\phi}{\partial x^{2m-2r}\partial y^{2r-2s}\partial z^{2s}} \quad . \quad . \quad . \quad . \quad . \quad 3.51$$

To obtain this series from our general solution we write as before

$$v=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{ix\alpha+iy\beta+iz\gamma}\left\{1-t(\alpha^2+\beta^2+\gamma^2)+\frac{t^2}{2!}(\alpha^2+\beta^2+\gamma^2)^2\ .\ .\ .\right\}$$

$$f(\alpha,\beta,\gamma)d\alpha d\beta d\gamma.$$

Then

$$(v)_{t=0}=\phi(x,y,z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{ix\alpha+iy\beta+iz\gamma}f(\alpha,\beta,\gamma)d\alpha d\beta d\gamma,$$

and the coefficient of t^n in the expansion for v is

$$\frac{1}{n!}(\mathrm{D}_1^2+\mathrm{D}_2^2+\mathrm{D}_3^2)^n\phi(x,y,z)$$

$$=\frac{1}{n!}\sum_{r=0}^{n}\sum_{s=0}^{r}\frac{n!}{(n-r)!\,(r-s)!\,s!}\,\frac{\partial^{2n}\phi}{\partial x^{2n-2r}\partial y^{2r-2s}\partial z^{2s}}\ .$$

Thus we are led to the same expansion, 3.51.

§ 15. SERIES SOLUTION.

Another series solution might be written

$$v=\sum_{m=0}^{\infty}\frac{t^m}{m!}f_1^{2m}(x)\sum_{r=0}^{\infty}\frac{t^r}{r!}f_2^{2r}(y)\sum_{s=0}^{\infty}\frac{t^s}{s!}f_3^{2s}(z)\quad .\quad .\quad .\quad 3.52$$

where initially

$$v=f_1(x)f_2(y)f_3(z).$$

Again this is, on expansion,

$$v=\left\{f_1(x)+tf_1''(x)+\frac{t^2}{2!}f_1^{\mathrm{iv}}(x)+\ .\ .\ .\right\}\left\{f_2(y)+tf_2''(y)+\frac{t^2}{2!}f_2^{\mathrm{iv}}(y)+\ .\ .\ .\right\}$$

$$\{f_3(z)+tf_3''(z)+\ .\ .\ .\}$$

$$=f_1(x)f_2(y)f_3(z)+t\{f_1''f_2f_3+f_1f_2''f_3+f_1f_2f_3''\}+\ .\ .\ .$$

$$=\phi(x,y,z)+t(\mathrm{D}_1^2+\mathrm{D}_2^2+\mathrm{D}_3^2)\phi(x,y,z)+\ .\ .\ .\ +\frac{t^n}{n!}(\mathrm{D}_1^2+\mathrm{D}_2^2+\mathrm{D}_3^2)^n$$

$$\phi(x,y,z)+\ .\ .\ .$$

which is the same as 3.51.

The series solution corresponding to 1.6 will be

$$v=\sum_{n=0}^{\infty}\left\{\frac{x^{2n}}{2n!}f^n(t)+\frac{x^{2n+1}}{(2n+1)!}g^n(t)\right\}\cdot\sum_{r=0}^{\infty}\left\{\frac{y^{2r}}{2r!}h^r(t)+\frac{y^{2r+1}}{(2r+1)!}k^r(t)\right\}\cdot$$

$$\sum_{s=0}^{\infty}\left\{\frac{z^{2s}}{2s!}l^s(t)+\frac{z^{2s+1}}{(2s+1)!}m^s(t)\right\}\quad .\quad 3.6$$

where, for $x=y=z=0$, we have the boundary conditions,

$$v=f(t)h(t)l(t),$$
$$\frac{\partial v}{\partial x}=g(t)h(t)l(t),$$
$$\frac{\partial v}{\partial y}=f(t)k(t)l(t),$$
$$\frac{\partial v}{\partial z}=f(t)h(t)m(t).$$

Expand 3.6

$$\begin{aligned}
v &= \{f(t)+xg(t)+\ .\ .\ .\ \}\ \{h(t)+yk(t)+\ .\ .\ .\ \}\ \{l(t)+zm(t)+\ .\ .\ .\ \} \\
&= [f(o)+tf'(o)+\ .\ .\ .\ +x\{g(o)+tg'(o)+\ .\ .\ .\ \}+\ .\ .\ .\ ] \\
&\times [h(o)+th'(o)+\ .\ .\ .\ +y\{k(o)+tk'(o)+\ .\ .\ .\ \}+\ .\ .\ .\ ] \\
&\times [l(o)+tl'(o)+\ .\ .\ .\ +z\{m(o)+tm'(o)+\ .\ .\ .\ \}+\ .\ .\ .\ ] \\
&= \left[f(o)+xg(o)+\frac{x^2}{2!}f'(o)+\ .\ .\ .\ +t\{f'(o)+xg'(o)+\ .\ .\ .\ \}+\ .\ .\ .\ \right] \\
&\times \left[h(o)+yk(o)+\frac{y^2}{2!}h'(o)+\ .\ .\ .\ +t\{h'(o)+yk'(o)+\ .\ .\ .\ \}+\ .\ .\ .\ \right] \\
&\times \left[l(o)+zm(o)+\frac{z^2}{2!}l'(o)+\ .\ .\ .\ +t\{l'(o)+zm'(o)+\ .\ .\ .\ \}+\ .\ .\ .\ \right] \\
&= \sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{F}^{2n}(x)\sum_{r=0}^{\infty}\frac{t^r}{r!}\mathrm{H}^{2r}(y)\sum_{s=0}^{\infty}\frac{t^s}{s!}\mathrm{L}^{2s}(z),
\end{aligned}$$

where

$$\mathrm{F}(x)=f(o)+xg(o)+\frac{x^2}{2!}f'(o)+\frac{x^3}{3!}g'(o)+\ .\ .\ .$$
$$\mathrm{H}(y)=h(o)+yk(o)+\frac{y^2}{2!}h'(o)+\frac{y^3}{3!}k'(o)+\ .\ .\ .$$
$$\mathrm{L}(z)=l(o)+zm(o)+\frac{z^2}{2!}l'(o)+\frac{z^3}{3!}m'(o)+\ .\ .\ .$$

Thus the three series solutions 3.6, 3.51, and 3.52 are equivalent as long as the arbitrary functions involved are consistent.

§ 16. INTEGRAL EQUATION SOLUTION.

We can again apply the integral equation method of solution to the equation III. From the general solution 3.1, we have

$$\phi(x,\ y,\ z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{ix\alpha+iy\beta+iz\gamma}f(\alpha,\ \beta,\ \gamma)d\alpha d\beta d\gamma.$$

To solve this explicitly we must make certain assumptions on the forms of the functions ϕ and f. Assume we may write

$$\phi=\phi_1(x)\phi_2(y)\phi_3(z),$$
$$f=f_1(\alpha)f_2(\beta)f_3(\gamma),$$

then, as before,

$$\phi_1(x)\phi_2(y)\phi_3(z) = \int_{-\infty}^{\infty} e^{ix\alpha} f_1(\alpha)d\alpha \int_{-\infty}^{\infty} e^{iy\beta} f_2(\beta)d\beta \int_{-\infty}^{\infty} e^{iz\gamma} f_3(\gamma)d\gamma.$$

$$\therefore\ \phi_1(x) = \int_{-\infty}^{\infty} e^{ix\alpha} f_1(\alpha)d\alpha, \quad \therefore\ f_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixs}\phi_1(s)ds\,;$$

$$\phi_2(y) = \int_{-\infty}^{\infty} e^{iy\beta} f_2(\beta)d\beta, \quad \therefore\ f_2(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iyu}\phi_2(u)du\,;$$

$$\phi_3(z) = \int_{-\infty}^{\infty} e^{iz\gamma} f_3(\gamma)d\gamma, \quad \therefore\ f_3(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-izw}\phi_3(w)dw.$$

$$\therefore\ v = \frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ix\alpha+iy\beta+iz\gamma-t(\alpha^2+\beta^2+\gamma^2)}$$

$$\times\left[\int_{-\infty}^{\infty} e^{-i\alpha s}\phi_1(s)ds\right]\left[\int_{-\infty}^{\infty} e^{-i\beta u}\phi_2(u)du\right]\left[\int_{-\infty}^{\infty} e^{-i\gamma w}\phi_3(w)dw\right];$$

$$v = \frac{1}{8\pi^3}\int_{-\infty}^{\infty} e^{-t\alpha^2}d\alpha\int_{-\infty}^{\infty} e^{-t\beta^2}d\beta\int_{-\infty}^{\infty} e^{-t\gamma^2}d\gamma\int_{-\infty}^{\infty}\phi_1(s)\cos\alpha(x-s)ds\int_{-\infty}^{\infty}\phi_2(u)\cos\beta(y-u)du$$

$$\times\int_{-\infty}^{\infty}\phi_3(w)\cos\gamma(z-w)dw \quad . \ . \ . \quad 3.7$$

By changing the order of integration, and performing the integrations with respect to α, β, and γ, this becomes the known form of solution 3.3.

(*Issued separately June* 10, 1925.)

XIX.—Note on Professor Whittaker's Atomic Model. By John A. Eldridge, University of Wisconsin. *Communicated by* Professor E. T. Whittaker, F.R.S. With further Notes by Professor E. T. Whittaker and Professor H. S. Allen.

(MS. received July 3, 1924. Read November 24, 1924.)

In the *Proceedings* of the Royal Society of Edinburgh for 1921–22[*] Whittaker suggests an atomic model designed to absorb and emit radiation by quanta. This model consists essentially of a system of radial magnets, set as the spokes of a wheel with like poles (say south) at the centre. The electron approaching the wheel along the axis of symmetry will, according to the author, set the wheel into rotation, giving a "magnetic current." The action of the approaching charge, giving a rotation to the North Pole and a "magnetic current," is quite the analogue to the North Pole approaching a non-axial electric charge, setting it into rotation, and producing an "electric current." The electron, with energy exceeding a definite value, succeeds in going through the atom, with the result that a definite amount of energy is permanently transferred from the electron to the atomic wheel, a result which resembles the absorption of electronic energy by the atom in spectrum excitation.

However, Professor Whittaker seems not to have taken into account the forces on the central pole; when this is done it is found that the total angular impulse given to the magnet wheel is zero—that the wheel will not in fact be set into rotation as Professor Whittaker assumes.

On page 132 of Whittaker's paper the expression for the force on a magnet pole μ due to an electron with charge e is given: $\frac{\mu e a \dot{x}}{r^3}$, where r is the distance from electron to pole, a the distance from axis to pole, and $\dot{x}$ the velocity of the electron along the symmetry axis. The moment of force about this axis is $\mathrm{L}=\frac{\mu e a^2 \dot{x}}{(a^2+x^2)^{\frac{3}{2}}}$, and the total angular impulse contributed by an electron which goes completely through the model is $\int_{-\infty}^{+\infty} \mathrm{L}\,dt$. L is taken as having a finite value for the outer pole but zero for the inner (for which $a=0$). This is true except at the point $x=0$

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[*] *Proc. Roy. Soc. Edin.*, 1921–22, p. 129.

where L is infinite. This makes the expression for impulse on the inner pole indeterminate. However, if we set a equal to an infinitesimal, we may compute the value of the angular impulse on the inner pole.

$$\text{Angular impulse} = \int_{-\infty}^{+\infty} \frac{\mu e a^2 \dot{x} dt}{(a^2+x^2)^{\frac{3}{2}}} = \int_{-\infty}^{+\infty} \frac{\mu e a^2 dx}{(a^2+x^2)^{\frac{3}{2}}} = 2\mu e.$$

This expression, being independent of a, maintains a constant value as the pole approaches indefinitely close to the axis, a value *exactly opposite* to the angular impulse on the outer pole. The electron then does not set the wheel into rotation, and the model is incompetent to absorb electronic energy in the way Whittaker assumed.

Professor Ewing pointed out that the model appears to violate the principle of conservation of momentum, which principle could be conserved only by calling on the assistance of the ether, a procedure of very doubtful significance. The real meaning of this violation is that Whittaker has in effect employed a mechanism involving a unipolar magnet, and the unipolar magnet is a conception abhorrent to physics.

NOTE BY PROFESSOR E. T. WHITTAKER, F.R.S., ON THE ABOVE PAPER.

1st *August* 1924.

The point to which attention is drawn in Mr Eldridge's note was raised at the time my paper was written by one of the friends (Sir Oliver Lodge, if I remember rightly) with whom I was in correspondence; and it was on this account that I added the footnote on page 139 of the paper (which Mr Eldridge has apparently overlooked), as well as the remarks on the lower half of page 135. In these places I was careful to point out that the essential thing in the paper is the "magnetic current," whose existence is inferred in § 2, and whose properties are expressed by the equations: as I say on page 139, we should "frankly assume that magnetic currents can exist in an atom, without laying stress on their realisation by means of bar-magnets." The magnetic current is, in fact, a new conception; and just because it is a new conception there must necessarily be some flaw in any attempt to represent it by means of old conceptions, such as bar-magnets. It is unfortunate that some readers have seized on the wheel whose spokes are bar-magnets, as if it were the important and characteristic feature of the paper, in spite of my express warnings to the contrary in the paper itself.

At the end of Mr Eldridge's note is a remark on momentum. As Sir

Alfred Ewing pointed out quite correctly, it is necessary in connection with the model to think of momentum as stored in the ether. This conception is necessary for the explanation of such phenomena as radiation-pressure (where momentum is transferred from the ether to material bodies) and is, so far as I know, accepted by all physicists. It is expounded by J. J. Thomson, *Phil. Mag.*, (6), **8**, (1904), pp. 331–356, and by H. A. Lorentz, *Theory of Electrons*, pp. 31–36. I do not understand why Mr Eldridge disbelieves in it.

NOTE ON PROFESSOR WHITTAKER'S QUANTUM MECHANISM. By Professor H. STANLEY ALLEN.

(MS. received February 21, 1925.)

The objection raised against Whittaker's quantum mechanism in the atom, viz. that it implies the existence of unipolar magnets, is valid only in the particular case considered, in which the electron moves along the axis of the magnetic wheel. In this case the electron has to pass through two rings of poles of contrary sign, and the angular impulse arising from the inner ring is exactly equal and opposite to the angular impulse from the outer ring.

There appears to be a possible way out of the difficulty if we suppose that the electron, instead of moving along the axis, is projected in such a manner that it passes through the ring composed of poles of one name, but not through the ring formed of poles of contrary name. It has been shown by B. B. Baker * and by H. A. Lorentz † that Whittaker's result may be generalised. When the motion of the electron is not restricted to the axis of the magnetic wheel, but is in *any* direction initially, the results deduced are precisely the same as those obtained by Whittaker in the particular case which he considered. The electron can permanently transfer a definite amount of energy to the magnetic wheel if, and only if, its velocity of projection is such that it penetrates the magnetic structure, and passes away out of the influence of the wheel without returning on its path.

To fix our ideas we may think of a ring of N. poles arranged round the upper rim of a conical waste-paper basket, the corresponding ring of S. poles being distributed round the lower rim, each bar magnet lying along a generating line of the conical surface. An electron may be projected in such a way that it passes through the top of the basket, but instead of reaching the bottom passes out through the side. In this way it is possible for the electron to thread through a ring of poles and then move away out of the influence of the magnetic structure without returning through the ring or passing through a ring of opposite sign.

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* B. B. Baker, *Phil. Mag.*, **44**, 777 (1922).

† H. A. Lorentz, *Proc. Amsterdam Acad.*, **25** (1922).

As illustrating the possibility of obtaining such rotations with ordinary bar magnets, we may recall the classical experiment of Faraday, who produced the continued revolution of one of the poles of a magnet round a vertical conducting wire. In this experiment the whole magnet, with the exception of one extremity, was immersed in mercury, and the action of the wire carrying the current was limited almost entirely to the exposed pole. In modifications of the experiment,* frequently used for lecture demonstration, the magnet is suspended on a pivot, and by means of mercury contacts the vertical current is made to pass near the upper pole only, the current being led off horizontally about the centre of the magnet. It was a model of this kind which suggested to me the possibility of meeting the difficulty raised by Mr Eldridge in the manner described above.

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* Francis Watkins, *A Popular Sketch of Electro-Magnetism and Electro-Dynamics*, 1828; P. M. Roget, *Treatise on Electro-Magnetism*, 1832.

(*Issued separately June* 10, 1925.

XX.—Unilateral Vasoligation on the Senile Male of the Domestic Fowl. By F. A. E. Crew.

(MS. received January 30, 1925. Read May 11, 1925.)

UNILATERAL ligation of the vas deferens in the mammal is a method now commonly employed in the surgical treatment of senility. The demonstrably beneficial results that follow the operation are explained on the assumptions that the testis is an endocrine gland; that if the gametogenic function is embarrassed, the interstitial tissue, regarded as the source of the sex-hormone, undergoes hyperplasia; and that associated with this increase in amount of tissue there is an increased elaboration of the internal secretion.

It is commonly assumed that the relation of the gonad and the rest of the sexual phenotype in the bird is precisely that which obtains in the case of the mammal. If this is so, it might be expected that vasoligation in the bird would be followed by results equivalent to those so amply demonstrated in the mammal. Vasoligation in the fowl has been performed, and the sequelæ recorded only by Shattock and Seligmann (1904), Massaglia (1920), and by Nonidez (1924), and all these have used young birds. Many of my conclusions agree with those of Nonidez, but since we approached this subject with very different interests some of them are complementary.

Three old (6, 6, and 8 years respectively) cocks—Scots Dumpie, Minorca, and Salmon Faverolle—were used. They were chosen because they exhibited the same general characterisation. They were markedly senile, with ragged plumage, decidedly weak sexual behaviour, fecundity that had become progressively reduced, and poor circulation. They were extremely emaciated, and had the excessively overgrown spurs and the reduced lopped comb of the aged. The operations were performed under ether. The vas on the right side was dissected away from neighbouring tissues and ligated in two places near the junction of the middle and caudal third, the piece (about 25 mm.) between the ligatures resected, and the cut ends displaced. The caudal third of the left testis was then removed and fixed in Bouin or Zenker, and later stained with hæmotoxylin and eosin or iron-hæmotoxylin. One hundred and eighty-two days later the birds were killed, both gonads were removed, fixed, and stained by the same methods. Accurate measurements of the gonads at the time of the first operation was impossible, but it could be seen that these were relatively small, being about 30 × 20 mm. in each case, the right being slightly but distinctly larger than the left.

The portions of the left testes removed at the first operation all presented the same histological structure. Mitosis was almost, but not quite, completely absent. Many of the tubules were filled with degenerated cells, others were entirely empty. The basement membrane was considerably thickened. The walls of the arteries were markedly sclerotic. No interstitial cells and no remnants of sex-cords (the so-called luteal cells) could be identified.

The birds were entirely unaffected by the operation of unilateral vaso-ligation; in no particular was their condition of senility improved—in fact, they became progressively more decrepit.

When one of the birds died about six months after the operation, the other two were killed and post-mortemed. Each was very emaciated, body-fat being almost entirely absent. No sign of organic disease was found. In each case the operation wound on the left testis had healed, but the site was clearly visible as a depression on the surface. The histological structure of the remains of the left testis and of the right was in all cases exactly as that of the portion of the left testis examined six months previously, save that sclerosis was more marked and that mitosis was completely absent.

	Standard Weight of Breed.	Weight at First Operation.	Weight at Autopsy.	Weight of Testes.			
				Right.		Left.	
	gms.	gms.	gms.	mm.	gms.	mm.	gms.
Scots Dumpie . .	3150	2554	2084	27 × 15	3·43	16 × 13	1·82
Minorca . . .	3375	2295	2144	30 × 19	4·56	21 × 17	2·78
Salmon Faverolle .	3375	2303	2225	25 × 14	3·84	20 × 14	·295

The facts that no interstitial cells and no fatty infiltrated remnants of sex-cords were found, and that the plumage was cocky-feathered, bear no exact relation to each other. The plumage was assumed months before the post-mortem examination was made, and the condition of the gonads at this time could only influence the characterisation of any part of the plumage that was being renewed. The sexual characterisation of a plumage is a reflection of the gonadic situation at the time when that plumage was assumed. The fact that these birds were cocky-feathered in spite of the fact that their gonads, judged by their histological structure, were apparently physiologically inert, is to be explained on the assumption that the plumage was not being renewed or/and that in spite of the histological findings there remained a residuum of functional activity in the gonads,

The results of this experiment, necessarily limited by the difficulty of obtaining really old fowls, suggest that unilateral vasoligation in the fowl is not attended by the rejuvenating effects that are associated with this operation in the male among mammals. It is possible that the birds used were too senile, that their gonadic tissues were exhausted; but none of them were so decrepit as the dog successfully treated by Sand (1922).

If the conclusion that this operation is not followed by rejuvenation phenomena in the fowl is warranted, then it must be contended that the physiological action of the testis in the bird is possibly, if not certainly, different from that of the testis of the mammal. The results obtained do not support the contention that the testis of the cock is the source of a specific sex-hormone; however, it cannot be claimed that they show that such a contention is unsound.

The expenses connected with this work were covered by a grant from the Royal Society: for this grant I wish to express my thanks.

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XXI.—Rejuvenation of the Aged Fowl through Thyroid Medication. By F. A. E. Crew.

(MS. received April 28, 1925. Read June 22, 1925.)

In order to examine the hypothesis, so ably enunciated by the Steinach school, that senility is a phase of life that is attended by, perhaps even caused by, the inefficient functioning of one or more of the components of the endocrine system, the writer set out upon a series of attempts to evoke rejuvenation in the aged fowl. In a previous communication (Crew, 1925) an account was given of an unsuccessful attempt to produce this result in the aged cock by means of unilateral vasoligation. In this paper the results of thyroid administration to aged birds of both sexes are reviewed.

Thyroid has been given to the fowl for purposes of experimentation by several investigators. Torrey and Horning (1922) reported that when dried thyroid was given to growing males these assumed hen-feathering. Crew and Huxley (1923), repeating this work, but with different material and methods, failed to obtain confirmatory results. However, it was noticed during the course of this work that the administration of thyroid was followed by a marked increase in the rate of plumage growth and replacement, and that in the birds used, Rhode Island Red (red ground with black areas) and Light Sussex (white ground with black areas), the birds receiving thyroid exhibited a pronounced tendency towards increased melanism, the black areas being markedly increased in size and intensity at the expense of the other colours. Since it is known that such a particoloured bird tends to become lighter as it ages, the suggestion presented itself that in senility the thyroid of the fowl becomes relatively less efficient in its functioning. Moreover, the observation that thyroid administration increased the rate of feathering in the growing chick and moulting adult gained in significance when Serebrovsky (1922)—and more recently Warren (1925)—showed that quick feathering as contrasted with slow is a typical sex-linked character. One who seeks to interpret genetic action in terms physiological is attracted by the notion that this sex-linked factor in its action determines the time during development when first the thyroid comes into action or the degree of the functioning of this gland. Cole and Reid (1924) also have repeated the work of Torrey and Horning, and have obtained results which show beyond all doubt that thyroid administration does indeed affect the plumage characters of the male. They found that new feathers grown by the birds receiving desiccated thyroid showed distinct modification towards the female type of structure and colour, and that the rate of growth of these new feathers was noticeably increased. There was a reduction of red pigment, the distribution of which was irregular, and also a reduction of that area of the feather in which the barbs lack barbules, the feathers of the hackle regions, instead of being pointed and elongate, coming to possess broad rounded ends closely resembling the feathers of these regions in the female.

Willier (1924) successfully implanted small pieces of thyroid gland of the fowl on to the chorio-allantoic membrane of the developing chick at the 7th to 10th day of incubation. When examined at the 17th to 18th day of incubation the body of the developing chick was found to be emaciated and considerably smaller, the segments of the wings and the legs shortened, and the chick's own thyroid and gonads reduced in size. Willier does not record whether or not thyroid administered in this way increases the rate of plumage development, though in the photograph of his experimental and control chicks there seems to be evidence that it does. It is worthy of note in this connection that those breeds in which the males are hen-feathered consist of comparatively small and active birds.

There are individuals and breeds which mature relatively early (5 to 6 months) and others that mature late (10 to 12 months). The former are light, active, nervous birds, the latter are heavy, slow, and placid. The former as a class are aged when four, the latter do not show the unmistakable signs and symptoms of senility until about seven. In old age the head furnishings of both sexes are diminished in size, venous in colour, and flabby; the eye is dimmed; the plumage loose and ragged; the moult is irregular and the tail-sickles are commonly not replaced; the skin is less smooth, and the legs more scaly. The bird is either emaciated or excessively fat, and exhibits sexual behaviour but rarely, if at all. Both sexes become very inactive and spend the greater part of their time drowsing in solitude. They sleep on the floor rather than fly to the perch. The hens lay infrequently; the cock crows very occasionally, takes no interest in hens, and hastily shuns the presence of a younger male.

Five cocks and seven hens, all known to be old (5 to 8 years) and all showing the classical signs of old age but not those of gonadic disease, formed the experimental pen. Since this included so great a variety of breeds, and since to obtain really old birds is a matter of considerable difficulty, no control pen was maintained. To each of these birds desiccated thyroid (for the gift of which I wish to thank Messrs Armour & Company) was administered daily for six months (May to October). The powder was incorporated in a small pellet of cold wet mash and given to each bird an hour or so before the morning feeding. During the first fortnight each bird received the equivalent of 0·2 mgm. of iodine each day, during the second fortnight 0·4 mgm., and thereafter 0·8 mgm. The results were striking.

From the table it may be seen that following upon the administration of thyroid the birds without exception promptly moulted and that the hens quickly started to produce eggs at a faster rate. These eggs were fertilised by the males in the pen. With the passing of the moult, the head furnishings became red and turgid, the plumage tight and close. All the birds became rejuvenated, looking fresh and being active. None of the hens became broody during the course of the experiment (12 months), though six out of the seven belonged to the sitting breeds. This observation may possess some significance in view of what is known concerning the profound disturbances in the metabolism of the broody hen.

The plumage characters of the males were affected both in colour and structure: those of the hens in colour only. The black and white Houdan and Campine had become increasingly whiter with age. They moulted out as black as a two-year old bird. The blue of the Old English Game had become lighter with succeeding moults. His smoky white plumage was replaced by a smooth blue. The gold black hackle of the Scots Dumpie was replaced by solid black. In the black and red Orloffs and in the silver Dorking hen the black was increased, whilst the barring of the Scots Dumpie hen became more distinct. No appreciable change was noticed in the colour of the Croad Langshan and Salmon Faverolle. The new neck hackle and tail of the Light Sussex hen carried far more black than those of a young bird of this breed. In the Houdan and the Orloff males the black was increased at the expense of the white or red in those feathers which are black on one side of the rachis, white or red on the other. The white and the red areas became peppered with black to form a condition similar to that ordinarily present in these feathers of the female.

The hackle feathers of all the cocks, previously elongate and pointed and with the barbs of the distal half free from barbules, were replaced by broad, blunt, gas-flame shaped feathers, with the barbule free area greatly reduced and very much like the hackle feathers of the female. None of the cocks could retain their new tail hackles. The Campine cock, hen-feathered in his earlier years, had become more and more cock-feathered with increasing age. He responded at once to the thyroid medication and became again hen-feathered.

The birds were kept under observation for six months after the

	At beginning of Experiment.				No. of Eggs during previous Six Months.	Period of Thyroid Administration.						Total No. of Eggs for Six Months.	Period following Thyroid Administration.						Total No. of Eggs for Six Months.	Remarks.
	Age.	Condition.	Weight in Gms.	Approximate Standard Weight in Gms.		No. of Eggs; M = Moult; weight in Gms.							No. of Eggs; weight in Gms.							
						May.	June.	July.	Aug.	Sept.	Oct.		Nov.	Dec.	Jan.	Feb.	Mar.	Apr.		
♂♂																				
Houdan . .	7	Emaciated	3150	3800	...	M	...	...	3370	...	3500	...	...	...	3450	...	...	3300	...	Died following Nov. from roup.
Old English Game (blue).	8	,,	2250	3300	...	M	...	...	2650	...	3250	...	...	...	2750	...	...	2700	...	Still alive; very senile.
Scots Dumpie .	7	,,	2550	3300	...	M	...	...	2850	3250	died	(roup)	...	...	...	...	...	...	...	
Orloff . .	7	Fat	4600	5000	...	M	...	...	4800 (died)	...	...	...	...	...	..	...	...	...	...	
Campine . .	7	Cock-feathered	3000	3150	...	...	M (hen-feathered)	...	3100	...	3100	...	...	...	3100	...	...	3000	...	Still alive; plumage intermediate.
♀♀																				
Silver Dorking .	8	Fat	3650	4250	2	0 M	0	2	6 3200 (died ruptured oviduct)	...	...	...	...	...	...	...	...	...	...	
Orloff . .	7	,,	3650	4000	4	0 M	1	0	5 3600	7	4 3580	17	died	(roup)	...	...	...	...	...	
Croad Langshan .	7	,,	2850	3100	6	2 M	4	9	6 2950	8	7 3280	36	6	7	5 3250	0	1	0 2950	19	Still alive; ovarian atrophy.
Scots Dumpie .	7	,,	2650	2900	5	1	0 M	3	6 2980	4	5 3150	19	5	4	0 2950	0	0	1 2800	10	Died following Nov. (pneumonia).
Salmon Faverolle	8	,,	2280	2400	4	0	2 M	4	6 2470	4	6 2470	22	5	1 3000	died	(ruptured oviduct)			6	
Light Sussex .	7	,,	2075	3000	13	3 M	5	8	7 2580	10	7 2580	41	9	6	2 2400	1	2	4 2500	24	Still alive; ovarian atrophy.
White Leghorn .	6	,,	1700	2000	8	1 M	12	10	18 1950	10	18 1950	69	15	15	10 1920	12	6	2 1850	60	Still alive; ovarian atrophy.

cessation of thyroid administration. During the first half of this period they remained in excellent condition. After this they became progressively senile again. Two fell easy victims to roup and pneumonia. Rupture of the oviduct led to fatal peritonitis in two of the hens; but since this is a very common cause of death among domestic poultry, no special significance can be attached to its incidence in the experimental pen. The Orloff cock died of what must be diagnosed as old age: no organic lesion was present. All the hens that are still alive (April 1925) are exhibiting the classical signs of ovarian atrophy.

These results entirely confirm the conclusions of Torrey and Horning, Cole and Reid. Further, they show that rejuvenation in the fowl can be evoked by thyroid medication. They indicate that in old age in the fowl the thyroid is deficient in its functioning, and that the physiological activity of this gland is essential to the reproductive function. In old age the gonad of the fowl becomes inactive, reduced in size, and fibrotic. Apparently thyroid medication, supplementing the action of the bird's own thyroid, now inefficient, evokes a new lease of activity in the gonad, and the endocrine chain is thus repaired.

It would seem, then, that the physiological activity of the thyroid is concerned in the establishment of the type of feathering of the individual. If this is so, it is necessary to examine the commonly held opinion that the plumage characterisation is but the reflection of the kind of gonadic tissue present in the individual. It is true that in the presence of functional testicular tissue the plumage of the male or of the ovariotomised female is typically male; that in the presence of ovarian tissue the plumage of the female and of the castrated male is typically female; that in the presence of both testicular and ovarian tissues in the male or female it is female; that the plumage of the capon and of the poularde is male-type in structure but more luxuriant and looser than that of the normal male; and that in certain breeds the functional male is hen-feathered, and that this hen-feathering in the male behaves as a dominant Mendelian character in inheritance. These facts are usually interpreted on the assumption that the ovary of the hen and the testes of hen-feathered cocks elaborate an internal secretion that inhibits cock-feathering, and Morgan (1920) suggested that the source of this hormone was the luteal tissue. However, this suggestion has been shown by Pease (1921), Fell (1923), Nonidez (1922), and Greenwood (1925) to be unwarranted, and with the destruction of this hypothesis the evidence in favour of the production of specific sex hormones by the sex glands of the fowl becomes scanty indeed.

The facts can be accommodated in a simpler way than by evoking the aid of specific gonadic internal secretions, the existence of which cannot be demonstrated. The gonads undoubtedly play an important rôle in determining the character of the general metabolic processes of the body, and the metabolic level in male and female is demonstrably different. It is suggested that sex differences in plumage characters are but reflections of corresponding differences in the general metabolic processes of the body, and that these are due to the different metabolic needs of ovarian and testicular tissues. The ovary in its functioning exerts a very considerable physiological demand upon the general economy of the individual, and the demand of the testis is comparatively less. These differences are illustrated in sexual differences in blood fat and blood calcium content. The response to the demand of the gonad is regulated by components of the endocrine system; by the thyroid among others. In the female the effect of the functioning of the ovary is such as to provoke in the thyroid an activity of such an order as to constitute the stimulus for hen-feathering. The thyroid of the cock-feathered cock is so stimulated by the physiological demands of the testes that its products are elaborated at a slower rate or in lesser quantity and so the plumage as it develops is as that of the typical male. Thyroid medication increases the quantity available for directing the growth of new feathers, and so the plumage of the cock becomes as that of the hen; that of the hen is unaffected. The effects of gonadectomy and gonad implantation on the plumage are not direct effects but are secondary to a change in the rate or quality of thyroid functioning. Gonadectomy is followed by degenerative changes in the thyroid.

In the present state of our knowledge of the endocrinology of the fowl it is as reasonable to explain hen- and cock-feathering on the assumption that these are based upon sexual differences in thyroid functioning as to assume that the ovary of the hen elaborates an internal secretion of a kind qualitatively different to that produced by the testis of a cock, and that the testes of cock- and hen-feathered cocks are different endocrinologically.

It is necessary to attempt the interpretation of the fact that hen-feathering in the cock behaves as a Mendelian character. Cunningham (1925) has recently hazarded the suggestion that this characterisation may be due to non-disjunction of the sex-chromosomes, but it will prove most difficult to secure any cytological support for this suggestion. It can be accepted that corresponding to any heritable character, physiological as well as morphological, there is a gene or gene-complex resident in the chromosomes. There is reason for holding that the rate or quality of the functioning of a gland of internal secretion is a character in the neo-Mendelian sense. It is suggested that a genetic difference in thyroid functioning is the basis of the difference between cock- and hen-feathered cocks; that the thyroid of those breeds in which the cocks are hen-feathered is provoked to a fuller activity by a lesser stimulus than is that of the breeds in which the hen only is hen-feathered, and that this difference is genetic in nature.

There are two classes of fowls: (1) those with a thyroid which in the presence of active testis responds to a degree that provides that concentration of thyroid secretion in the blood which directs the activity of the feather follicles to the production of cock-feathering, and which in the presence of active ovary responds to a fuller degree and leads to hen-feathering; and (2) those with a thyroid which responds fully to the lesser stimulus of active testis. The difference in the thyroid of the two classes is a genetic difference in the threshold of response on the part of the thyroid to a particular type of stimulus. It is suggested that there are genetic differences in the quality or degree of thyroid functioning, and that the feathered follicles are ambivalent, being capable of producing feathers of the cock- or the hen-type according to the kind or intensity of the stimulus that directs their activity. These suggestions are reasonable, since thyroid from any source can supplement the bird's own, and a bird can be hen-, cock-, and hen-plumaged with successive moults. Further, the plumage characterisations can be controlled experimentally by gonadic grafts.

If the difference between cock- and hen-feathering lay in the physiological activity of the two types of testicular tissue, then interchange of testes should be followed by reversal of plumage type, the cock-feathered should become hen-feathered, and *vice versa.* Pézard, Sand, and Caridroit (1924) claim to have shown that the implantation of a testis from a hen-feathered cock into a cock-feathered cock was followed by the replacement of cock- by hen-feathering. This would seem at first sight to be a conclusive demonstration of the endocrinal differences in the testes of the two sorts of males. Further consideration of their claim, however, shows that it is by no means so. They used a heterozygous cock-feathered silver bird out of a Brown Leghorn × Dorking mating, "dont la masse testiculaire dépasse a peine le minimum efficace," implanting subcutaneously fragments of a testis derived from a hen-feathered Sebright. New feathers of the saddle region resembled those of a Brown Leghorn hen. There is much to criticise in this experiment, but for the moment it is enough to point out that in it the testes of a cock-feathered cock were not replaced by testicular material from a hen-feathered male, but that the host's own gonadic tissue was supplemented with testicular material from another source. If the host had been a developmental poularde—a hen in which the ovary had not functioned sufficiently at the time when the plumage was assumed—the testis implant might be expected to evoke a fuller activity of the bird's own gonad, and so hen-feathering would follow. If, on the other hand, as is more probable, the host was a developmental capon, the testis implant would create by its functioning an environment in which the bird's own testicular material would function more completely, and the unusual amount of testicular material present in this bird would be equivalent in its effect upon the thyroid to a functional ovary. Be this as it may, Pézard, Sand, and Caridroit failed to control their results by the implantation of testis from a cock-feathered male. Until it has been shown that the implantation of testis from a hen-feathered cock into a completely castrated cock-feathered male leads to hen-feathering, and that the augmentation of the testes of a cock-feathered cock by testicular material obtained from cock-feathered males does not, it cannot be maintained that the testes of the two sorts are endocrinologically different, or shown that the suggestion that the two races differ genetically in the threshold of response on the part of the thyroid to gonadic stimulation is wrong. Experiments devised to answer these questions have been performed and the results will be communicated in a further paper.

The expenses incurred in the course of this investigation were met out of a grant made to me by the Government Grant Committee of the Royal Society, and for this I wish to express my gratitude.

SUMMARY.

The administration of desiccated thyroid to aged fowls of both sexes is followed by definite rejuvenation effects. Cock-feathered cocks when so treated assume a plumage that is distinctly henny in structure and coloration.

It is suggested that the effect of gonad on plumage is not direct, but that it is secondary to the effect of gonad on thyroid. It is suggested further that the thyroid of those races in which the cocks are hen-feathered is genetically different from that of the cock-feathered in that it exhibits a lower threshold of response to the gonadic stimulus.

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XXII.—L'Entretien des Pendules au Moyen de Cellules Photo-électriques. Par M. le Général G. Ferrié, Membre de l'Académie des Sciences de Paris, L.L.G., C.M.G. *Communicated by* Professor R. A. Sampson, F.R.S., General Secretary.

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Il est fréquemment nécessaire de mesurer, avec la plus haute précision possible, les intervalles de temps compris entre des signaux inscrits sur un appareil enregistreur (signaux horaires, tops d'observations astronomiques, etc.). Lorsque l'intervalle de temps à mesurer est de courte durée, on peut utiliser divers procédés, par exemple un diapason entretenu électriquement *sans contact* au moyen d'un amplificateur à lampes et dont on inscrit le courant d'entretien sur le même enregistreur. Il est aisé d'évaluer l'intervalle à mesurer en nombre de vibrations et fraction de vibration. Si cet intervalle dépasse quelques secondes, le procédé devient impraticable et il est nécessaire de subdiviser l'intervalle à mesurer en fractions d'une seconde par exemple, de compter le nombre entier de secondes intercalées entre les deux signaux et de ne mesurer en vibrations du diapason que les résidus compris entre le premier signal et la 1ère seconde, le dernier signal et la seconde la plus voisine de lui.

La division en secondes est obtenue d'ordinaire au moyen des contacts électriques donnés par une pendule astronomique de précision. On sait que ces instruments ont actuellement des marches entièrement satisfaisantes. Les pendules Leroy, des caves du Bureau International de l'Heure, par exemple, ont des marches de l'ordre de $\frac{1}{100}$ de seconde par jour.

Il faut remarquer toutefois que les mesures très précises faites notamment au B.I.H., ont montré que les contacts électriques de ces pendules n'étaient pas exactement isochrones et que des différences dépassant un centième de seconde existaient parfois entre les durées des secondes limitées par ces contacts. Le fait est dû au mécanisme qui commande ceux-ci, les roues dentées notamment ne pouvant pas être taillées d'une façon absolument parfaite.

Dans la mesure d'un intervalle de temps effectuée comme il vient d'être dit, si la seconde voisine du commencement de cet intervalle, et aussi celle voisine de la fin, comportent une telle erreur, le résultat de la mesure comportera une erreur totale de 2 ou 3 centièmes de seconde environ.

Nous avons cherché, en collaboration avec M. R. Jouaust, à faire disparaître cet inconvénient. Une première solution a consisté à utiliser les propriétés des cellules photoélectriques pour enregistrer les passages du balancier par la verticale et supprimer les contacts électriques.

La description complète de l'installation réalisée sur une pendule des caves du Bureau International de l'Heure, a été donnée dans le *Bulletin Astronomique*, de Mars 1925. Il suffit de rappeler ici très sommairement le dispositif:

Un miroir est fixé au balancier et réfléchit un rayon lumineux pendant un temps très court sur une cellule photoélectrique, au moment où le balancier passe par la verticale. Le courant ainsi engendré dans la cellule est considérablement amplifié par des lampes à 3 et 4 électrodes de manière à pouvoir agir sur l'appareil enregistreur.

Ce mode opératoire permet d'éviter les erreurs provenant des irrégularités du contact électrique mais laisse subsister celles qui sont dûes à l'entretien des oscillations du balancier. Pour les faire disparaître également, autant que possible, nous avons employé le procédé suivant:

Un pendule libre, sans mouvement d'horlogerie ni contact matériel extérieur d'aucune sorte autre que son point de suspension, porte un miroir sur sa tranche et un aimant en fer à cheval à son extrémité. Une des branches de cet aimant pénètre, à chaque oscillation, à l'intérieur d'une bobine portant un enroulement convenable. On utilise ce miroir, comme on l'a vu plus haut, à réfléchir un faisceau lumineux étroit sur une cellule photoélectrique, au moment du passage du balancier par la verticale. Le courant ainsi engendré par la cellule, et convenablement amplifié, parcourt la bobine et entretient les oscillations du balancier.

Il est nécessaire toutefois de ne laisser subsister qu'une seule action par oscillation double car, si l'impulsion donnée au moment de l'un des passages a pour effet d'entretenir les oscillations, l'impulsion donnée au moment du passage suivant en sens inverse produirait un effet contraire.

Cet inconvénient ne se produirait évidemment pas si on faisait agir la lumière sur la cellule au moment où le balancier atteint son élongation maximum, à gauche par exemple, mais l'impulsion donnée à ce moment altèrerait la période propre d'oscillation du pendule.

La suppression d'une action sur deux, au moment du passage du balancier par la verticale, a été réalisée de la manière suivante:

Le courant amplifié de la cellule photoélectrique traverse, non seulement la bobine d'entretien du pendule libre qui porte le miroir, mais aussi la bobine semblable d'un deuxième pendule qui se trouve ainsi synchronisé par le premier. Ce deuxième pendule porte un dispositif analogue à celui qui existe dans les pendules ordinaires à entretien électrique et qui a pour but de fermer le circuit d'entretien une fois par oscillation double. Il n'est pas nécessaire que les oscillations de ce deuxième pendule soient parfaitement isochrones, et il n'y a aucun inconvénient à ce que son balancier passe par la verticale un peu avant ou un peu après le passage du premier balancier par la verticale.

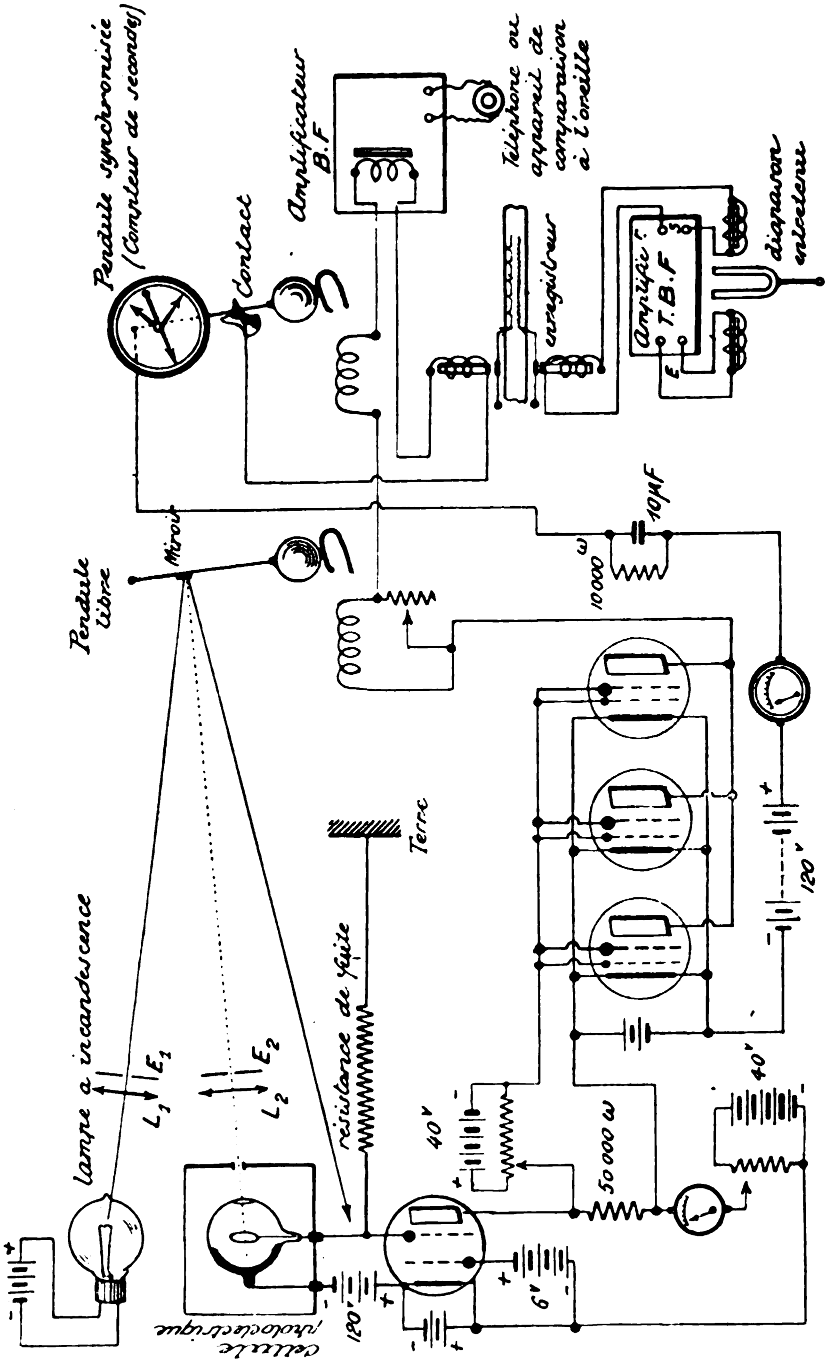
Le dispositif de fermeture du courant d'entretien, porté par la deuxième pendule, est donc organisé de telle sorte qu'il ferme le circuit un temps notable avant le passage par la verticale et ne le coupe qu'un certain temps après ce passage. Le courant photoélectrique amplifié trouvera ainsi fermé, au moment de sa production, le circuit d'entretien des pendules, et cela une seule fois par oscillation double. Signalons encore que la deuxième pendule peut sans inconvénient recevoir un mouvement d'horlogerie ordinaire, actionné par les oscillations de son balancier et agissant sur les aiguilles d'un cadran indiquant les heures, minutes et secondes ; c'est à dire, fonctionnant comme compteur totalisateur.

Le problème posé serait donc résolu sans difficultés si l'emploi de lampes à 3 ou 4 électrodes pour amplifier le courant photoélectrique ne venait apporter une complication nouvelle. Les deux bobines excitatrices des pendules sont en effet placées dans le circuit plaque des lampes du dernier étage d'amplification, et ce circuit est normalement parcouru en permanence par un courant. Le fait de fermer ce circuit sur les bobines excitatrices aura donc pour effet de lancer dans celles-ci un courant qui tendrait alors à entretenir les oscillations des pendules, indépendamment du courant photoélectrique. Il est donc nécessaire de prendre des dispositions pour supprimer entièrement tout courant permanent dans le circuit plaque des lampes. Ce résultat a pu être aisément obtenu.

Ces remarques générales étant faites, il nous reste à décrire avec quelques détails le montage d'ensemble que nous avons réalisé et dont le schéma est donné par la figure 1.

Sur le balancier du pendule libre, battant la 1/2 seconde, est fixé un petit miroir plan, perpendiculaire au plan d'oscillation. Le balancier se déplaçant d'une angle θ, les rayons réfléchis par le miroir tourneront d'un angle 2θ.

L'image du filament rectiligne d'une lampe à incandescence est produite sur le miroir par une lentille L_1. Les rayons réfléchis par le miroir sont recueillis par une lentille L_2 qui donne une nouvelle image du filament sur la cellule (cellule photoélectrique à l'hydrure de potassium, remplie d'argon).


Pendule synchronisée
(Compteur de secondes)
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10μF
Terre
lampe à incandescence
L1
E1
L2
E2
résistance de fuite
cellule
photoélectrique
40v
50000 ω
6v
180v
120v
40v
+
-

FIG. 1.

En avant des deux lentilles L_1 et L_2 sont disposés des écrans E_1 et E_2 munis de fentes de 1 à 2 millimètres de largeur.

Dans ces conditions, le flux lumineux ne peut évidemment agir sur la cellule que pour une seule position du balancier ou tout au moins pour une série très courte de positions successives. On règle le dispositif optique de manière que cette position du balancier soit la verticale, afin que le pendule soit entretenu par le courant photoélectrique avec sa période propre, autant que possible.

Le dépôt de potassium dans la cellule est réuni à une batterie de piles de 120 à 140 volts, tandis que l'anode est reliée à la grille extérieure d'une lampe à deux grilles, dont le filament est réuni au pôle positif de la batterie de 120 volts. On voit que la résistance interne de l'espace filament-grille extérieure de la lampe est ainsi placée en série avec la cellule sur cette batterie. Le réglage de la tension appliquée sur la plaque de la lampe à deux grilles est effectué au moyen d'un potentiomètre.

Dans ces conditions, lorsque la cellule au potassium est éclairée, elle dégage des électrons, et ceux-ci viennent charger l'anode de la cellule et par suite la grille extérieure de la lampe, qui lui est reliée. Cette grille devient alors négative et provoque une diminution du courant plaque.

Pour un éclairement donné, le courant photoélectrique a une valeur i. La tension V atteinte par la grille extérieure au bout du temps dt est donnée par la relation

$$i\,dt = CV,$$

C étant la capacité de l'ensemble : anode de la cellule-grille extérieure de la lampe.

Il y a évidemment intérêt à rendre cette capacité aussi faible que possible de manière que V puisse atteindre une valeur élevée. On doit donc réduire au minimum la longueur du fil qui joint l'anode à la grille extérieure ; c'est à dire, placer les deux instruments côte à côte.

Pour amplifier la variation du courant plaque dans la lampe à deux grilles, celle ci est suivie d'un amplificateur à trois lampes montées en parallèle. Nous avons trouvé avantage à employer encore des lampes à deux grilles, celles ci étant montées en parallèle.

Dans le circuit plaque de la première lampe, associée à la cellule, est intercalée une résistance de 50,000 ohms dont une extrémité est reliée aux filaments des trois lampes en parallèle, l'autre extrémité étant connectée aux grilles de ces mêmes lampes.

Le sens des connexions est choisi de telle sorte que la chute de tension aux bornes de la résistance de 50,000 ohms, au repos c'est à dire la cellule non éclairée, rende les grilles des trois lampes fortement négatives. Dans ces conditions le courant dans le circuit plaque de ces lampes est rigoureusement nul. Nous avons fait remarquer plus haut que cette condition était indispensable pour entretenir régulièrement l'oscillation du pendule libre.

Lorsque la cellule est éclairée, le courant plaque de la 1ère lampe diminue, et la chute de tension aux bornes de la résistance de 50,000 ohms diminue également. Les grilles des trois lampes en parallèle deviennent moins négatives et il peut s'établir un courant dans leur circuit plaque. Pour que ce courant soit assez intense pour l'entretien des pendules, il faut que les grilles deviennent positives. Dans ce but, une tension réglable par un potentiomètre est intercalée sur le fil de connexion des grilles des trois lampes avec la résistance de 50,000 ohms. Les réglages sont effectués de telle façon que, la cellule n'étant pas éclairée, les grilles des trois lampes en parallèle soient à -6^{v} par rapport à leurs filaments, le courant plaque étant alors rigoureusement nul, tandis que l'éclairement de la cellule a pour effet de porter les grilles à $+6^{v}$ par rapport à leurs filaments et d'engendrer un courant d'environ 1 milliampère dans les circuit des plaques.

On a évidemment tout intérêt à obtenir que la production de ce courant plaque par l'éclairement de la cellule, donne une impulsion aussi brusque que possible. Dans ce but une batterie de condensateurs de 10 à 12 microfarads, shuntée par une résistance de 10,000 ohms, a été intercalée dans le circuit plaque en série avec les bobines excitatrices des pendules.

Il n'est pas seulement nécessaire que le courant d'excitation des pendules s'établisse très rapidement, il faut également que sa durée soit très courte. Or la grille extérieure de la 1ère lampe reliée à la cellule est parfaitement isolée. La charge qui lui est apportée par les électrons de la cellule, se dissipe donc lentement. Pour obtenir une décharge rapide, et par suite une très courte durée du courant d'excitation, on a créé un défaut d'isolement en réunissant la grille extérieure de la 1ère lampe à la terre par une résistance élevée, constituée par une feuille de papier recouverte de plombagine. On obtient ainsi pour les courants d'excitation une durée de quelques centièmes de seconde. Il ne semble pas utile de chercher à diminuer encore davantage cette durée.

Signalons aussi que la bobine excitatrice du pendule libre a été shuntée par une résistance réglable, sur laquelle on agit au début des opérations, pour amener l'amplitude des oscillations du pendule libre à la valeur voulue.

L'enregistrement ou l'écoute à l'oreille, des courants d'excitation des pendules est obtenu aisément en intercalcant l'appareil d'enregistrement et l'appareil d'écoute en série avec les bobines excitatrices des pendules. L'énergie disponible est largement suffisante pour le fonctionnement simultané de tous ces organes.

Nous avons fait usage pour réaliser l'ensemble du dispositif, de deux pendulettes battant le 1/2 seconde. L'une d'elles construite par M. Leroy comme pendulette électrique, avait été dépouillée de son mouvement d'horlogerie et de tous ses organes. Seuls, le balancier avec son aimant, sa suspension et la bobine d'entretien avaient été conservés.

Le balancier de l'autre pendulette portait un doigt qui entraînait, dans son mouvement oscillant, une pièce en forme de V, dont une branche était conductrice, l'autre étant en matière isolante. Suivant le sens du mouvement du balancier, le doigt était en prise avec l'une ou l'autre des branches du V, fermant ou ouvrant ainsi le circuit d'excitation des bobines. Ce dispositif, connu sous le nom de Fabre-Bull, est d'ailleurs couramment employé dans certaines pendules électriques.

Mentionnons encore un autre dispositif que nous avons également expérimenté avec succès :

Pour ne laisser qu'une seule action du courant photoélectrique par oscillation double, nous avons fait commander par le contact en V actionné par la deuxième pendule, un relais agissant sur un obturateur placé sur le trajet des rayons lumineux. Une seule fois par oscillation double, l'obturateur s'ouvre avant le passage du balancier par la verticale. Il est alors possible de laisser le circuit plaque des trois lampes en parallèle, fermé en permanence sur les bobines excitatrices, le courant plaque au repos étant rigoureusement nul.

Le réglage de ce dispositif est moins délicat que le premier. Il exige en revanche deux organes de plus, le relais et l'obturateur.

Il est possible cependant qu'il soit préféré lorsque nous réaliserons des appareils définitifs, ceux qui ont été décrits n'étant que des appareils d'étude.

Resultats d'Expériences.—Les courants photoélectriques d'entretien ont été enregistrés pendant plusieurs heures consécutives, et aucune variation de période n'a été constatée à aucun moment, les accumulateurs étant bien chargés au début.

Aucune différence n'a pu être trouvée entre la période des oscillations du pendule auto-entretenu comme il a été dit, et la période des oscillations du même pendule, fonctionnant en pendule entièrement libre sans entretien. La différence entre les deux périodes était en tous cas inférieure à $\frac{1}{10000}$. Le courant photoélectrique, dans le deuxième cas, n'était utilisé que pour l'enregistrement des oscillations.

L'appareil enregistreur employé était un Boulitte à noir de fumée, avec oscillographe Abraham petit modèle. La vitesse de déroulement du papier n'était pas rigoureusement constante, aussi enregistrait-on en même temps les vibrations d'un diapason de $\frac{1}{50}$ de seconde (période de double vibration), entretenu électriquement sans contact matériel, au moyen d'un amplificateur à lampes.

Dans toutes les mesures faites on a trouvé parfois des différences de 1 à 2 millièmes de seconde entre les durées de quelques unes des oscillations, la moyenne demeurant rigoureusement constante.

Il est donc extrêmement probable que ces petites différences ne peuvent pas être attribuées au mode d'entretien et sont dûes à l'imperfection de l'enregistrement graphique (frottement du style, déformation du papier par le laminage pour fixer le noir de fumée), à moins qu'elles ne soient dûes à des influences extérieures (trépidations, petits séismes). La pendule était en effet fixée à un mur de l'Observatoire de Paris, sans précautions spéciales. D'autres mesures seront effectuées, notamment sur des enregistrements photographiques.

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XXIII.—On the Cardinal Function of Interpolation-Theory.

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INTRODUCTION.

THE function which is discussed in the greater part of this paper was introduced by Professor E. T. Whittaker in vol. xxxv (1915), pp. 181–194 of these *Proceedings*.* A one-valued function, analytic in the finite part of the plane save possibly for poles, is taken and a table made of the values of $f(x)$ for $x=a,\ a+w,\ a-w,\ a+2w,\ a-2w,\ \ldots$ The function

$$C(x)=\frac{w}{\pi}\sin\frac{\pi}{w}(x-a)\sum_{r=-\infty}^{\infty}\frac{(-1)^r f(a+rw)}{x-a-rw}, \qquad . \quad . \quad . \quad (A)$$

called the Cardinal Function, is then shown to be the simplest function satisfying the table of values so obtained, in that it is a function which, when analysed into its periodic constituents, has no rapid oscillations.

There are, in general, many functions which satisfy the same table of values, among them the function given by the Gauss interpolation formula

$$f(a+xw)=f_0+x\delta f_{\frac{1}{2}}+\frac{x(x-1)}{2\,!}\delta^2 f_0+\frac{(x+1)x(x-1)}{3\,!}\delta^3 f_{\frac{1}{2}}+\ \ldots, \qquad . \quad (B)$$

where

$$f_0=f(a),\ \delta f_{\frac{1}{2}}=f(a+w)-f(a),\ \text{etc.}$$

Professor Whittaker showed, assuming the convergence of both series, that $C(a+xw)$ is identical with the function given by (B). Subsequent writers have shown, by particular examples, that the series for $C(a+xw)$ may diverge when (B) converges. The present paper (§§ 6, 6.1) examines more closely the relation between the two series and shows that, when the series of type (A) converges, (B) also converges and has the same sum. Further (§ 6.2) it is established that, if the Gauss series is convergent, the cardinal function series is either convergent or has a "generalised sum" (in the sense used by de la Vallée Poussin for Fourier series) equal to the sum of the Gauss series.

The relation of the cardinal function to other types of expansion is considered, and (§ 3) an expression is obtained, subject to certain convergence conditions, for the Gauss interpolation formula as a definite integral.

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* References to this in the present paper are denoted by (I).

In §§ 4.1, 4.3 it is shown that given *any* set of numbers $t(r)$, $r=0, \pm 1, \pm 2, \ldots$, however derived, there is in general an integral function $f(x)$ such that $f(r)=t(r)$. Moreover, the conditions for the existence of $f(x)$ are required, not to deal with curious distributions or wide oscillations of the numbers $t(r)$, but to limit, in *some* definite manner, their order of magnitude for large values of r.

§ 1. *The cardinal function as a limiting case of Lagrange's formula.*

The polynomial of degree n in x which, at the points $x=x_0, x_1 \ldots x_n$, has the values $y_0, y_1 \ldots y_n$ is, by Lagrange's well-known formula,

$$\phi(x)=\sum_i y_i \mathrm{A}_i,$$

where

$$\mathrm{A}_i=\frac{(x-x_0)(x-x_1)\ \ldots\ (x-x_{i-1})(x-x_{i+1})\ \ldots\ (x-x_n)}{(x_i-x_0)(x_i-x_1)\ \ldots\ (x_i-x_{i-1})(x_i-x_{i+1})\ \ldots\ (x_i-x_n)}. \quad . \quad (1)$$

Suppose that $x_0=0$, and that a further set of points $x_{-r}=-x_r$ and a corresponding set of values y_{-r} are introduced. If, further, n be increased indefinitely, the limiting form of A_i is $(i \neq 0)$.

$$\lim_{n\to\infty}\frac{x(x+x_i)(x^2-x_1^2)(x^2-x_2^2)\ \ldots\ (x^2-x^2_{i-1})(x^2-x^2_{i+1})\ \ldots\ (x^2-x_n^2)}{x_i(2x_i)(x_i^2-x_1^2)(x_i^2-x_2^2)\ \ldots\ (x_i^2-x^2_{i-1})(x_i^2-x^2_{i+1})\ \ldots\ (x_i^2-x_n^2)}.$$

Put $x_r=rw$: then $\mathrm{A}_i=\lim\limits_{n\to\infty}(\mathrm{N}/\mathrm{D})$, where

$$\mathrm{N}=x(1-x^2/1^2w^2)(1-x^2/2^2w^2)\ \ldots\ (1-x^2/n^2w^2)/(1-x/iw)$$

provided that $x \neq iw$, and

$$\mathrm{D}=2iw(1-i^2/1^2)(1-i^2/2^2)\ \ldots\ \{1-i^2/(i-1)^2\}[\{1-i^2/(i+1)^2\}\ \ldots\ (1-i^2/n^2)].$$

Now

$$\lim_{n\to\infty}\mathrm{N}=\frac{iw}{iw-x}\left(\sin\frac{\pi x}{w}\right)\frac{w}{\pi},$$

and it is at once indicated that

$$\lim_{n\to\infty}\mathrm{D}=iw(-1)^{i-1}.$$

This latter deduction, resting as it does on a "double limit problem," is somewhat doubtful as it stands: it can, however, be established directly by using the properties of the gamma function, thus:—

The part of D in square brackets may be written as

$$\left[\left(1-\frac{i}{i+1}\right)e^{i}\left(1-\frac{i}{i+2}\right)e^{i/2}\ \ldots\right]\left[\left(1+\frac{i}{i+1}\right)e^{-i}\left(1+\frac{i}{i+2}\right)e^{-i/2}\ \ldots\right];$$

and, using the result of example 3, p. 236, Whittaker and Watson, *Modern Analysis* (3rd edition), it is easily shown that

$$\lim_{n\to\infty} \mathrm{D} = iw(-1)^{i-1}.$$

Thus, when $i \neq 0$, the limiting form of A_i is

$$\frac{w}{\pi}\sin\frac{\pi x}{w}\frac{(-1)^i}{x-iw}.$$

The corresponding limiting form of A_0 is readily shown to be

$$\frac{w}{\pi}\sin\frac{\pi x}{w}.$$

The *cardinal function, i.e.*

$$\frac{w}{\pi}\sin\frac{\pi x}{w}\sum_{r=-\infty}^{\infty}\frac{(-1)^r f(rw)}{x-rw} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

is therefore the limiting form* of Lagrange's parabolic interpolation formula when the values of a function are known at a set of points 0, ± rw.

It is this property which distinguishes it from other possible cotabular functions. The absence of constituents of period less than $2w$, a characteristic of $\mathrm{C}(x)$ prominent in its original mode of derivation, is equally shared by the cotabular functions $\mathrm{C}(x)+\mathrm{K}\sin\frac{\pi x}{w}$: the additional term appears naturally when the method of § 2 is used to obtain this and similar functions.

§ 1.1. If in (1) we put $x_0=0$, $x_r=rw$, and let n increase indefinitely without introducing the points $x_{-r}=-rw$, the limit process is no longer a rigorous one. N/D, when suitably prepared, consists of two factors; the first of these, as $n\to\infty$, tends to the limit

$$(-1)^i\frac{w}{\pi}\sin\frac{\pi x}{w}\Gamma\left(1+\frac{x}{w}\right)\Big/(x-iw)i\,!,$$

while the second is exp. $\{(i-x/w)\log n\}$.

It is somewhat curious, therefore, to find that a total disregard of the factor which tends rapidly to infinity when $iw>x$ gives a highly convergent† series, representing a function which satisfies the given table of values at the points 0, rw. This series is

$$\mathrm{G}(x)=\frac{w}{\pi}\sin\frac{\pi x}{w}\Gamma\left(1+\frac{x}{w}\right)\sum_{r=0}^{\infty}\frac{(-1)^r f(rw)}{r\,!\,(x-rw)}. \quad . \quad . \quad . \quad (3)$$

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[*] This fact was pointed out, but not discussed fully, in a paper by T. Arnold Brown, *Proc. Edin. Math. Soc.*, xxxiv (1915–16).

[†] y_r is assumed to be bounded as $r\to\infty$.

§ 2. *A method of deriving these and similar formulæ.*

Series of type (2) and (3) are closely allied to the class of expansions considered in Whittaker and Watson, *loc. cit.*, p. 134.

Suppose that $f(z)$ is an analytic function of z, with no singularities in the finite part of the plane. Then the function

$$F(z)=f(z)\operatorname{cosec}(\pi z/w)$$

has poles at some or all of the points $z=0, \pm rw$.

Let C_m be the circle $|z|=(m+\theta)w$, where $0<\theta<1$. Then, if a sequence of circles $C_1, C_2, \ldots$ can be found on whose circumferences $|F(z)|$ is uniformly bounded,

$$\frac{1}{2\pi i}\int_{C_m}\frac{F(z)}{z-x}dz=\frac{1}{2\pi i}\int_{C_m}\frac{F(z)}{z}dz+O(1/mw).$$

By the theory of residues, $F(x)$ may be represented by a series which is uniformly convergent, except in the neighbourhood of points $0, \pm rw$, for $|x|<a$, where a is an arbitrary positive number.

If, further, the series

$$\sum_{r=1}^{\infty}(-1)^r\frac{w}{\pi}\left\{\frac{f(rw)}{rw}+\frac{f(-rw)}{-rw}\right\}$$

is convergent and has the sum Sw/π, we obtain from the series for $F(x)$,

$$f(x)=\sin\frac{\pi x}{w}\left[\frac{f(0)}{x}\cdot\frac{w}{\pi}+\sum_{r=1}^{\infty}\frac{w}{\pi}(-1)^r\left\{\frac{f(rw)}{x-rw}+\frac{f(-rw)}{x+rw}\right\}\right]$$
$$+\sin\frac{\pi x}{w}\cdot\frac{w}{\pi}\{f'(0)+S\}.$$

Writing now

$$C(x)=f(x)-\frac{w}{\pi}\sin\frac{\pi x}{w}\{f'(0)+S\},$$

we obtain the series for the cardinal function. The change in the order of the terms from that given in (I) is essential when the series is obtained in this manner.

§ 2.1. Replacing $F(z)$ by $f(z)\Gamma\left(-\frac{z}{w}\right)\Big/\pi$, and making suitable assumptions as to the behaviour of $F(z)$ on circles C_m, we obtain the formula (3).

This method of obtaining formulæ may be extended by using various forms of $F(z)$. The actual formulæ are readily arrived at by putting suitable functions of x outside the summation sign and corresponding numerical factors in the separate terms of the sum.

A useful formula is

$$C_p(z)=\frac{w}{\pi}(z-b)^p\sin\frac{\pi z}{w}\left[\frac{f(0)}{(-b)^p z}+\sum_{r=1}^{\infty}(-1)^r\left\{\frac{f(rw)}{(rw-b)^p(z-rw)}+\frac{f(-rw)}{(-rw-b)^p(z+rw)}\right\}\right] \quad (4)$$

By choosing b, p suitably the series can be made to converge fairly rapidly.

§ 3. *The expression of the cardinal function as a definite integral.*

For simplicity of working we take $a=0$, $w=1$. This is always attainable by the use of the transformation $Z=(z-a)/w$.

Let $a_r=(-1)^r f(r)$, $a_{-r}=(-1)^r f(-r)$, and assume further that Σa_r and Σa_{-r} are both convergent.

If $v_r=1/r$, $\Sigma\,|v_r-v_{r+1}|$ is convergent, and therefore* $\Sigma a_{-r}/r$ is convergent. Hence the series

$$B(x)=\frac{a_0}{x}+\frac{a_{-1}}{x+1}+\ .\ .\ .\ +\frac{a_{-r}}{x+r}+\ .\ .\ .$$

is convergent throughout the x plane, with the exception of points $x=0,\ -n$, while the power series

$$\phi(t)=a_0+a_{-1}t+a_{-2}t^2+\ .\ .\ .$$

has a radius of convergence not less than unity. Moreover,

$$B(x)=\int_0^1 t^{x-1}\phi(t)dt,\quad R(x)>0,$$

since the latter is, under the given hypothesis, integrable term by term.†

Similarly, if

$$B_1(x)=\frac{a_1}{x-1}+\frac{a_2}{x-2}+\ .\ .\ .\ +\frac{a_r}{x-r}+\ .\ .\ .$$

$$\phi_1(t)=a_1+a_2t+\ .\ .\ .\ +a_rt^{r-1}+\ .\ .\ .$$

$$B_1(x)=-\int_0^1 t^{-x}\phi_1(t)dt,\quad R(x)<1.$$

Thus, if Σa_r and Σa_{-r} are convergent,

$$C(x)=\sin\pi x\{B(x)+B_1(x)\}/\pi$$

converges throughout the x plane, and its value is given by

$$C(x)=\frac{\sin\pi x}{\pi}\int_0^1\{t^{x-1}\phi(t)-t^{-x}\phi_1(t)\}dt,\quad 0<R(x)<1\quad .\quad .\quad (5)$$

Thus, in (I) § 5, example 1,

$$B(x)=\int_0^1 t^{x-1}\log(1-t)dt,\qquad B_1(x)=\int_0^1 t^{-x-1}\log(1-t)dt\,;$$

example 2,

$$B(x)+B_1(x)=\int_0^1\{(t^x+t^{-x})(1-t)/(1-t^3)\}dt.$$

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* Bromwich, *Theory of Infinite Series* (1908), p. 205.

† *Cf.* N. Nielsen, *Handbuch der Theorie der Gammafunktion* (B. G. Teubner, Leipzig, 1906), p. 127. In further references this work is denoted by (II).

The integrals are at once evaluated by the standard formulæ given in (II), p. 172 (4), p. 170 (8) respectively.

The method is, of course, applicable to $G(x)$ or any other of the forms suggested in §2.*

§ 4. *An existence theorem and some convergency considerations.*

So far, in this and in most other work on similar topics, an integral function $f(z)$ has been assumed:—its values at $z=a+rw$ give rise to a certain table. The interpolation formulæ of this table have then been considered.

If, however, A_r is a set of values corresponding to the points $x=a+rw$, where the A_r set is derived from *any* given collection of data, it is not at once obvious that an integral function satisfying the given table does exist. In the examples of (I) § 5 such a function is given by the cardinal function. If, say, $a=0, f(rw)=(-1)^r$, when r is positive or zero, but $=0$ when r is negative, the series which formally defines the cardinal function is

$$\frac{w}{\pi}\sin\frac{\pi x}{w}\sum\frac{1}{x-rw},$$

and is certainly divergent for $x=w/2, 3w/2, \ldots$. If an integral function with the given values at the given points does exist, it cannot be represented by the cardinal function series of (I).

The following simple analysis provides a necessary fundamental existence theorem of interpolation-theory when formulæ similar to those in (I) and § 2.1 of the present paper are used.

§ 4.1. THEOREM.—Given an arithmetic sequence extending in both directions $a+rw(r=0, \pm 1, \pm 2, \ldots)$, and a set of values $a_r=f(a+rw)$, we can find an analytic function of z which has no poles in the finite part of the plane and assumes the values a_r for $z=a+rw$, provided that $|a_r|=O(r^{p-\delta})$, where p is some finite real number and δ is positive.

No generality is lost by assuming $a=0$, $w=1$, since we can always effect the transformation $Z=(z-a)/w$.

We first establish that, if $p \geqslant 0$,

$$D_p(z)=(1/\pi)(z-b)^p\sin\pi z\left[\frac{a_0}{(-b)^p z}+\sum_{r=1}^{\infty}\frac{(-1)^r a_r}{(r-b)^p(z-r)}\right] \quad . \quad . \quad (6)$$

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* K. Ogura, *Tôhoku Math. Journal*, vol. xvii, p. 141, has shown that Newton's formula is also expressible as an integral, viz. $-\frac{\sin\pi z}{\pi}\int_0^1 t^{-z-1}(1-t)^z\phi(t)dt$, $-1<R(z)<0$ where $\phi(t)=f_0-t\delta f+t^2\delta^2 f_1-\ldots$. The present integral is, essentially, an expression for the Gauss formula in terms of the tabulated values.

is an analytic function throughout any finite part of the z plane, has the required values at 0, 1, 2, . . ., and the value zero at -1, -2, ...

The $(r+1)^{\text{th}}$ term of (6) may be written

$$\frac{1}{\pi}(z-b)^p \sin \pi z \cdot \frac{r-b}{z-r} \cdot \frac{a_r}{(r-b)^{p+1}} .$$

On the circle $|z| = R$, $|(\sin \pi z)(z-b)^p/\pi|$ is continuous and bounded; let M be its upper bound. Moreover, if $r > 2R$,

$$\left| \frac{r-b}{r-z} \right| \leqslant \left| \frac{r}{r-R} \right| < 2,$$

where b is taken to be real, positive, $< R$.

Thus, from and after a fixed term of the series (6), the terms are, for every z on $|z| = R$, less in absolute value than those of the series

$$\Sigma 2M \, |a_r| / (r-b)^{p+1} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The terms of (7) are $O(1/r^{1+\delta})$, and so (7) is convergent.

Hence (6) converges uniformly on the circle $|z| = R$. Each of its terms is an analytic function in the interior of the circle, and therefore the series represents a function which is analytic throughout the circle and its interior.

Similarly, the series

$$E_p(z) = \frac{1}{\pi}(z-b)^p \sin \pi z \sum_{r=1}^{\infty} \frac{(-1)^r a_{-r}}{(-r-b)^p (z+r)} \quad . \quad . \quad (6, a)$$

represents an analytic function throughout any finite part of the z plane, has the tabular values at -1, -2, . . . and the value zero at 0, 1, 2, . . .

Hence the function

$$C_p(z) = D_p(z) + E_p(z) \quad . \quad . \quad . \quad . \quad . \quad (6, b)$$

has the tabular values at 0, ± 1, ± 2, . . . and is analytic throughout any finite part of the plane.

Corollary.—If the set of values a_r is bounded, we can put $p = 1$; if $a_r = O(1/r^\epsilon)$ where ϵ is positive, we can put $p = 0$ and $b = 0$, when $C_p(z)$ becomes the cardinal function.

§ 4.2. *The convergence of the cardinal function when* a_r *is bounded.*

It is not of course *necessary* that $a_r = O(1/r^\epsilon)$ for the cardinal function series to converge, since the above analysis employs only the roughest of known tests for uniform convergence.

Theorem.—If $\sum_{r=1}^{\infty}(-1)^r a_r$ and $\sum_{r=1}^{\infty}(-1)^r a_{-r}$ are both either convergent or finitely oscillating series, the cardinal function series represents an analytic function.

In this case

$$\left| \frac{1}{z-r} - \frac{1}{z-(r+1)} \right| = O\left(\frac{1}{r^2}\right).$$

The series

$$\left| \frac{1}{\pi} \sin \pi z \right| \sum_{r=0}^{\infty} \left| \frac{1}{z-r} - \frac{1}{z-(r+1)} \right|$$

is uniformly convergent on $|z| = \mathrm{R}$, and by a known theorem (*cf.* G. H. Hardy, *Proc. London Math. Soc.* (II), 4 (1907), 250–1) it follows that

$$\sum_{r=0}^{\infty} (-1)^r a_r \sin \pi z / \pi (z-r)$$

is also uniformly convergent on $|z| = \mathrm{R}$.

As in § 4.1 we can complete the argument and show that $\mathrm{C}(z)$ is analytic throughout $|z| = \mathrm{R}$ and its interior.

We shall show (§ 6) that the Gauss interpolation formula, when suitably treated, certainly converges if the cardinal function series does, so that the above theorem provides a useful *sufficient* condition that the Gauss formula, constructed from a given table of values, should be a convergent series.

§ 4.3. *Some remarks on the existence theorem.*

The theorem of § 4.1 is only typical and may be extended to most modes of behaviour of a_r, a_{-r} as r tends to infinity. Thus a series $\mathrm{C}_1(z)$ whose typical term is, $p > 0$,

$$\frac{1}{\pi} e^{pz} \sin \pi z (-1)^r a_r / e^{pr} (z-r),$$

has the values a_r at points $r (\geqslant 0)$, the value zero at $r (< 0)$, and is a convergent expansion if $|a_r| = O(e^{(p-\delta) r})$ or $O(e^{pr}/r^{\delta})$ where δ is positive. While $\mathrm{C}_2(z) = \mathrm{E}_q(z)$, defined as in § 4.1, is convergent if $|a_{-r}| = O(r^{q-\delta})$, has the value 0 at points 0, r, and the values a_{-r} at points $-r$.

$\mathrm{C}_1(z) + \mathrm{C}_2(z)$ represents an analytic function satisfying the table a_{-r}, a_0, a_r, when the above conditions govern their mode of behaviour as r becomes great.

Finally, such series might be used when a_r is bounded and a highly convergent expansion is required, *e.g.*

$$\mathrm{F}_p(z) = \frac{w}{\pi} e^{pz} \sin \frac{\pi z}{w} \sum_{r=0}^{\infty} \frac{(-1)^r f(rw)}{e^{rpw}(z - rw)}$$
$$+ \frac{w}{\pi} e^{-pz} \sin \frac{\pi z}{w} \sum_{r=1}^{\infty} \frac{(-1)^r f(-rw)}{e^{rpw}(z + rw)}.$$

§ 5. *The Newton interpolation formula in terms of the observed values of the function instead of their differences.*

In attempting to obtain an expansion equivalent to Newton's formula of a type similar to that which defines the cardinal function, difficulties are at once encountered by reason of the behaviour of the quotient

$$\frac{x(x-1)(x-2)\ .\ .\ .\ (x-r)}{z(z-1)(z-2)\ .\ .\ .\ (z-r)} \qquad (8)$$

as r tends to infinity. In the Gauss formula investigation, the quotient there in question tends to the value $\sin \pi x / \sin \pi z$ (*vide* (7) *et seq.* I. p. 191), and an infinite series is obtained which is equivalent to the Gauss formula when they both converge. In the case of the Newton formula all we can do is to obtain a finite series (9), of a type resembling (2), which gives the first $(r+1)$ terms of the Newton expansion. Let

$$B_r(z) = (-1)^r \frac{z(z-1)\ .\ .\ .\ (z-r)}{r!} \sum_{k=0}^{r} (-1)^k \binom{r}{k} \frac{f(k)}{z-k} \qquad (9)$$

Then (9) represents the first $(r+1)$ terms of the Newton formula. It is a polynomial of degree r, and is therefore analytic throughout a circle C_r, $|z| = r + \theta$, $0 < \theta < 1$; also, if $k \leqslant r$, $B_r(k) = f(k)$.

Again, if $B(z)$ be *any* analytic function without poles in the circle C_r, and have the values $f(k)$ at points $z = k$, while x is any point within this circle,

$$B(x) = \frac{1}{2\pi i}\int_{C_r} \frac{B(z)}{z-x}dz$$

$$f_0 = B(0) = \frac{1}{2\pi i}\int_{C_r} \frac{B(z)}{z}dz$$

$$\delta f_1 = f_1 - f_0 = \frac{1}{2\pi i}\int_{C_r} \frac{B(z)}{z(z-1)}dz$$

$$\delta^2 f_1 = \delta f_2 - \delta f_1 = \frac{2!}{2\pi i}\int_{C_r} \frac{B(z)}{z(z-1)(z-2)}dz, \text{ etc.}$$

Hence, from the algebraical identity

$$\frac{1}{z-x} = \frac{1}{z} + \frac{x}{z(z-1)} + \frac{x(x-1)}{z(z-1)(z-2)} + \ .\ .\ .\ + \frac{x(x-1)\ .\ .\ .\ (x-r+1)}{z(z-1)\ .\ .\ .\ (z-r)}$$
$$+ \frac{x(x-1)\ .\ .\ .\ (x-r)}{z(z-1)\ .\ .\ .\ (z-r)(z-x)},$$

$$B(x) = f_0 + x\delta f_1 + \frac{x(x-1)}{2!}\delta^2 f_1 + \ .\ .\ .\ + \frac{1}{2\pi i}\int_{C_r} \frac{x(x-1)\ .\ .\ .\ (x-r)B(z)}{z(z-1)\ .\ .\ .\ (z-r)(z-x)}.$$

Now, since B(z) is assumed to have no poles within C_r,

$$\frac{1}{2\pi i}\int_{C_r}\frac{r!\ B(z)}{z(z-1)\ .\ .\ .\ (z-r)(z-x)}dz$$
$$=\frac{r!\ B(x)}{x(x-1)\ .\ .\ .\ (x-r)}+(-1)^r\sum_{k=0}^{r}(-1)^k\binom{r}{k}\frac{f(k)}{(k-x)},$$

and this becomes zero when we put $B_r(x)$ for B(x). Hence $B_r(x)$ is the sum of the first ($r+1$) terms of the Newton interpolation formula. In fact, $\{B_r(x)-B_{r-1}(x)\}$ is a term of that formula.

The formula (9) is not new but appears to be the only analogue to the cardinal function's relation to the Gauss interpolation formula. In its generalised form the series (9) was, implicitly at least, used by Gauss in deriving the Newton from the Lagrange formula (see *Enc. des Sci. Math.*, Tome I, vol. iv, fasc. 1, p. 135).

Since $r^z(-z)(-z+1)\ .\ .\ .\ (-z+r)/r!$ tends to $1/\Gamma(-z)$, Newton's formula may be written

$$\lim_{r\to\infty}\frac{-r^{-z}}{\Gamma(-z)}\sum_{k=0}^{r}(-1)^k\binom{r}{k}\frac{f(k)}{z-k}.$$

§ 6. *The relation between the cardinal function and the Gauss formula.*

Ogura, *loc. cit.*, p. 138, and Okada, *Tôhoku Math. Journ.*, **20**, 91, have pointed out the insufficiency of the proof of this relationship given in (I). The following analysis shows that if C(z) be an integral function, then the Gauss formula and the cardinal function, constructed from the table satisfied by C(z), are equivalent whenever the cardinal function series converges. The existence theorems of §§ 4 *et seq.* show that there is, in general, an integral function which satisfies any given table of values: we make use of the formulæ of Professor Whittaker's work but denote by C(z) this integral function and not the cardinal function.

That the Gauss formula, when convergent, does not necessarily represent the cardinal function has been shown by Ogura, *loc. cit.*, by means of the example $C(z)=e^{hz}$, $|h|<1{\cdot}762$—the Gauss formula represents C(z) itself, while the cardinal function series does not converge.

§ 6.1. If we write $a=0$, $w=1$ in (7) of (I),

$$C(x)=f_0+x\delta f_{\frac{1}{2}}+\frac{x(x-1)}{2!}\delta^2f_0+\frac{(x+1)x(x-1)}{3!}\delta^3f_{\frac{1}{2}}+\ .\ .\ .$$
$$+\frac{x(x^2-1^2)(x^2-2^2)\ .\ .\ .\ \{x^2-(r-1)^2\}(x-r)}{2r!}\delta^{2r}f_0$$
$$+\frac{1}{2\pi i}\int_\gamma\frac{x(x^2-1^2)\ .\ .\ .\ (x^2-r^2)C(z)dz}{z(z^2-1^2)\ .\ .\ .\ (z^2-r^2)(z-x)}\quad .\qquad .\qquad .\ (10)$$

Take γ to be the circle $|z|=\rho$, $r<\rho<r+1$. Then if $C(z)$ is analytic and has no poles within or on γ, whose radius is supposed sufficiently large to ensure that $z=x$ is within the circle,

$$\frac{1}{2\pi i}\int_\gamma \frac{C(z)\,dz}{z\left(1-\frac{z^2}{1^2}\right)\left(1-\frac{z^2}{2^2}\right)\cdots\left(1-\frac{z^2}{r^2}\right)(z-x)} \quad . \quad . \quad . \ (11)$$

$$= \frac{C(x)}{x\left(1-\frac{x^2}{1^2}\right)\cdots\left(1-\frac{x^2}{r^2}\right)} + \sum_{k=1}^{r} \frac{C(k)}{2\left(1-\frac{k^2}{1^2}\right)\cdots\left\{1-\frac{k^2}{(k-1)^2}\right\}\left\{1-\frac{k^2}{(k+1)^2}\right\}\cdots\left(1-\frac{k^2}{r^2}\right)(x-k)}$$

$$-\frac{C(0)}{x} + \sum_{k=1}^{r} \frac{C(-k)}{2\left(1-\frac{k^2}{1^2}\right)\cdots\left\{1-\frac{k^2}{(k-1)^2}\right\}\left\{1-\frac{k^2}{(k+1)^2}\right\}\cdots\left(1-\frac{k^2}{r^2}\right)(x+k)},$$

the evaluation being effected by the usual method of residues.

The coefficient of $C(k)/(x-k)$ in the above reduces to

$$(-1)^{k-1}\frac{(r!)^2}{(r-k)!\,(r+k)!}.$$

Thus (11) may be written

$$C(x)/x\left(1-\frac{x^2}{1^2}\right)\cdots\left(1-\frac{x^2}{r^2}\right) - C(0)/x - \sum_{k=1}^{r} v_k(r),$$

where

$$v_k(r) = (-1)^k \frac{(r!)^2}{(r-k)!\,(r+k)!}\left\{\frac{C(k)}{x-k}+\frac{C(-k)}{x+k}\right\}.$$

Hence, if $C(x)$ be *any* analytic function with no poles in γ, and $C_r(x)$ denote the first $(2r+1)$ terms of the Gauss formula obtained from the table satisfied by $C(x)$,

$$C_r(x) = x\left(1-\frac{x^2}{1^2}\right)\cdots\left(1-\frac{x^2}{r^2}\right)\left[\frac{C(0)}{x} + \sum_{k=1}^{r}(-1)^k\frac{(r!)^2}{(r-k)!\,(r+k)!}\left\{\frac{C(k)}{x+k}+\frac{C(-k)}{x+k}\right\}\right] \quad (12)$$

and the Gauss formula is the limit of this as r tends to infinity.

It is easily established by means of the extension of Tannery's theorem given by Hardy and Chapman in *Quart. Journ. of Math.*, 42 (1911), 188, that this limit is also equivalent to

$$\frac{\sin \pi x}{\pi}\left[\frac{C(0)}{x} + \sum_{k=1}^{\infty}(-1)^k\left\{\frac{C(k)}{x-k}+\frac{C(-k)}{x+k}\right\}\right] \quad . \quad . \quad . \ (13)$$

whenever this series is convergent.

Or directly, if we write $a_k(r)=(-1)^k\{C(k)/(x-k)+C(-k)/(x+k)\}$ when $r>k$ and $=0$ when $r<k$, and $f_k(r)=(r!)^2/(r-k)!\,(r+k)!$;

(*a*) for a fixed k, $\lim_{r\to\infty} f_k(r)=1$,

(*b*) for a fixed r, $f_k(r)$ is a positive decreasing monotonic sequence,

(*c*) $f_0(r)=1$.

We can, when (13) converges, find a definite N such that when $m \geqslant N$,

$$|\, a_{m+1}(r) + \ . \ . \ . \ a_{m+p}(r) \,| < \epsilon \text{ for every } r,$$

and so, by Abel's lemma,

$$|\, a_{m+1}(r) f_{m+1}(r) + \ . \ . \ . \ a_r(r) f_r(r) \,| < \epsilon \text{ for every } r.$$

The proof that the limit of (12) is (13) may now be completed by the same argument as Bromwich * uses to establish Tannery's original theorem.

The series (13) is the cardinal function of the table satisfied by $C(x)$ but with its terms suitably grouped in pairs; so that we have obtained the theorem "whenever the cardinal function series, arranged as in (13), converges, its sum is equal to the Gauss series when pairs of terms after the first are bracketed together."

§ 6.2. The relationship between the cardinal function and the Gauss interpolation formula retains considerable theoretical interest when the series for the first of these diverges, or oscillates, and its "generalised sum" replaces the ordinary sum of a convergent expansion.

In 1908, de la Vallée Poussin (*Bull. de l'Acad. Belge, Classe des Sci.*, p. 233) introduced

$$\lim_{n \to \infty} \left[\tfrac{1}{2} a_0 + \sum_{k=1}^{n} \frac{n(n-1) \ . \ . \ . \ (n-k+1)}{(n+1)(n+2) \ . \ . \ . \ (n+k)} (a_k \cos kx + b_k \sin kx) \right]$$

as the "generalised sum" of a Fourier series. If this limit is definite and finite, the series may be said to be summable (VP). Césaro's method of summation is less extensive in that a series summable (C, r) is summable (VP) to the same sum, but a series summable (VP) is not necessarily summable (C, r).† [A]

Consider now the cardinal function series

$$\frac{\sin \pi x}{\pi} \left[\frac{C(o)}{x} + \sum_{k=1}^{\infty} (-1)^k \left\{ \frac{C(k)}{x-k} + \frac{C(-k)}{x+k} \right\} \right].$$

By the formula (12) its sum (VP) is

$$\lim_{r \to \infty} \left[\sin \pi x \cdot C_r(x) / \pi x \left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right) . \ . \ . \left(1 - \frac{x^2}{r^2}\right) \right],$$

where $C_r(x)$ is equivalent to the first $(2r+1)$ terms of the Gauss formula. Hence the sum (VP) of the cardinal function series is the Gauss interpolation formula with its terms grouped in pairs, and the former series has a sum (VP) equal to the Gauss interpolation formula whenever the latter is a convergent expansion. [B]

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* *An Introduction to the Theory of Infinite Series* (1908), p. 123.

† R. D. Carmichael, *Bull. Amer. Math. Soc.* (2), vol. xxv (1918–19), p. 113; or T. H. Gronwall, *Comptes Rendus* (June 1914), vol. clviii, p. 1664.

The combination of statements [A] and [B] leads at once to the result:

"If the cardinal function series is summable by Césaro's mean [or, more generally, is summable (C, r)], the Gauss interpolation formula, when terms after the first are grouped in pairs, is a convergent expansion equal to the 'sum' of the cardinal function series."

From the point of view of the theory of divergent series, the existence of a direct expansion for the sum (VP) of any series of the type

$$\frac{\sin \pi x}{\pi}\left[\frac{a_0}{x}+\sum\left(\frac{a_r}{x-r}+\frac{a_{-r}}{x+r}\right)\right],$$

and so, putting $a_{-r}\equiv 0$, and multiplying by $\pi/\sin \pi x$, for any series of the type

$$\Sigma a_r/(x-r),$$

is in itself remarkable.*

The limitation that the Gauss series should be bracketed in pairs is perhaps worth accentuating in view of the fact that the conditions which govern the possibility of the expansion of an integral function as a Gauss series are more stringent than those which govern its expansion as a Gauss series with its consecutive terms bracketed.† The Gauss series as usually written need not converge because the cardinal function series does; the bracketed Gauss series must do so.

§ 7. *A well-known expansion of Cauchy's.*

In the *Exercices de Mathématiques* (1827) "Sur le développement des fonctions d'une seule variable en fractions rationnelles" (Cauchy; Œuvres, sér. ii, tome 7, 326), appears a theorem which shows the true character of the cardinal function series when regarded from the point of view of the theory of residues. "If the finite singularities of a uniform function $f(z)$ are poles only, and we can find a sequence $\rho_1, \rho_2, \ldots$ tending steadily to infinity, such that, as $|z| = \rho_m \to \infty$, $f(z)$ tends to zero (save possibly for certain isolated values of arg. z for which $f(z)$ remains finite), then $f(z)$ is the sum of its polar elements."‡ The cardinal function series is merely this expansion as applied to the function $f(z)/\sin \pi z$ when $f(z)$ is an integral function (assuming that the quotient satisfies the conditions of the theorem), and then multiplying throughout by $\sin \pi x$ to get the cardinal series for $f(x)$.

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* I hope to publish shortly a note dealing with this point.

† Norlünd, *Annales Sci. de l'Ecole Normale* (3), vol. xxxix (1922), p 365.

‡ Dougall, *Proc. Edin. Math. Soc.*, vol. xviii (1900), p. 78, has used this theorem to obtain some useful expansions in the theory of spherical harmonics.

If $f(z)/(z-b)^p \sin \pi z$ satisfies the conditions of the theorem, and $f(z)$ have the values a_r, a_o, a_r, at points $r, 0, -r$ respectively, we see that, in § 4.1,

$$\text{Series } (6, b) = f(x) - \mathrm{R}(b)(x-b)^{p-1} \sin \pi x,$$

where $\mathrm{R}(b)$ is the residue of $f(z)/(z-b)^p \sin \pi z$ at the point $z=b$.

§ 7.1. The following general theorem connects this type of expansion with the suitable Gauss interpolation expansion.

If $f(z)/\phi(z)$ be an integral function of z such that $f(z)/\phi(z) \sin \pi z$ satisfy the conditions of Cauchy's theorem (given above), then, in any region not containing the poles of $\phi(z)$,

$$f(x) = \frac{\phi(x) \sin \pi x}{\pi} \left[\frac{f(0)}{x\phi(0)} + \sum_{k=1}^{\infty} (-1)^k \left\{ \frac{f(k)/\phi(k)}{x-k} + \frac{f(-k)/\phi(-k)}{x+k} \right\} \right],$$

and the series on the right is $\phi(x)$ multiplied by the bracketed Gauss interpolation formula for $f(x)/\phi(x)$; if the summation stops at $k=r$, the sum is given by taking only $(2r+1)$ terms of the Gauss formula.

The relationship with the Gauss formula is readily established by the aid of § 6.1. If $f(z)/\phi(z)$ have poles in the finite part of the plane, there will be additional terms due to the residues of $f(z)/\phi(z) \sin \pi z$ at those points, and the relationship with the Gauss formula no longer holds.

(*Issued separately August* 26, 1925.)

XXIV.—The Electrosynthesis of n-Duotriacontane Dicarboxylic Acid. By **David A. Fairweather,** B.Sc., Ph.D., Chemistry Department, Edinburgh University. Communicated by Professor Sir JAMES WALKER, F.R.S.

(MS. received May 15, 1925. Read June 8, 1925.)

THE highest member of the series of normal dibasic acids hitherto obtained by electrosynthesis is hexadecane dicarboxylic acid, $HOOC.(CH_2)_{16}.COOH$.* In attempts to synthesise by this means dicarboxylic acids of still higher molecular weight, the following difficulties are encountered:—(1) The temperature of the liquid undergoing electrolysis must be kept above the melting point of the synthetic di-ester, in order that the latter may not solidify on the anode and prevent the passage of the current; (2) the alkali ethyl salts of the higher dibasic acids are of a soapy nature, and cannot readily be dealt with in concentrated solution. Two of the conditions favourable to electrosynthesis, high concentration of the electrolyte and low temperature, are thus unattainable.

Petersen † synthesised paraffin hydrocarbons by electrolysis of salts of the fatty acids: *e.g.* potassium stearate gave tetratriacontane:—

$$2CH_3 . (CH_2)_{16} . COOK = 2K + 2CO_2 + C_{34}H_{70}.$$

The difficulty of electrolysing a soapy solution was met by using aqueous alcohol as solvent, and avoiding alkalinity by adding from time to time an alcoholic solution of the free acid. A repetition of this work readily gave good yields of hydrocarbons; but when a solution of sodium ethyl hexadecane dicarboxylate was electrolysed under identical conditions, no synthesis took place.

By reducing the quantity of alcohol to the minimum necessary to avoid frothing, the synthesis was ultimately effected as represented in the following equation—

$$2EtOOC . (CH_2)_{16} . COOK = 2K + 2CO_2 + EtOOC . (CH_2)_{32} . COOEt.$$

EXPERIMENTAL.

Diethyl hexadecane dicarboxylate was obtained by electrolysis of sodium ethyl sebacate. This was half saponified with sodium hydroxide in cold dilute alcoholic solution, an operation requiring great care if the formation of a preponderance of disodium salt is to be avoided. A complication is introduced by the low solubility of this ester in cold alcohol (2·4 grms. at 15°, and less than 1 grm. at 5° per 100 c.c.). A solution of 30 grms. in 3 litres of rectified spirits was cooled to 5°, and sufficient alcoholic sodium hydroxide added to half saponify only 20 grms. The temperature was kept at 5° for three days, and then at 10–15° for a week, after which time neutrality was reached. The disodium salt (5 grms.) was filtered off, and the sodium ethyl salt (10 grms.) isolated in the usual way.

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* Brown and Walker, *Proc. Roy. Soc. Edin.*, xvii, 297.

† *Zeit. für Electrochem.*, xii, 144.

Hydrogen Ethyl Hexadecane Dicarboxylate.—The acid was precipitated from a solution of the sodium salt on addition of hydrochloric acid. It was purified by several recrystallisations from aqueous methyl alcohol, and was found to melt at 71°. The silver salt was prepared and analysed. Found Ag = 23·82; $C_{20}H_{37}O_4Ag$ requires Ag = 24·00 per cent.

Electrolysis.—A solution of 10 grms. of the sodium salt in 45 c.c. of water sets to a soapy mass at about 40°. It was not found possible to electrolyse this, on account of the frothing which took place. Alcohol (5 c.c.) was therefore added, the temperature raised to 70–75°, and a current of 2–3 amps. passed for three hours. Water and alcohol were added from time to time to replace loss by evaporation.

The operation was conducted in a beaker, the electrodes consisting of a flat spiral of thick platinum wire (anode) and a sheet of platinum foil (cathode), with a spiral glass cooling coil sandwiched between them.

The oil which had collected on the surface of the liquid set to a white solid (2·5 grms.) on cooling. This was separated from the soap which, however, retained a further quantity (2 grms.) of product in solution. It was extracted by evaporating to complete dryness, powdering, and treating with ether in a Soxhlet apparatus.

The dried solid ester melted at 62°, 20 grms. being obtained. A separation into two portions was effected by taking advantage of the smaller solubility in ether of one constituent.

1. Melting point, 30–35°; weight, 15 grms. This was not unsaturated, and appeared to be a mixture: its nature was not determined.

2. Melting point, 80° (corr.); weight, 5 grms. A molecular weight determination by Rast's camphor method gave a value of 610, showing that this was the expected synthetic ester (m.w. 595). Found C = 76·55, H = 12·58; $C_{38}H_{74}O_4$ requires C = 76·70, H = 12·55 per cent. It is sparingly soluble in cold alcohol, petroleum ether, and also in ether, from which it crystallises in microscopic rhombohedral plates.

The ester was saponified by boiling with alcoholic potash, the potassium salt separating out after a short time. It is soluble in water, giving a soapy solution. Found K = 12·5; $C_{34}H_{64}O_4K_2$ requires 12·7 per cent.

The acid separates out when an aqueous solution of the potassium salt is saturated with carbon dioxide. It is insoluble in water, and only very sparingly so in most of the organic solvents. It can be recrystallised from chloroform, when it appears as a white crystalline powder melting sharply at 123° (corr.).

(*Issued separately August* 26, 1925.)

XXV.—Some Points in the Anatomy of Dicksonia. By Samuel Williams, M.Sc., Lecturer on Plant Morphology, Glasgow University. *Communicated by* Professor F. O. Bower, F.R.S. (With Two Plates and Four Text-Figures.)

(MS. received May 12, 1925. Read May 25, 1925.)

Our knowledge of the anatomy of *Dicksonia* is mainly based on the description of *D. Barometz* given by Gwynne-Vaughan.* The stem of this species possesses a radial dictyostele, which, on account of the shortness of the leaf gaps, is not far removed from the solenostelic condition. The leaf trace arises as a broad ribbon of tissue with inwardly folding margins, which soon, however, breaks up to form a horse-shoe of isolated strands.

The anatomy of the larger dendroid species, such as *D. antarctica* and *squarrosa*, does not appear to have been described in any detail. Professor Bower † states that "the structure, even in the large dendroid species, such as *D. squarrosa, antarctica*, and *Schiedei*, appears to be the same as that in *D. Barometz*, but on a larger scale." De Bary ‡ mentions *D. antarctica* as having a similar type of vascular structure to that seen in *Dryopteris Filix-mas*.

Professor Bower recently placed in my hands material of *D. antarctica* and *squarrosa*,§ with the suggestion that the anatomical construction might prove to be of interest.

Dicksonia antarctica.

Dicksonia antarctica is a large dendroid species attaining a height of 30 35 ft. In the case of the material examined the stem was 14 cms. in diameter (excluding the outer covering of roots and old leaf bases, which brought the total diameter up to 23 cms.). In transverse section (Plate I, 1 and text-fig. 1) the stele appears as a curiously corrugated structure surrounding a pith 7·5 cms. in diameter. The latter is entirely devoid of vascular elements, and, apart from a few irregularly disposed sclerotic strands, it is composed entirely of parenchymatous cells which contain

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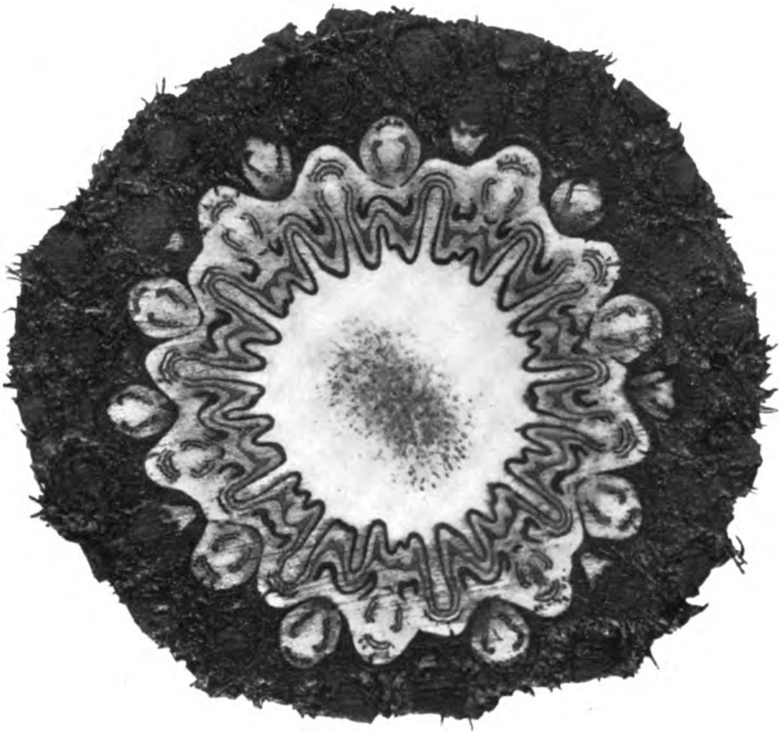
* Gwynne-Vaughan, "Observations on the Anatomy of Solenostelic Ferns," Part II, *Ann. Bot.*, 1903.

† Bower, *Origin of a Land Flora*, p. 595.

‡ De Bary, *Comparative Anatomy*, p. 286.

§ Obtained through the courtesy of Mr Banks, of the Glasgow Botanic Gardens.



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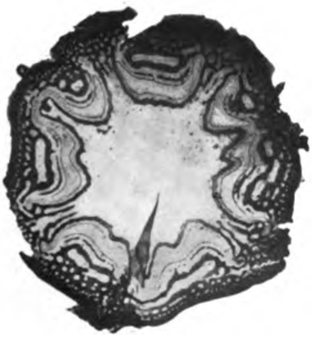

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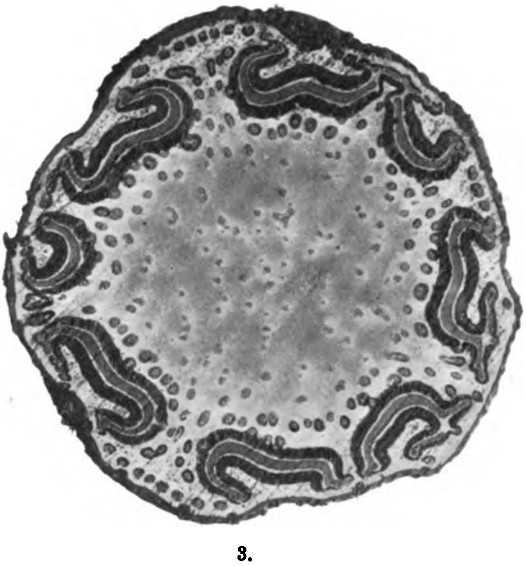

3.

large quantities of starch. The text-fig. 1 indicates without further detailed description the method of leaf-trace departure. It differs only slightly from that described by Gwynne-Vaughan as occurring in *D. Barometz*, the chief difference being that the leaf trace is broken in the peripheral region before its complete departure from the axial strand At a higher level the petiolar strands divide to form a horse-shoe curve of strands, with the open end of the horse-shoe to the adaxial side. The leaf gaps are extremely short, as is indicated by a consideration of the leaf traces *c* and *d* in text-fig. 1, and by the drawing of the dissection (Plate II, 1). They are indeed so short that in some sections no leaf gaps are to be seen, but in the majority of sections one or two leaf gaps are present, and the vascular structure must be regarded as dictyostelic. It appears that, in spite of the close insertion of a large number of leaves round the axis, there is, just as in *D. Barometz*, only a slight advance on the solenostelic condition. An interesting feature is the marked thickening of the xylem ring at the margins of the leaf gaps. In these regions the xylem forms comparatively massive inwardly projecting flanges, the significance of which will be considered later.

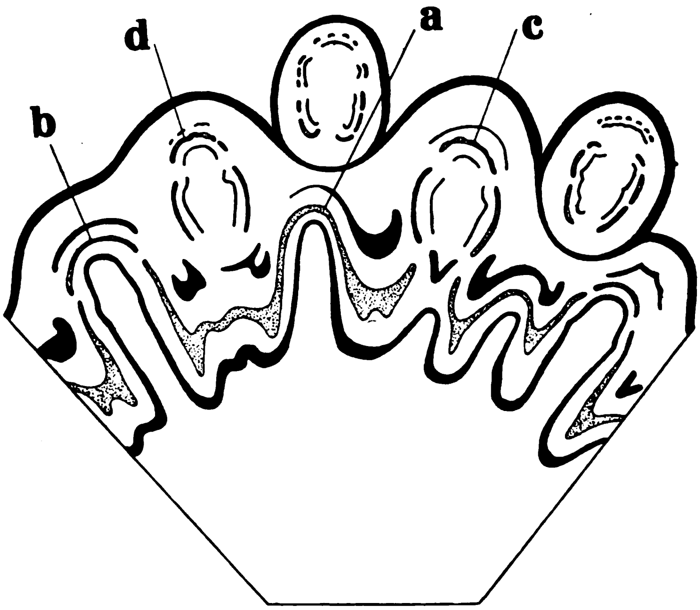

FIG. 1.—*Dicksonia antarctica.* Sector of stem (natural size); *a*, *b*, *c*, and *d*, leaf traces at successive levels. Vascular tissue, stippled; sclerenchyma, black.

The detailed structure of the stele need not be described, since it presents no new features of particular interest.

D. squarrosa.

D. squarrosa is another dendroid species, though somewhat smaller than *D. antarctica.* The material available for examination had a stem 4·5 cms. in diameter (excluding the roots and leaf bases).

The general stelar structure was found to be very similar to that of *D. antarctica,* only on a much smaller scale. The method of leaf-trace departure differs in detail from that seen in the latter, as is shown by the figs. A, B, and C in text-fig. 2. The thickening of the xylem ring at the leaf-gap margins described in the case of *D. antarctica* is again present, though the flanges so formed are by no means so prominent as is the case in the latter species.

Text-fig. 3 indicates the nature of the petiolar trace at successively higher levels in the frond. At the base of the petiole the vascular supply is arranged in the form of a somewhat involuted horse-shoe of separate strands, each strand being surrounded by a sheath of sclerenchyma. There is a broad peripheral band of sclerenchyma, which is, however, interrupted in a lateral position by pneumathodes which give lateral ventilation. The centre of the petiole is occupied by sclerotic tissue (text-fig. 3, A). At higher levels in the petiole the separate strands move together to form a continuous, much corrugated horse-shoe (text-fig. 3, B, C, D, E). The pinna traces depart in an extra-marginal fashion. The detailed structure of the petiolar strand (Plate II, 2) shows several features of interest. The protoxylem groups are very distinct and occupy the grooves of the corrugated horse-shoe. On the adaxial side of each protoxylem mass a group of large thin-walled cells is present, forming what Russow termed the "cavity parenchyma." De Bary describes this as occurring in the petiolar strands of a number of Ferns,* but its function does not appear to be known. In the phloem there are present small mucilage ducts, which will be mentioned again presently.

Mucilage Ducts.

The presence of mucilage ducts in the Cyathaceæ and Dicksonieæ has been noted for a considerable time. Gwynne-Vaughan † described "remarkable mucilage ducts or vessels that are found in the external phloem of the petiolar strands of *Dicksonia Barometz.*" Schlumberger ‡ states that the

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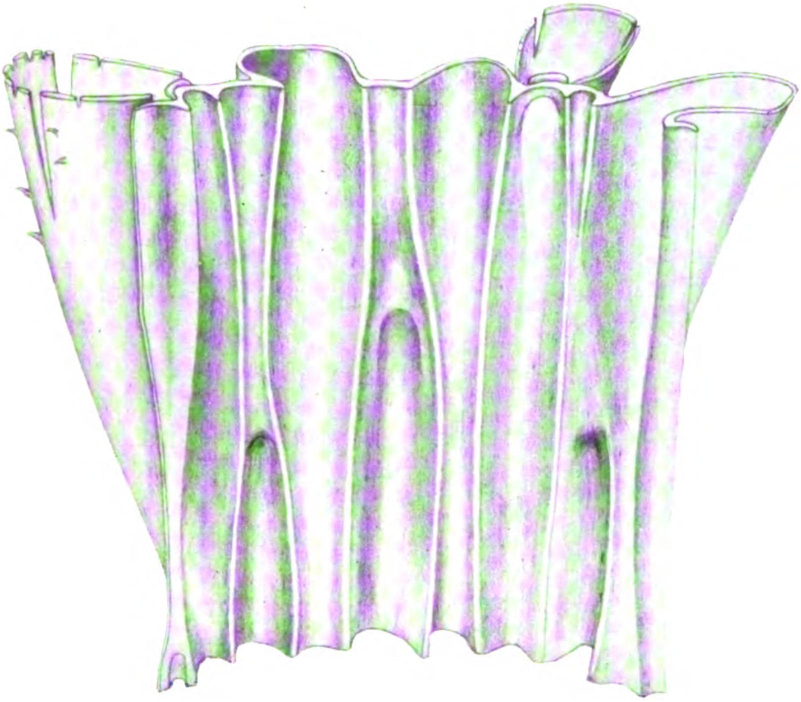
* De Bary, *Comparative Anatomy,* p. 346.

† Gwynne-Vaughan, *loc. cit.,* p. 731.

‡ Schlumberger, Otto, *Flora,* N.F. 2, 1911. Earlier literature on the subject is quoted here.



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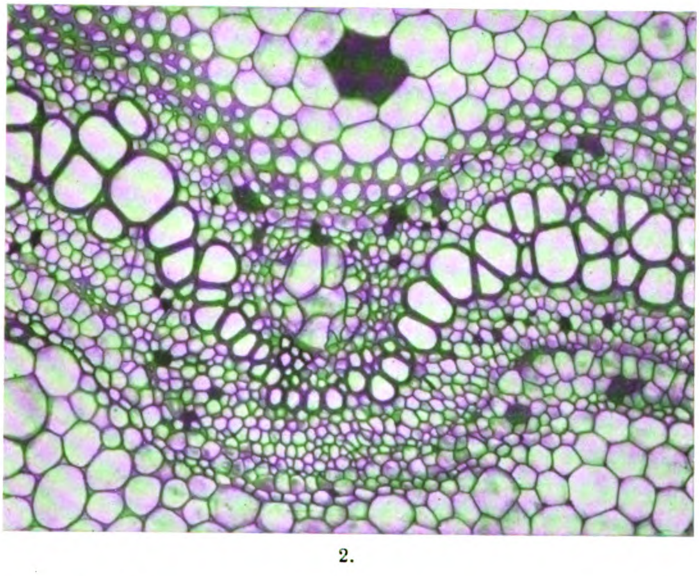

2.

MR SAMUEL WILLIAMS. [Plate II.

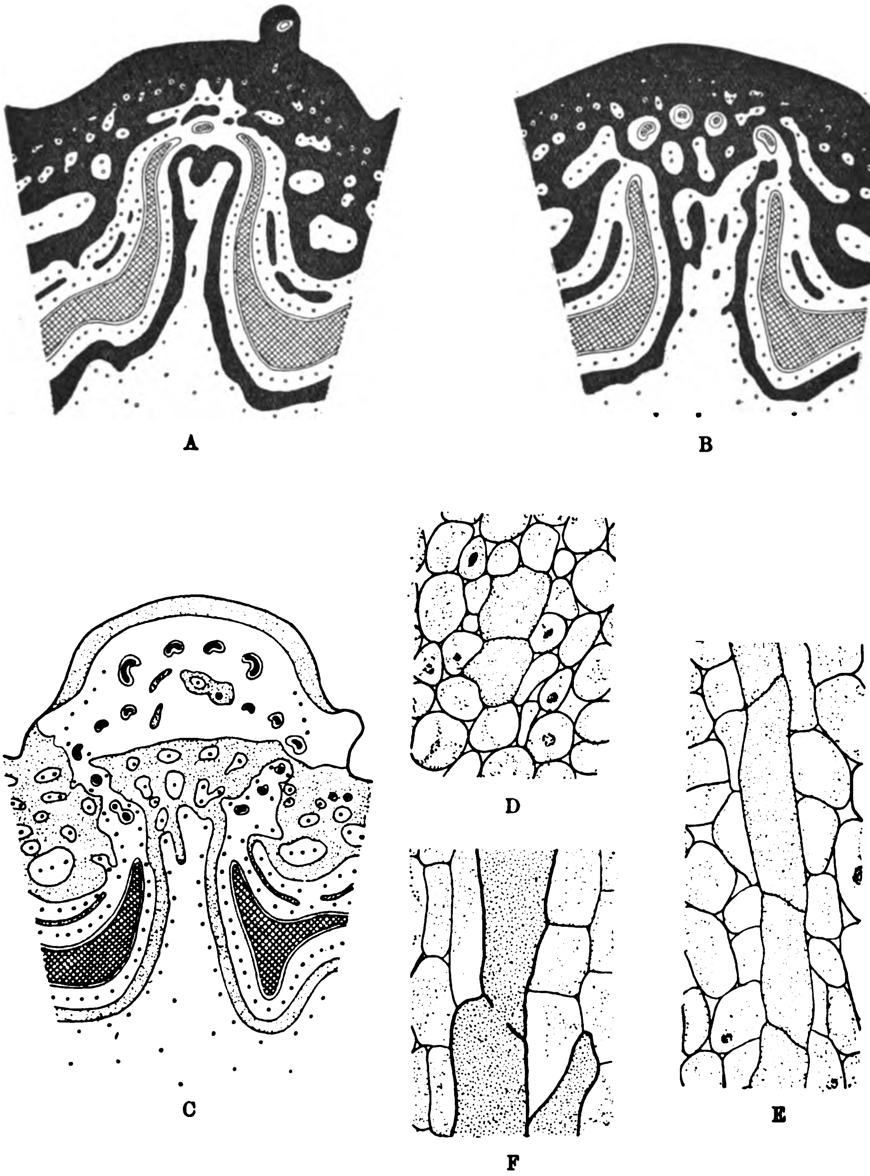

Fig. 2.—*Dicksonia squarrosa.* A, B, and C, leaf traces at successive levels (× 4). Xylem, cross hatched; sclerenchyma, black; in C the imperfectly differentiated sclerenchyma is stippled. Mucilage ducts are shown as small circles. D, mucilage ducts seen in T.S. near the apex; E, the same in L.S.; F, mucilage duct in L.S. in old part of stem, showing transverse wall broken down in central region. (D, E, and F × 60.)

presence of mucilage ducts in the vascular strands and parenchyma is characteristic for the Cyatheaceæ and Dicksonieæ generally, and he describes their structure and development in *D. antarctica*. In the case of *D. squarrosa* they are very abundant and occur in the islands of parenchyma which are present in the otherwise sclerenchymatous cortex, in the parenchyma round about the vascular strand, and, to a less extent, in the pith (text-fig. 2, A, B, and C). In a transverse section a little back from the apex they appear in the situations mentioned as single cells, or small groups of cells, which can be distinguished by their large size and densely staining contents (text-fig. 2, D). In longitudinal section these cells are seen to be arranged in longitudinally running rows which anastomose with one another at intervals (text-fig. 2, E). Further back from the apex the transverse walls become swollen and mucilaginous, and may occasionally break down in their central region to give an open, mucilage-filled duct (text-fig. 2, F). In the case of *D. antarctica*, and in the Cyatheaceæ, this

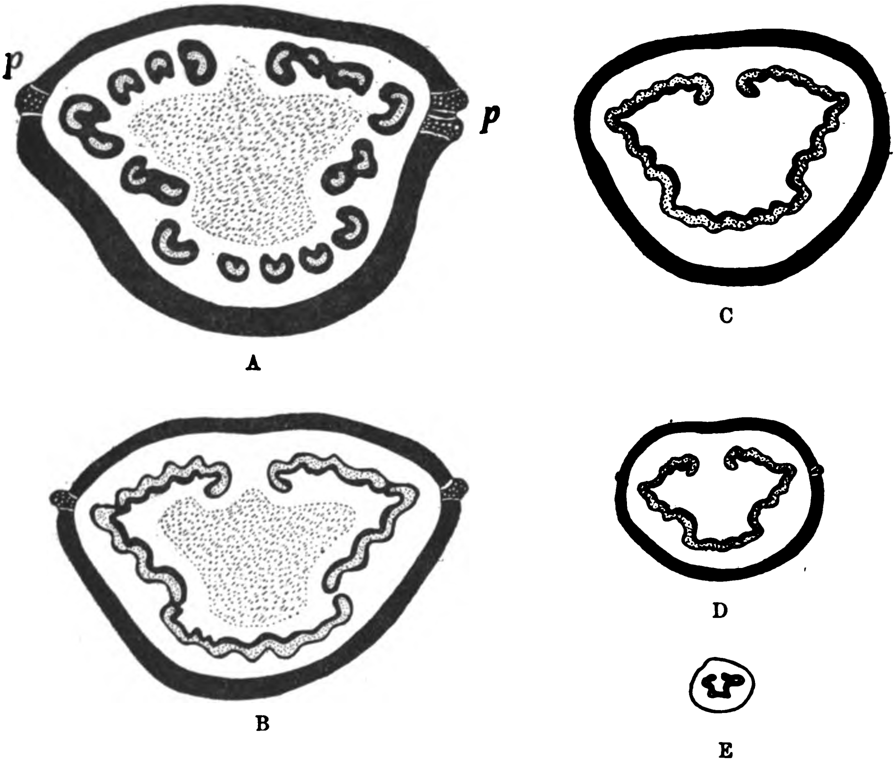

Fig. 3.—*D. squarrosa*, series of sections at successive levels in the petiole (× 4). Vascular strands, stippled; sclerenchyma, black; in A and B the central shaded area represents sclerotic tissue; *p*, pneumathodes.

appears to occur with regularity, but in the sections of *D. squarrosa* examined by me the transverse walls frequently remain intact so that no open duct is formed.

Similar mucilage ducts are present in the parenchyma of the petiole, and smaller ones are present, as in the case of *D. Barometz,* in the phloem of the petiolar strand (Plate II, fig. 2).

Superficial Appendages.

The surface of the stem and petiole of *D. antarctica* and *squarrosa* is clothed with simple hairs. In the case of those of the former there are no features of particular interest. The hairs of *D. squarrosa,* however, which are much stouter than those of *D. antarctica,* show the "equisetoid" structure, first described in the case of the stout hairs occurring on the stem and leaves of *Botryopteris forensis.* Externally there appears to be a ring of small teeth at the distal end of each cell, but further examination * shows that the explanation given by Professor Bower † of this appearance in the hairs of *Botryopteris forensis* applies here also. The "equisetoid" appearance is actually due to the frilling of the peripheral region of each transverse septum.

THE OPERATION OF THE SIZE FACTOR.

The large size of the two species described above leads one to consider how the Size Factor ‡ operates in such extreme cases.

Professor Bower § has described a corrugated stele as occurring in *Pteris incisa,* and has shown how in this case the corrugation is due to the operation of the Size Factor, arising from the necessity for an increase in the proportion of surface to bulk in large steles. In the case of *D. antarctica* and *squarrosa* there is a corrugated stele bearing a superficial resemblance to the structure described for *Pteris incisa.* In the case of these *Dicksonia* species it is clear, however, that the corrugation is immediately due to the very oblique passage of the leaf traces through the cortex, and not to the operation of the Size Factor. In *Pteris incisa* the peculiar configuration of the stele is in no way related to the method of leaf-trace departure, so that the facts for *Dicksonia* do not in any way vitiate Professor Bower's explanation of the structure of this Fern.

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* Williams, "Note on 'Equisetoid' Hairs," *Ann. Bot.*, 1925.

† Bower, F. O., *The Ferns*, vol. i, p. 199.

‡ The operation of the Size Factor is fully discussed by Professor Bower in his paper on "Size, a Neglected Factor in Stelar Morphology," *Proc. Roy. Soc. Edin.*, vol. xli.

§ *Loc. cit.*, p. 14.

The slow passage out of the leaf traces in the case of *Dicksonia,* leading to the corrugation described, can, however, be regarded as an indirect manifestation of the operation of the Size Factor, and its effect on the general stelar structure is indeed much the same as in the examples given by Professor Bower, leading, as it does, to a much increased ratio of surface to bulk. In relation to this question the inwardly projecting flanges of the stele already mentioned are of significance, since they still further increase the surface of interchange between the stele and the surrounding living tissues. It is interesting to note that in the case of *D. squarrosa,* where the diameter of the stele is comparatively small, the inwardly projecting flanges are much less markedly developed than in the very much larger stele of *D. antarctica.* Such local thickenings are known in a number of Ferns, and Gwynne-Vaughan regarded them as being the first indication of the tendency to form the internal accessory strands which are present in the stems of a number of large Ferns. *Dicksonia antarctica* and *Cyathea medullaris* (Plate I, fig. 3) present an interesting contrast in the way in which they have solved the problem of maintaining an adequate ratio of surface to bulk in the case of their very large steles; in the former by corrugation and the formation of inwardly projecting flanges, and in the latter by the curving of the meristeles and the formation of accessory strands.

Although in the stem of *Dicksonia* the issue is somewhat confused by the method of leaf-trace departure, the influence of the Size Factor is very clearly seen in the structure of the petiolar strand, which falls into line with the cases already described by Professor Bower, such as *Metaxya* and *Thyrsopteris.* The corrugation of the strand greatly increases the surface of interchange between the vascular strand and the surrounding tissues.

The operation of the Size Factor is seen also in the detailed structure of the bulky xylem masses present in the two species of *Dicksonia* examined. An examination of these masses shows that the xylem is not homogeneous, but that practically every tracheid comes into contact, at some point or other of its surface, with one or more living parenchymatous cells.* This fact seems to have some physiological significance, especially in the light of recent research on the problems of translocation of water and elaborated food-stuffs within the plant. C. W. Wardlaw † has recently demonstrated very clearly that in *Psilotum, Tmesipteris,* and *Lycopodium* the distribution of the xylem in the various species can be explained by the physiological necessity for the tracheids to remain in contact with living thin-walled tissue. In small steles the xylem forms a little compact mass; in steles of increasingly larger size the xylem is disintegrated in such a way as to maintain the greatest possible surface of contact between the tracheids and the living parenchymatous cells.*

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* The companion cells (Geleitzellen) of Russow, *Vergl. Unter.*

† Wardlaw, C. W., "Size in Relation to Internal Morphology. No. 1—Distribution of the Xylem in the Vascular System of Psilotum, Tmesipteris, and Lycopodium," *Trans. Roy. Soc. Edin.*, vol. liii.

The condition of the xylem in the *Dicksonia* species examined leads to an inquiry as to whether the same physiological necessity is operative in the case of Ferns. De Bary † has already pointed out, mainly on the basis of the observations of Russow, that the xylem masses of Ferns are of two types: those in which the xylem is homogeneous, and those in which the tracheids are intermingled with parenchyma. He states that "the two conditions are distributed according to species and perhaps genera, not according to the form of the bundle." It is therefore of interest to inquire whether the homogeneity or heterogeneity of the xylem is indeed merely a specific or generic character, or whether it is not, as in the case of *Lycopodium,* a result of the operation of the Size Factor.

Homogeneous xylem is stated by De Bary to occur in—

A. *Marsilia* and *Pilularia.*—The xylem is here in the form of a "one to three layered ring of tracheids."

B. *Polypodium vulgare* and *Lingua.*—My own observations on *P. vulgare* show that, except in the case of the smallest meristeles, the xylem mass consists of tracheids intermingled with starch-containing parenchyma (text-fig. 4, B). In the case of the smallest meristeles the xylem mass is small and only about two tracheids thick, similar to the condition seen in *Neocheiropteris,* where the meristeles are also very small (text-fig. 4, C).

C. The petioles of *Asplenium auritum* and *Scolopendrium vulgare.*—I have not examined the petiolar strand of the former, but in the case of *Scolopendrium vulgare,* where the petiolar trace is in the form of two C-shaped strands at the base of the petiole and an X-shaped strand at a higher level, the xylem mass is never more than two or three tracheids in thickness. It may be noted in the case of branching varieties of *Scolopendrium vulgare* that the xylem mass often becomes bulky in the region just below the branching, and in such bulky masses the tracheids do become intermingled with parenchyma.

It is clear from the above examples, and from a number of others which

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* Wardlaw, *loc. cit.*, in which the bearing of these observations on the physiology of translocation is discussed and need not be repeated here. The literature bearing on the subject is also there quoted.

† De Bary, *Comparative Anatomy*, p. 344.

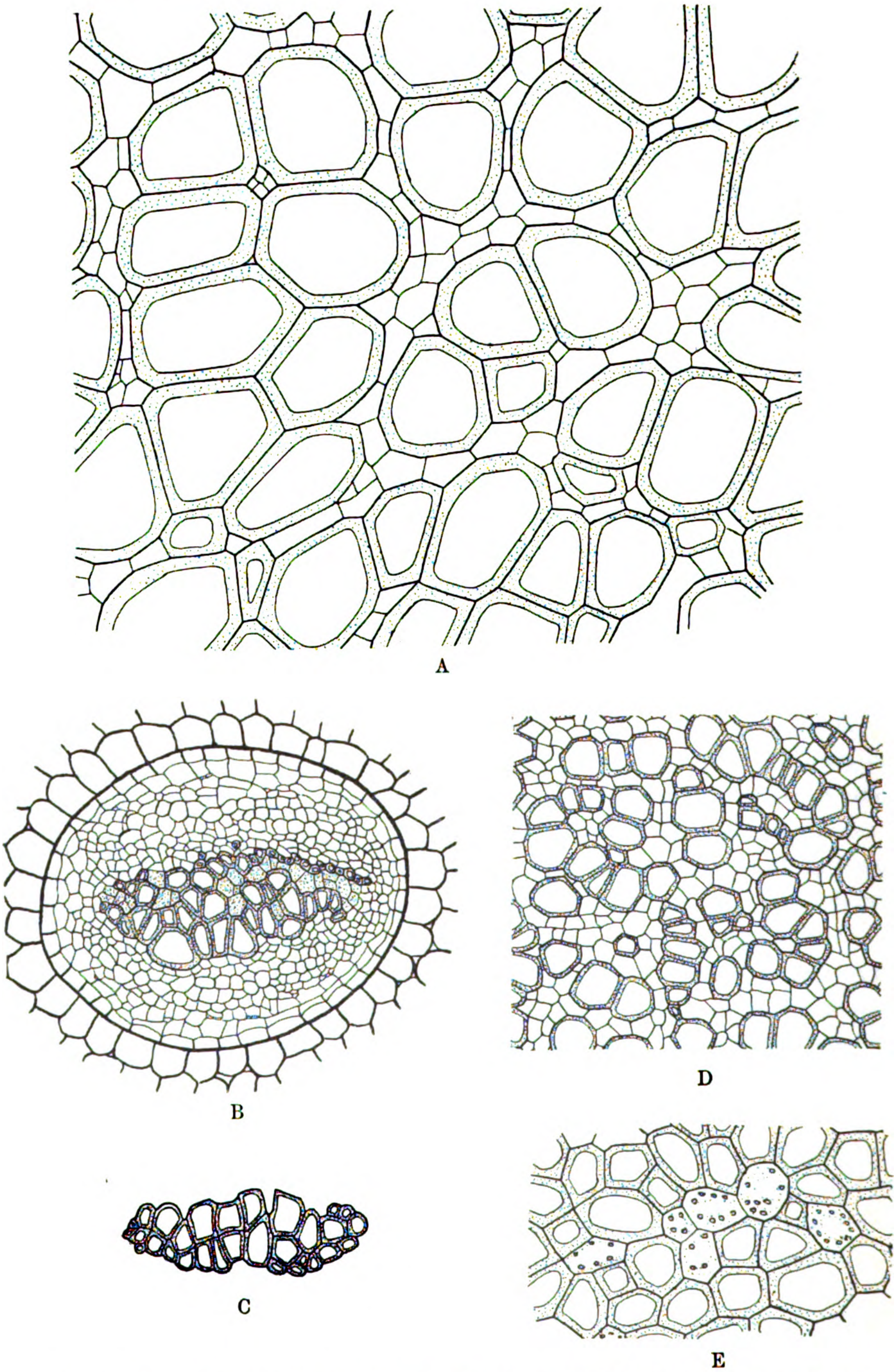

Fig. 4.—A, portion of xylem of *D. antarctica;* B, meristele of *Polypodium vulgare*; C, xylem mass of a small meristele of *Neocheiropteris* sp.; D, portion of the xylem in the protostele of *Gleichenia flabellata*; E, portion of the xylem in the solenostele of *Metaxya rostrata*. (All × 80.)

have been examined, that wherever homogeneous xylem is present, the xylem mass is either small or the distribution of the tracheids is such that practically all the tracheids are in contact with living tissue. Wherever the xylem mass becomes at all bulky, and this occurs to a greater or less extent in the vast majority of living Ferns, the xylem mass becomes heterogeneous (text-fig. 4, A, B, D, E). It appears, then, that it is the size of the xylem mass which is the factor determining the homogeneous or heterogeneous nature of the xylem, and that the latter is not, as De Bary thought, a specific or generic character.

Point is given to the above argument by the structures present in various species of *Cheilanthes* and *Pellaea* described by Marsh.* In *Cheilanthes Fendleri* and *gracillima* there is a narrow band of tracheids, not broken up by parenchyma; in *C. lanuginosa* there is a wider band of wood which is somewhat broken up by parenchyma; finally, in *C. persica* and *Pellaea andromedaefolia* there is a broad band of wood and the tracheids are largely intermingled with parenchyma. Facts such as these, taken from closely allied forms, point clearly to the conclusion that increase in the bulk of the xylem mass in modern Ferns is accompanied by an intermingling of parenchyma with the tracheids.

Apparently the only exceptions to this rule are to be found in certain of the fossil Ferns, and in a few of the primitive forms of living Ferns. Comparatively bulky masses of homogeneous xylem are found in *Botryopteris cylindrica* and *ramosa*; in *Asteropteris*; and in the fossil Osmundaceae. This also appears to be the case in *Osmunda* itself, where the bulky xylem masses have little or no parenchyma. The interest of these exceptions lies in the fact that they are all ancient or primitive types.

From the foregoing it is clear that increase in size of the stele in the case of the vast majority of Ferns has been accompanied by—

(*a*) Adaptations to increase the surface of interchange between the stele and the surrounding tissues.

(*b*) Modifications of the xylem mass to ensure constant contact between the tracheids and living parenchymatous elements.

The physiological bearing of these facts has already been dealt with by Professor Bower and Wardlaw and need not be repeated here.

I desire to express my thanks to Professor F. O. Bower for much helpful criticism, and to the Carnegie Trust for their assistance in the production of the illustrations.

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* Marsh, A. S., "The Anatomy of some Xerophilous Species of Cheilanthes and Pellaea," *Ann. Bot.*, 1914, p. 677. *Vide* figs. 7, 8, and 9.

DESCRIPTION OF PLATES.

PLATE I.

1. *D. antarctica*, transverse section of the stem (half natural size).
2. *D. squarrosa*, T.S. stem (natural size).
3. *Cyathea medullaris*, T.S. stem (half natural size).

PLATE II.

1. Drawing of a dissection of a portion of the dictyostele of *D. antarctica.*
2. T.S. portion of petiolar strand of *D. squarrosa* (× 80).

(*Issued separately September* 24, 1925.)

XXVI.—On the Vertical Force Changes during the "Sudden Commencement" of a Magnetic Storm. By A. Crichton Mitchell, D.Sc. (With One Plate.)

(MS. received May 25, 1925. Read June 22, 1925.)

About six years ago I communicated to the Society an account* of some measurements made at Eskdalemuir Observatory of the more minute changes which are almost continually occurring in the vertical component of the earth's magnetic field. The method adopted was that of connecting the terminals of a coil of wire, whose axis is vertical, to a suitable galvanometer on which was measured the current induced in the coil by changes in the vertical component of the terrestrial field.

Since then, the apparatus has been improved and a large number of observations have been made with it. These are now under examination, but some time must elapse before any general results are published. Meanwhile, a record obtained of the "sudden commencement" of a very moderate magnetic storm which occurred on the 4th September 1924 is regarded as being of sufficient interest to warrant separate publication.

The coil consists of a four-cored insulated armoured cable laid on the surface of the ground to the west of Eskdalemuir Observatory. Approximately, the cable was laid in a horizontal plane, and enclosed an area of about 139,300 square metres. The exact position of the cable has not as yet been determined by survey, but it is believed that its area projected on a vertical plane is negligibly small compared with the projected horizontal area. Each core of the cable consists of copper wire, 0·122 cm. diameter, and the four cores are placed in series. The armouring (galvanised iron wire) is cut at two points in order to minimise the effects of possible currents which might flow in it. The terminals of the cable are connected by leads to an Ayrton-Mather galvanometer within the Observatory. The period of the galvanometer in open circuit is 7 seconds. It is of the utmost importance that the galvanometer should be adjusted to a periodicity, and this has been effected by the inclusion in the circuit of a resistance of 480 ohms, the cable resistance being 116 ohms, that of the galvanometer 91 ohms.

The deflections of the galvanometer are recorded on a strip of photographic paper, which moves at a rate of about 1 centimetre per minute.

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* Not published.

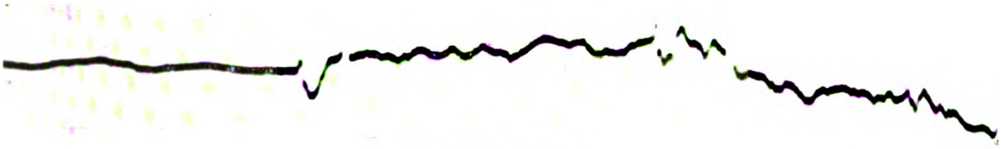
Time marks are made on the record by means of a shutter cutting off the beam of light to the galvanometer mirror, the minute marks being 2 seconds long, the hour marks 16 seconds. The shutter is actuated by an electro-magnet, the current for which is switched on by the Observatory clock, regulated to Greenwich Mean Time.

The current in the galvanometer is proportional to the rate of change of the vertical component of the earth's field, to the area of the coil in a horizontal plane, and to the number of cores in the cable. It is inversely proportional to the sum of the resistances in the circuit. From the value of these quantities and from the known sensitivity of the galvanometer, it follows that a deflection of 1 centimetre on the record represents a rate of change in V equal to 0·0164 γ per second ($1\,\gamma = 10^{-5}$ C.G.S.). From such a record of $\frac{dV}{dt}$ in terms of the time it is possible by graphical processes of integration to obtain another record of V in terms of the time.

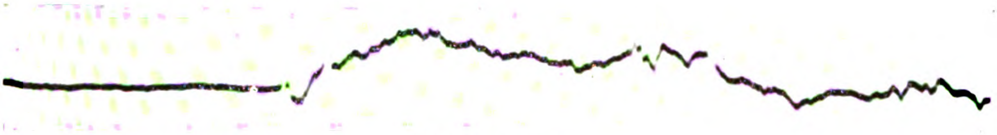
Fig. 1 is a reproduction of the records obtained from the ordinary magnetographs of the Observatory between 4 h. and 10 h. G.M.T. on 4th September 1924. These instruments show changes which take place in the values of the three geographical components (North N, West W, and vertically downwards V) of terrestrial magnetic force at the Observatory. The sudden commencement now under discussion is shown as a sharp movement, shortly before 6 h., in each of the three co-ordinate directions, the change in V being small, but just perceptible. In this case, as in all other cases hitherto recorded at Eskdalemuir (*i.e.* since 1st January 1911), the change in V has been a diminution in value during the earliest part of the phenomenon. In other words, a sudden commencement, at least at this station, involves the action of an upward vertical force during a very short interval of time.

Fig. 1 (Plate) is given here in order to show the sources of information which have been available hitherto for the study of this phenomenon. It is clear that from these records little can be obtained with regard to (1) the exact time of occurrence, (2) the magnitude of the change in V. (In the case of the horizontal components the changes are generally larger, and have been measured in a considerable number of cases.) For more detailed information other means must be employed.

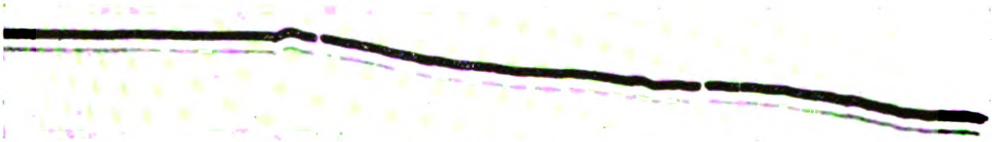
Fig. 2 (Plate) is a photographic reproduction of the cable record taken between 5 h. 42 m. and 6 h. G.M.T. on 4th September 1924. For some hours before 5 h. 45 m. the trace had only shown slight fluctuations. It will be noticed that at 5 h. 45·3 m. a sharp downward movement begins, which may be taken as the time of the sudden commencement.



North Component.



West Component.



Vertical Component.

G.M.T., 4th September 1924.

Fig. 1.

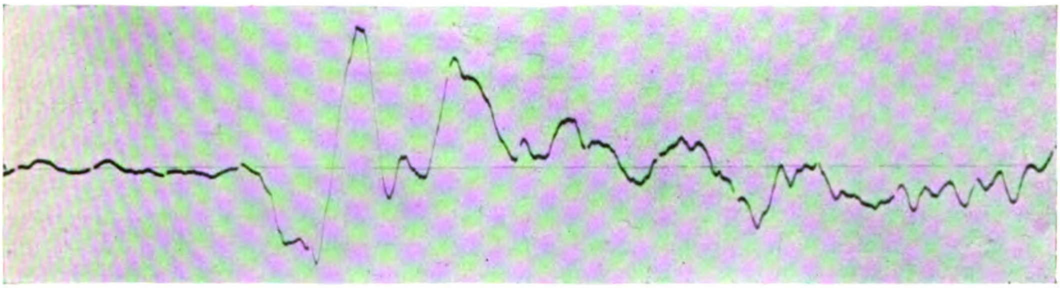
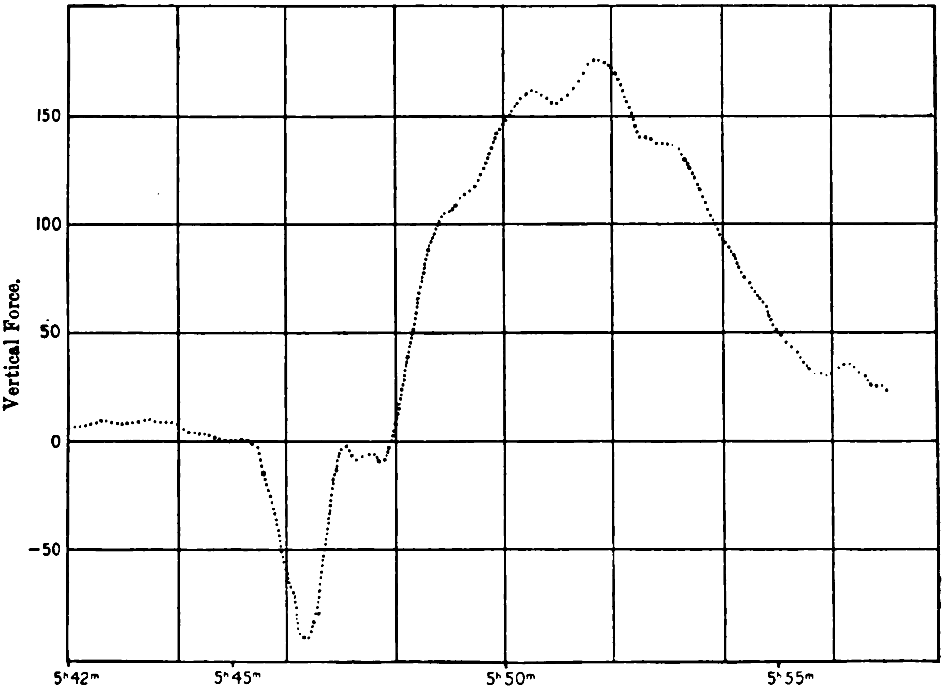


Fig. 2.

Dr A. Crichton Mitchell.

Fig. 3 is a curve whose ordinates represent changes in V as deduced from the values of $\frac{dV}{dt}$ in fig. 2. The integration was carried out by measuring the record in fig. 3 at intervals of 0·1 minute from 5 h. 42 m. to 5 h. 57 m., plotting these readings on squared paper, counting the squares between each tenth of a minute, and adding them successively.

Fig. 3 contains several features of interest. First, it is a reproduction on an enormously enlarged scale of the barely perceptible movement on the ordinary V magnetogram. Second, it shows that this particular sudden commencement consisted of two parts; the first part being an increase of upward directed vertical force, followed by a return to the undisturbed value, the increase occupying 1 minute, the recovery about 0·6 minute; the second part is an increase of downward directed vertical force, lasting nearly 4 minutes, followed by a decrease for nearly the same time. These two parts are separated by a tolerably quiet interval of about 1 minute, during which the value of V was nearly normal and steady.



G.M.T., 4th September 1924.

Fig. 3.

The record of this sudden commencement raises more questions than it settles, and to some of these a brief reference may be made.

1st. As far as is known, this is the first occasion on which the time of occurrence of a sudden commencement has been determined with an accuracy of one-tenth of a minute. Several attempts have been made to show that sudden commencements are not simultaneous all over the earth, and that they are propagated with a certain velocity. But to those accustomed to deal with magnetograms it is at once obvious that any such attempt must fail in so far as the determination of the time is obtained from the ordinary slow-running magnetograph, whose time-scale is generally about 15 millimetres per hour. Extending the time-scale is not of itself sufficient, for it is also necessary that the magnetograph should be very much more sensitive to change. Last year I brought this matter before the British National Committee for Geophysics, and it was resolved to recommend to the General Assembly at Madrid that observations with specially designed apparatus should be made at different stations in order to ascertain whether sudden commencements are or are not simultaneous. The Assembly has taken action in the direction indicated, and it is hoped that by these means this question will be brought nearer solution.

2nd. The first part of the sudden commencement consists of a bay-like movement downwards on the curve of fig. 3, and it is significant that this movement is not symmetrical about its centre, the recovery to normal being quicker than the departure from it. There is abundance of room for speculation here, and one is tempted to the making of theoretical bricks without the necessary evidential straw. The possibilities are a change in position of a current-carrying medium, a change in the strength and/or direction of a current in a more or less fixed medium, or a combination of these. The phenomena now under discussion seem to point to the first of these rather than either of the others.

3rd. It is not always the case that a magnetic storm follows immediately after the sudden commencement. In the case here recorded, it did so follow; but if it had not, the question arises whether the increase in V shown after 5 h. 47·8 m. would have occurred, or would have occurred at that time. In other words, while it is tolerably certain that the initial drop and recovery in V is an essential part of the sudden commencement phenomenon, it is not so certain that what followed immediately after is also a part of it, especially in view of the separation between the two parts by the quiet interval from 5 h. 47 m. to 5 h. 47·8 m. Further observation is required to settle this point.

4th. Another important question is whether the nature (apart from the extent) of the vertical force changes during a sudden commencement varies with the position of the observing station. It is easy to understand that for three stations—one on the Equator, another, such as Eskdalemuir, not far south of the zone of maximum auroral frequency, and a third well inside that zone—the changes in question might be very different. It is here that we have another reason for deploring the absence of a permanent magnetic observatory in some high northern latitude.

5th. Lastly, the local time of occurrence of a sudden commencement may have an important bearing on the character assumed by the changes taking place during that phenomenon.

It is hoped that further observations which it is intended to make at Eskdalemuir, and possibly elsewhere, may assist towards a solution of these interesting questions.

The actual observations on which this paper is founded were taken by Mr H. W. L. Absalom, B.Sc., A.R.C.S., Assistant Superintendent, Eskdalemuir Observatory. Mr Absalom also prepared the photographs and curves which are now published. I am further indebted to him for other assistance in the preparation of the paper.

Dr G. C. Simpson, F.R.S., Director, Meteorological Office, Air Ministry, has kindly given me permission to utilise the records from Eskdalemuir Observatory.

(*Issued separately September* 24, 1925.)

XXVII.—A Colour-Vision Spectrometer. By Professor W. Peddie, D.Sc.

(MS. received June 8, 1925. Read June 8, 1925.)

It is now a thoroughly well-established fact that a satisfactory match to any coloured light observable in nature can be made by a combination, in suitable proportions, of three standard lights suitably chosen from a spectrum. It is customary to choose for the standards a red, a green, and a blue light of definite average wave-length in each case. The greatest deviation from the average is made so slight that there is no visible difference in colour between the extreme components present in any one standard. A simple way (used by Maxwell) of obtaining these standards would be by means of an ordinary spectroscopic arrangement in which light is focussed on a slit, and is parallelised by a collimating lens, after which it passes through a prism and the object glass of a telescope, on whose focal plane a spectrum is thus formed. A screen placed in that plane, and in which three slits are cut, allows the three standard lights only to pass; and these can, by suitable means of reflection or refraction, be superposed on another screen, or at the focus of an eyepiece used for observation. In this method the relative intensities of the standards can be altered by opening or closing, more or less, the three focal slits. This raises practical difficulties.

In other forms of the apparatus (for example, in Helmholtz's spectrophotometer), the light which passes through the primary slit then traverses a double image prism. In this way two equally bright images of the slit are formed, and their angular distance apart can be altered by change of the position of the double image prism. Consequently two spectra are superposed at the focus of the observing telescope, and their relative brightnesses are alterable in any desired ratio by means of a second double image prism. A single slit placed at that focus will therefore transmit two standard spectrum lights, whose relative intensity and whose wave-lengths can be varied at will. A rough estimate of the intensities is got simply from the angular positions of the two double image prisms; but, as the two images furnish rectangularly polarised beams, the two beams are usually differently acted upon at the refracting surfaces, so that an elaborate correction becomes necessary if very accurate results are needed. A third standard may be supplied by means of a side collimator and a glass reflector placed in front of the prism.

But the fact stated in the first sentence above shows that a match to any spectrum light of wave-length intermediate to the two standards, with a suitable amount of white light added to it, can be obtained by a mixture of the two standard lights, provided that these be appropriately selected. From measurements of the amounts of the two standards, and of the quantity of white required to complete the match, the amounts of the three standards required to make a match for the intermediate light can readily be found. This is the method used in Helmholtz's spectrophotometer, the white light being furnished by a side collimator and reflected off the face of the prism. Different pairs of the three standards are employed in different parts of the spectrum.

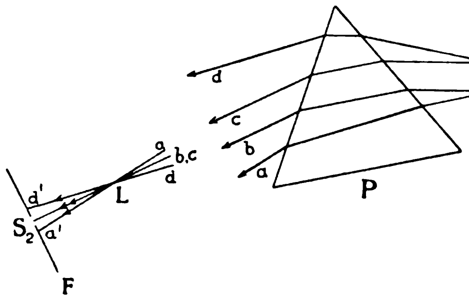
The form of spectrometer now described can also be used in accordance with either of the above methods. The chief aims in its design are similar treatment of the light which forms the different spectra and ease of determination of the intensities.

Whereas, in the form just described, two virtual images of the main slit (S in the diagrammatic sketch) are produced by double refraction, in the present form two real images (I_1, I_2) are produced by a lens, which, for that purpose, is split into semicircular halves by a diametral plane. These half lenses (L_1, L_2) can be slid relatively to each other across the line of vision, as in the heliometer. Thus the angular separation of the images can be varied so as to adjust the amount of overlap of the spectra which are finally formed at the focal plane (F) of the observing telescope. A similar split lens ($L_1{}^1$, $L_2{}^1$) is placed at its own focal distance from I_1 and I_2, and so gives beams of light, each of zero divergence, which traverse the prism, P, and the objective, L, of the observing telescope, in slightly different directions. Thus spectra are formed on the focal plane so that, in general, light of different wave-lengths is focussed in the respective spectra, at one and the same position, S_2, which represents a slit in an opaque screen.

If L_1 is the lower half lens, the light from it will, in the arrangement described so far, illuminate the upper half lens $L_2{}^1$ only, and conversely in the case of L_2 and $L_1{}^1$. Thus two spectra only are superposed at the focal plane of the telescope. So far, therefore, there is no advantage in using two split lenses. A single split lens, with S_1 as its focus, would be sufficient. This simplest form, or the more complicated arrangement as so far described, are suitable for use with the two-standard method of colour matching referred to above.

Instead of a single split lens, an unsplit lens, with a biprism placed behind it, could be used.

For use with very intense sources of light, the images I_1 and I_2 are formed on an Ilford diffusion plate, which scatters light so effectively that the light furnished by both L_1 and L_2 illuminates effectively both halves, L_1^1 and L_2^1, of the second lens. Thus four spectra are in general superposed at S_2; but, if the centres of the semilenses L_2^1 and L_1^1 are at a distance apart equal to I_1 I_2, two of these four will be exactly superposed at S_2, and so the three standard lights required for the first method of colour matching can be obtained. In the sketch this condition is shown for the two parallel rays marked *b*, *c*, the semilenses being represented by their respective centres. The point on the focal plane, F, of the telescope (whose objective is centred at L) to which these rays proceed can be formed by drawing a line through L in their common direction. They are represented as passing through the slit, S_2, placed at the focus. So, if we presume that the slit S_1 is illuminated by green light of a definite wave-length, a green image of S_1 is formed at S_2, due to light which has passed through the lens combinations L_1, L_2^1, and L_2, L_1^1. Green light of the same wave-length which has passed through the combination L_2, L_2^1 is focussed at d^1, while that which has passed through L_1, L_1^1 is focussed at a^1. Consequently, if S_1 is now illuminated by white light, the combination L_2, L_2^1 may focus blue light on S_2, whilst the other focusses red light on it. Thus the light passing through S_2 is compounded of red, green, and blue lights of approximately definite wave-lengths. So, as stated, the apparatus can be employed both for the three-standard and the two-standard methods of determining the colour sensation curves.



When the source of light is not strong, the use of the diffusion plate is practically debarred. In that case, if it is desired to use the three-standard method, a biprism of suitable power is mounted behind each of the semilenses L_1^1 and L_2^1, and is movable with it. These two half lenses with their attached biprisms give the four spectra as formerly, and two of these can be superposed by adjustment of L_1^1 and L_2^1 so as to supply the slit S_2 with compound light of the three-standard wave-length. Strictly speaking, the first pair of semilenses, L_1 and L_2, are not now necessary, and if they were removed the slit S_1 would be placed at the focal plane of L_1^1 and L_2^1, that is, the plane I_1 I_2. But the retention of L_1 and L_2 gives a ready means of ensuring the proper illumination of L_2^1 and L_1^1 without loss of light, when they are slid apart.

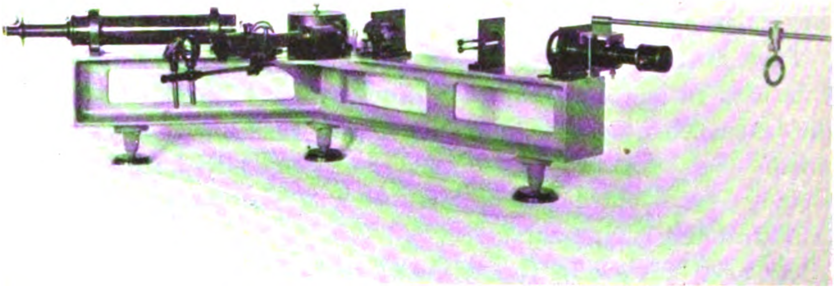
Measurable variation of the intensities of the standards is secured by adjustable opaque circular sectors, mounted centrally behind each semilens and sliding sideways with it. Each sector is double, both segments being rotatable in its own plane, and one can pass behind the other; so that if, for example, the angle of each segment is 30°, any angular area between 30° and 60° can be utilised, the effective area of the semilens being correspondingly reduced. To avoid scattering of light from the cut surfaces of the semilenses the full area of 180° is slightly diminished by a stop attached to each. The stop cuts out a small central semicircular area and a sectorial area of small angle on each side of it. Thus, if the full effective angular area is 170°, while a sector of 45° is used in one quadrant of a semilens, the ratio of the amounts of light transmitted by the two quadrants is 40 : 85. If, in this case, the biprisms were not in use, so that the two-standard method of colour matching was adopted, the effective area of the one semilens would be 125 : 170 of the maximum.

The field of view in the telescope is limited by means of four sliding edges placed at the focus S_2 of the objective. Two horizontal shutters, placed close to the focal plane I_1 I_2 of the front semilenses, can serve the purpose of the vertical limitation of the field of view, but are used merely as screens against stray light, since the adjustments are more readily made at the eye-piece. Correct centering of the eye on the visual line can be aided by a cap, pierced by a small opening, placed over the eye-piece; but any need for this, which is practically non-existent, can be obviated by placing a weak diffusion plate at the focus S_2. The opening serves the purpose of an artificial pupil.

A side collimator mounted in a pair of ring clamps is used to obtain a direct spectrum of the light under investigation for comparison with the compounded light passing through S_2. Light from the source is focussed upon its slit by means of a mirror and lens mounted on a rod attached to the collimator, and is then reflected by a strip of silvered glass so as to pass through the prism and the telescope to the eye.

In this way the slit S_2 is filled in part by the compounded light and in part by light from the comparison spectrum, the two portions being in close vertical juxtaposition. This arrangement suits whether the two-standard or the three-standard method of obtaining colour matches is used. The comparison colour can be varied by rotation of the strip.

When undispersed light from the source is desired for comparison, the strip of silvered glass is mounted behind the prism, and the side collimator is mounted in another pair of ring clamps aligned so that the light is reflected from the silvered strip, and, reaching the slit S_2, is seen in juxtaposition with the compounded light. This undispersed light is also available for use with the two-standard method, in which a match with a spectrum colour intermediate between those of the two standards can only be got by adding some undispersed light to that intermediate colour. In this case a biprism would be fitted to one of the half lenses L_2^1 and L_1^1 only. This half lens would therefore supply the superposed standard lights to S_2; the other would supply the light of intermediate wave-length. The side collimator would add undispersed light to the latter, the intensity of which could be regulated most conveniently by the usually employed method of a rotating sector.



When differential sensitivity is to be investigated, the semilenses L_1, L_2, are rotated through a right angle so that their line of sliding is vertical instead of horizontal, and the biprisms are both removed. L_1^1 and L_2^1 now each supply similar simple spectra of the source, which, by appropriate sliding of L_1 and L_2, can be placed in exact juxtaposition one over the other, so that the eye can detect no difference in the illumination of the upper or lower parts of S_2. L_1^1 may then be moved till a difference is just detectable. Then L_2^1 may be moved until the difference first disappears and then just reappears; and so on successively throughout the spectrum. It is evident also that differential sensitivity with respect to variation of intensity, of purity, and so on, is readily investigable.

The simplest form of the apparatus would be one employing a single biprism, with its obtuse angle truncated symmetrically, and unsplit lenses in an ordinary spectrometer furnished with a side collimator and reflecting strips. This would give three spectra, but the amount of their overlap would be limited, so that the standards would be fixed except in so far as they could be altered by rotation of the telescope. The two-standard method could be used by covering one face of the truncated biprism. This plan can, of course, be utilised in the apparatus described above, so as to block out one of the three standards. With the truncated biprism the estimation of intensities would not be so simple, and the possibilities for research in connection with colour vision would not be so complete.

The accompanying photographic reproduction of the apparatus will make the general construction clear. The total length is about five feet, A wooden cover fits over the whole space between the two split lenses, so as to make it practically light-tight.

I desire to acknowledge the generosity of the Department of Scientific and Industrial Research, and of the Council of University College, Dundee, who shared equally the cost of the apparatus.

(*Issued separately September* 24, 1925.)

XXVIII.—A Ball-and-Tube Flowmeter. By Sir J. Alfred Ewing, K.C.B., F.R.S., President. (With Three Plates and Six Text Figures.)

(Read June 8, 1925. MS. received June 26, 1925.)

About 1876 the author had occasion to devise a method of indicating the rate of flow of a liquid through a pipe system, without interfering with the flow. For this purpose he used a ball, heavier than the liquid, in a slightly tapered glass tube, through which the liquid flowed from the narrower to the wider end. In the first instance the tube was set vertical, with the wider part above. The ball fitted more or less closely at the lower end: it was carried up by the current till the clearance round it became such as to suit the particular rate of flow. Its height was read off on a scale fixed alongside of the tube, and served to indicate the rate of the flow, once the relation of scale-reading to flow had been experimentally determined. It was observed, however, that, when the tube was vertical, the ball was liable to acquire a vibration which disturbed the readings. This instability could be escaped by giving the tube a suitable inclination. The device was not then further examined—other work having intervened—but the author has recently revived it as a means of indicating the rate of circulation of the working substance in refrigerating machines, in connection with an inquiry on which the Engineering Committee of the Food Investigation Board is engaged under his chairmanship, and has made it the subject of a number of experiments, the results of which will be described in this paper. Other obvious uses are to indicate the flow in the supply of liquid fuel to an internal-combustion engine, or the supply of oil in a lubricating circuit. The paper, however, is concerned, not with applications but with the physics of the device.

Tubes with a suitable taper are readily procured from glass-blowers. The author is indebted to Messrs John Moncrieff, Limited, of the North British Glass Works, Perth, for supplying him with several such tubes of gauge-glass quality, capable of safe use under pressures such as are found in boilers. In most of the experiments steel balls such as are made for ball-bearings have been used; they are to be had in closely graded sizes, to suit any tube, and are very accurately spherical.

When the tube is set in a vertical or nearly vertical position, and a gentle stream is turned on, the ball in rising preserves a more or less central position and the flow is distributed in the surrounding annular space, but the effect of casual inequalities is apparent in giving the ball some rotation about a horizontal axis the direction of which suffers frequent and apparently capricious change. With increasing flow, the ball develops instability: a tap on the side of the tube will set the ball into lively vibration in a horizontal plane: it then makes a chattering noise by hitting the sides of the tube in the course of a horizontal rotary movement. The character of this movement is seen by observing the reflection in the polished surface of the ball of a distant luminous point (such as a lamp). The reflected point, viewed from above, appears as a circle or an ellipse. By applying a magnet to the outside of the tube the chattering may be stopped, but when the magnet is withdrawn the ball will again start chattering spontaneously, or will do so when the tube is jarred. A change from the steady to the chattering state affects the scale-reading: the ball rises (sometimes several inches) when it begins to chatter, though the flow is maintained constant, and the ball falls again when the chattering is suppressed. With a fairly strong current the vibration may become violent, and the more violent it is, the higher does the ball rise. The rotation about a horizontal axis—which has been referred to above and is an entirely distinct type of motion from that associated with chattering—also affects the scale-reading, but to a much less degree. It too can be checked by applying a magnet, and the scale-reading may (though with difficulty) be obtained which corresponds to a really steady state on the part of the ball. The readings, however, are so liable to be affected by vibration or rotation of the ball, when the tube is vertical, that in the practical use of the device as a flowmeter the vertical position should be avoided.

The lateral instability which is observed when the tube is vertical or only a little inclined is probably due to differences of velocity in the stream at different parts of the annulus. It appears that in the chattering vibrations the ball comes with a sharp impact against the side of the tube because the pressure on one side has fallen as a consequence of relatively high velocity there. When the ball touches the side the current is stopped at the place of contact, and the hydrostatic pressure there is restored. The ball is accordingly forced towards another position on the tube's side, where the same process repeats itself.

By giving the tube a considerable slope, the tendency to instability on the part of the ball is removed: the readings become definite and satisfactory. The ball then rests quietly on the under side of the tube's bore, keeping a steady position so long as the flow remains constant, and rolling up or down to a new position when the flow is increased or reduced. A slight tendency to wobble sideways (cradle fashion) may sometimes be detected, but this is easily checked, and has no serious effect on the scale-reading. With a tube inclined at any angle θ to the horizontal, the longitudinal force which the current exerts on the ball is W sin θ, and the pressure of the ball against the side is W cos θ, where W is the weight of the ball in the liquid. A slope of 30° to the horizontal answers well, and is recommended for practical applications.

The range of the instrument may readily be extended by placing in the tube two or more balls of different sizes, so selected that when the largest ball is nearing the top of its range the next ball takes up the duty of indicating the flow, and so on. In the apparatus shown there are three balls, resting initially against a stop at the bottom of the tube, so placed that the largest ball (which is above the other two) begins to indicate for a very small flow, but comes against another stop at the top of the tube when the flow has reached a moderate limit. At a flow slightly less than this, the second ball has begun to indicate: it continues, as the flow is further increased, to serve until it comes against the upper stop, and the third ball has then come into action. The balls are all visible, whether in transit or resting against the stops, so that the observer is in no doubt as to which ball is in action. The effect is to secure an extended range and an open scale with a tube of moderate length.

Fig. 1 illustrates a calibration curve, obtained in this case with a tube inclined at 30°. The ball, whose diameter was 14·68 mm., fitted the tube tightly near the small end, and the scale was set with its zero at the place where the ball jammed. The relation of the current to the scale-reading of the ball's position, which is shown by the curve, was observed up to a scale-reading of 27·4 inches, at which the diameter of the tube was 20·0 mm. It will be seen that the observed points lie well on a curve which is smooth except for variations in the rate at which the tube tapers. Direct measurements of the diameter, made by dropping balls of known sizes into the tube and observing where they stuck, confirmed the indication which the curve gives that the taper is relatively small for the first 11 inches or so, and then increases more rapidly. When the ball stood at 27·4 inches the flow was 107 cub. cm. per sec. In this tube balls of about 11 and 8 mm. diameter were found to be suitable for the second and third members of a sequence of three, when it was desired to deal with a larger range of flow.

To examine the nature of the supporting action on the ball, recourse was had to Osborne Reynolds' method of exhibiting stream-line flow by admitting a filament of coloured liquid as part of the stream. Fig. 2 shows the tube and scale supported on a frame that can be set (by help of a clinometer) to any desired slope. The current enters by the pipe A and leaves by B. The upper T-piece carries a cork with a straight wire projecting into the tube to serve as a stop. The lower T-piece allows a small glass

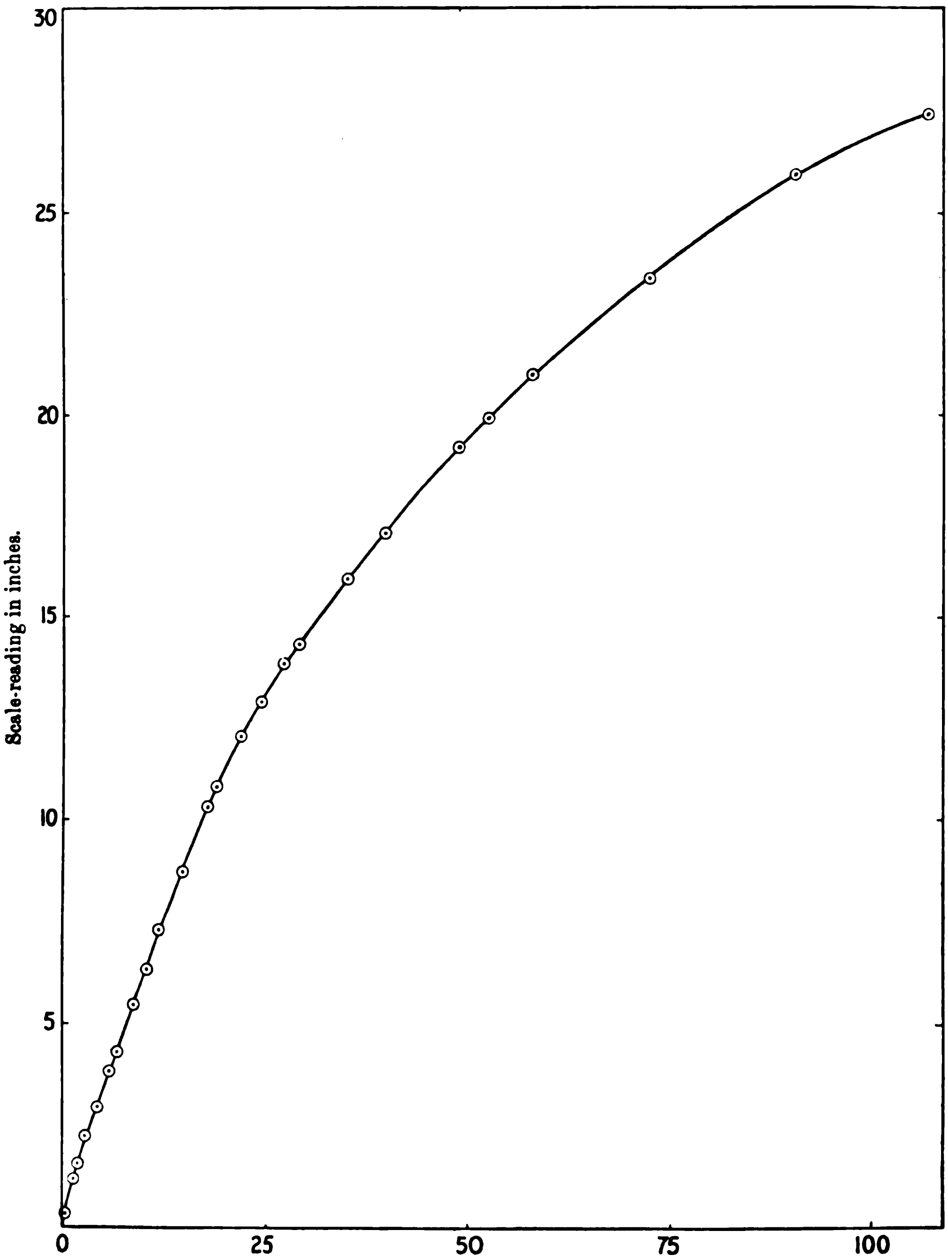


FIG. 1.—Flow in c.cms. per second.

tube to enter, which delivers a coloured filament into the smooth stream a little way up the tube, below the ball. The small tube which delivers the coloured liquid is drawn to a fine point, and the connection to the T-piece is flexible enough to let the delivery be arranged to take place anywhere in the section of the main stream. The coloured liquid (usually water with a little potassium permanganate in solution) comes from an elevated aspirator through the tube C. It was found desirable to take the main stream from a cistern, to avoid fluctuations of delivery.

Under these conditions, if the ball be removed, it is easy to adjust the flow so that a coloured filament extends over the whole length of the tube. When the ball is replaced, and has taken up its steady position, the filament will be seen to bend rather sharply as it approaches the ball, and then to shoot with a greatly accelerated velocity over the top or over a shoulder, preserving, however, a mainly stream-line character for some distance beyond the ball (often 2 or 3 inches), after which the motion becomes turbulent. In the space immediately beyond the ball there is accordingly rapid stream-line motion on the upper side of the tube (with some turbulence at the edges of this rapid stream), and dead-water on the under side of the tube. Beyond that there is general turbulence.

These features are illustrated by photographs in which the camera was so placed as to give a full-size image. Fig. 3 shows a coloured filament approaching the ball just above the axis of the tube: it bends up and becomes attenuated by the acceleration of the stream; shoots over the top, and is clearly seen for a distance of 1 or 2 inches beyond, after which it disappears in the general turbulence.

Fig. 4 shows the effect when the coloured stream-line is directed against a lower part of the ball. It then curls down and sideways, comes back and turns up, following a spiral course, after which it passes over the shoulder of the ball, and is, as before, seen for some distance beyond. The curling of the lower stream-lines before they pass the ball is also shown in fig. 5, where a larger jet of coloured liquid has been admitted. The coloured jet develops into a sheet which can be traced in the photograph as it streams over the shoulder of the ball. The same features will be seen in figs. 6 and 7, which show the action with a thicker tube and smaller ball. Here the continued stream-like movement is seen to extend a greater distance beyond the ball, but general turbulence developed itself a little further on. In these photographs it will be noticed that there is denser shading of the region beyond the ball, due to diffusion of some colour in the dead-water as a result of the turbulence above.

A corresponding experiment with the tube set vertically showed that there was stream-line motion in the region below, up to the ball. This was seen even when the ball was chattering, but in that case there was an immediate development of turbulence when the stream had passed the ball.

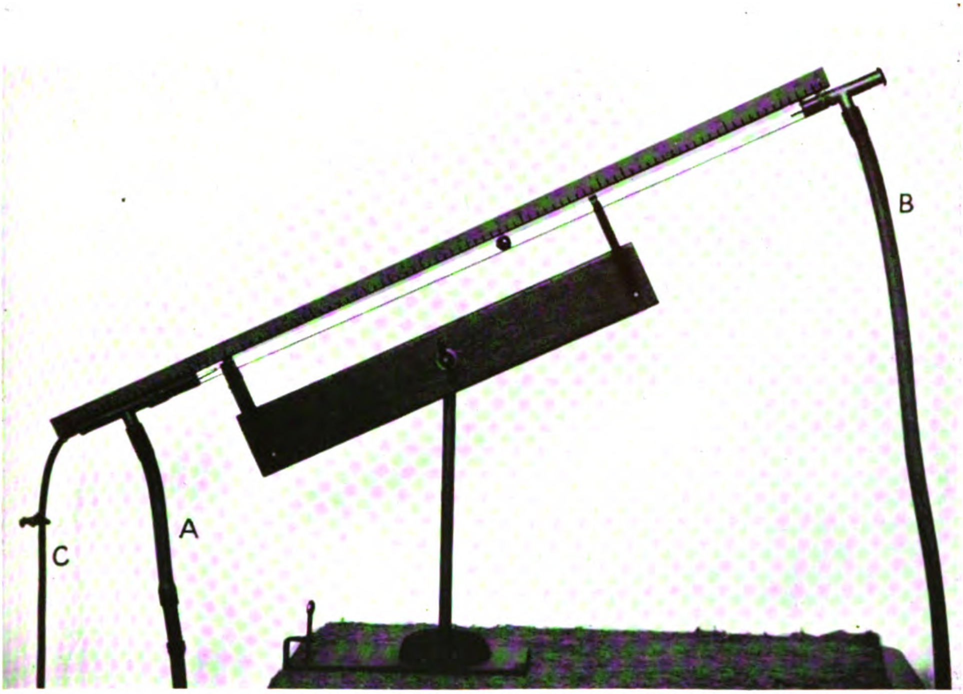


Fig. 2.

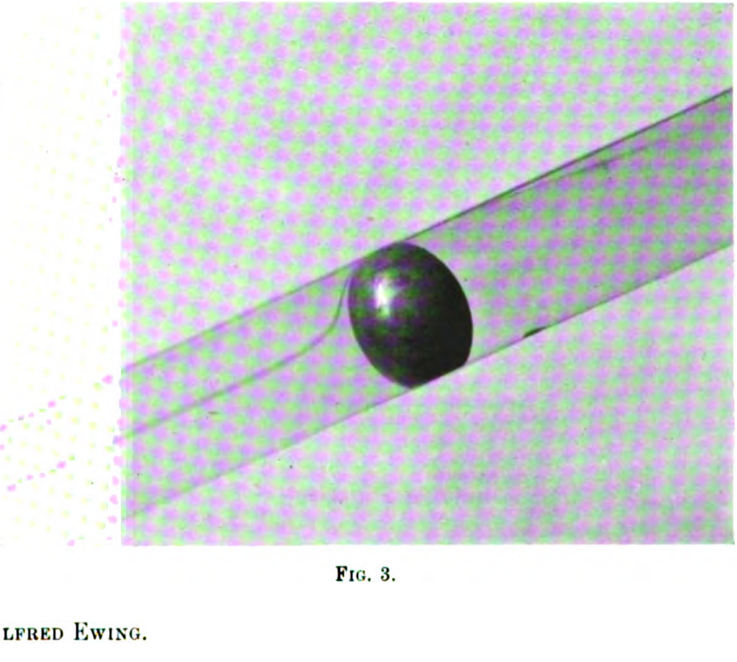


Fig. 3.

Sir J. Alfred Ewing.

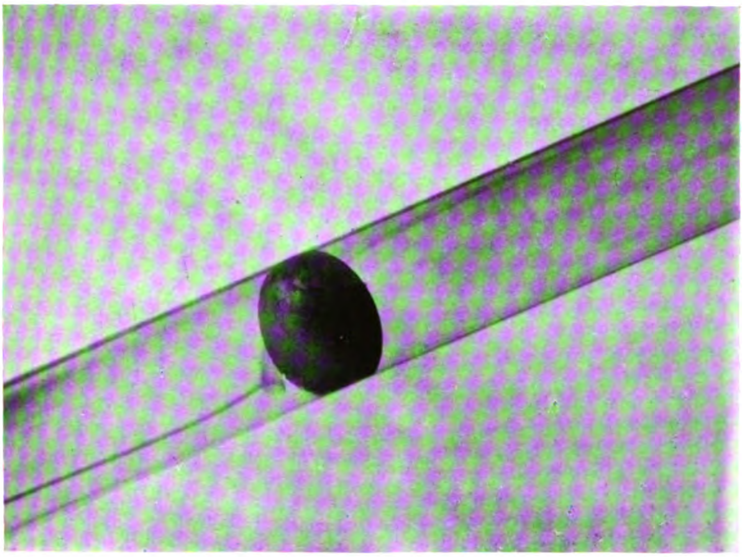


Fig. 4.

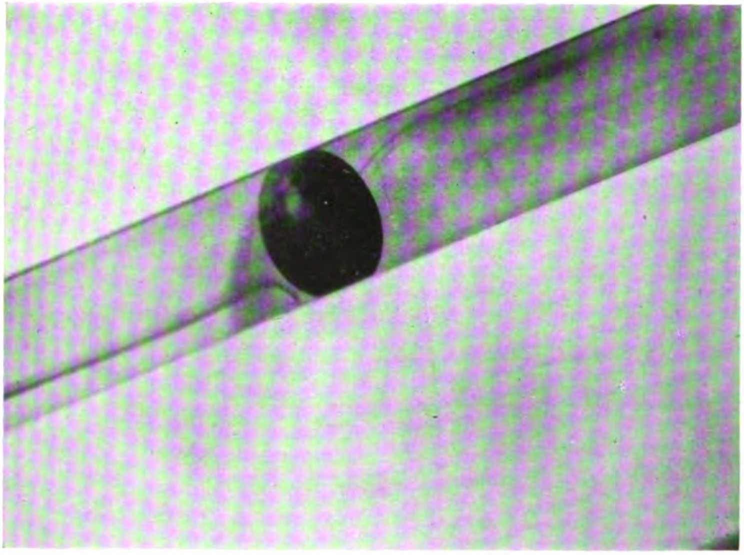


Fig. 5.

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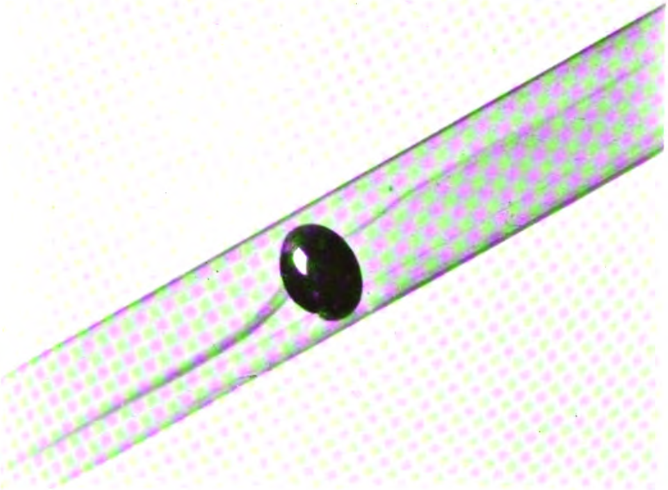

Fig. 6.

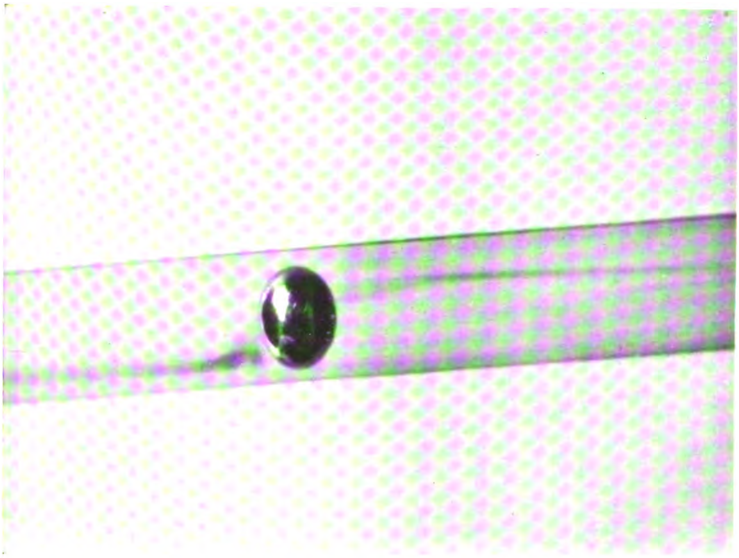

Fig. 7.

Sir J. Alfred Ewing.

With a wide tube and a strong flow there may be some turbulence in the region below the ball, whether the tube is sloping or upright. But in any case the flow past the ball, with acceleration in the narrows and subsequent loss of velocity in the enlarged passage, involves a further development of turbulence in the region above the ball.

The fact that there is, in all circumstances, a development of turbulence as a result of the stream's passing the ball, shows that the sustaining force which the stream exerts on the ball is, in part, an *inertia effect.* The dissipation of energy through turbulence implies that in the region above the ball the hydrostatic pressure is less than it would be if there were no turbulence. If the acceleration in the narrows had been followed by a slowing down without turbulence, that is to say, if steady motion had been maintained throughout the whole action, there would have been a recovery of pressure which does not occur. Hence the existence of turbulence contributes a sustaining force which, *cæteris paribus*, is proportional to the density ρ of the fluid.

An additional sustaining force is contributed by the viscosity. As the fluid streams past the ball it exerts a tangential drag, tending to carry the ball up the tube. Under given conditions of the speed of flow and the geometry of the passage, this force should be proportional to the viscosity μ. It is distinct from the sustaining force already described as due to inertia.

Since F, the total sustaining force acting in the direction of the tube's length, is equal to W sin θ, we may write

$$\mathrm{W} \sin \theta = \mathrm{A}\rho + \mathrm{B}\mu,$$

where A and B are factors depending on the rate of flow and the geometry of the passage, namely the dimensions of tube and ball. $\mathrm{A}\rho$ expresses the part of the force that is due to inertia, and $\mathrm{B}\mu$ the part that is due to viscosity.

Suppose now we examine separately the flow of two fluids, differing in viscosity but not in density (or differing in density but not in viscosity), and determine the angles θ and θ' which will give the same position of the ball along with the same rate of flow in both cases—the only variable being the slope. Under these conditions A and B have the same values for the two fluids, and consequently the relative amounts of the inertia and viscosity effects in each of the two fluids become determinate.

To carry out this idea, the author has made a complete series of observations with two liquids, differing greatly in viscosity. The same tube and the same ball were used throughout the series, and the tube was set at various slopes from 15° upwards. One of the liquids was water: the other was a mixture of glycerine and methylated spirit adjusted to have the same density as water. The viscosity of the mixture (at the temperature of the tests) was 0·137 or twelve times that of water.*

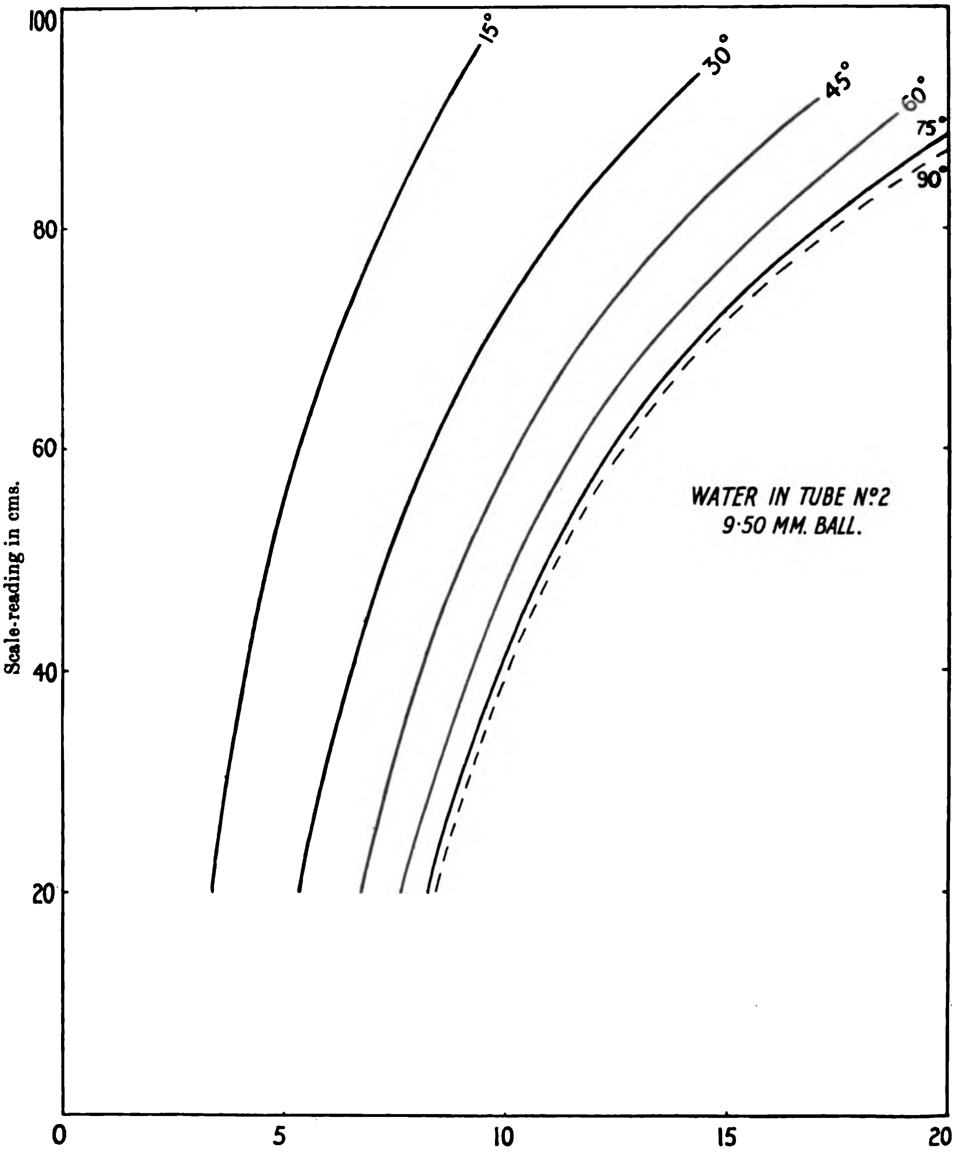

Fig. 8.—Flow in c.cms. per second (water).

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* The author is indebted for the measurement of the viscosity to Dr Ezer Griffiths of the National Physical Laboratory, who, along with his assistant, Mr J. H. Awbery, has taken part in developing and testing the device. Their experiments confirm the author's, which were carried out in Edinburgh.

If we write θ for the angle of slope at which, with water flowing at a particular rate, the ball assumed a particular scale position, and θ' for the angle of slope at which with the viscous mixture the scale position of the ball and the rate of flow were the same, then

$$\frac{\sin\theta}{\sin\theta'} = \frac{A\rho + B\mu}{A\rho + B\mu'},$$

where μ is the viscosity of water and μ' that of the mixture.

Writing r for $\sin\theta/\sin\theta'$ this gives

$$\frac{A\rho}{B\mu} = \frac{r\frac{\mu'}{\mu} - 1}{1 - r}.$$

Values of r were found experimentally by making a series of observations of the flow and scale-reading, at various slopes, separately for water and for the mixture, using the same tube and the same ball. In this series a tube of small taper was used (Tube No. 2), the bore of which increased from 10 mm. to 11·1 mm. in a length of 1 metre. The ball had a diameter of 9·5 mm. The zero of the scale-readings was arbitrary; it did not correspond to zero flow.

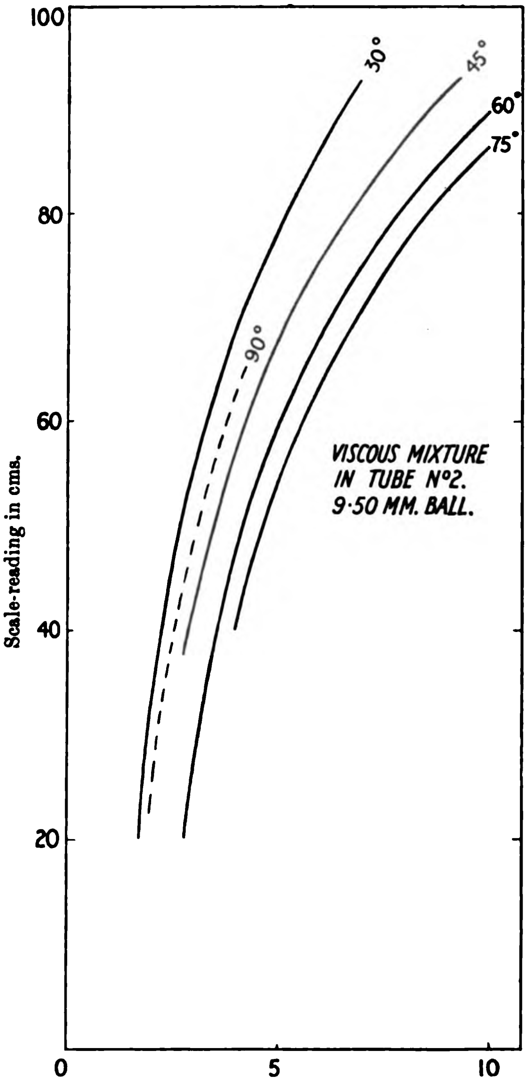

Fig. 9.—Flow in c.cms. per second (viscous mixture).

Fig. 8 gives a series of curves of flow and scale-reading for this tube and ball, at slopes of 15°, 30°, 45°, 60°, 75°, for water, and fig. 9 gives a corresponding series for the mixture, at slopes of 30°, 45°, 60°, and 75°. [Readings in the vertical position, shown by broken-line curves, were also taken, the anomalous character of which will be referred to later.] From these data the curves of fig. 10 have been drawn. They exhibit for several values of the scale-reading (80, 70, 60, and 50 cms.) the relation of the flow to the sine of the slope for water (sin θ), and for the mixture (sin θ')· Curves of this type should, if prolonged backwards, pass through the origin.

On comparing the curves in fig. 10 for water and for the mixture, it will be found, for example, that at scale-reading 80, and with a flow of 8·5 cub. cms. per sec., sin θ' (for the mixture) is 0·966, and sin θ (for water) is 0·332, making the ratio $r = 0{\cdot}344$. At the temperatures of the experiments the viscosity of the mixture was 0·137, and that of water 0·0115. Hence μ'/μ was 12, and it follows that for this particular position of ball and rate of flow the proportion of the inertia effect to the viscosity effect is in water

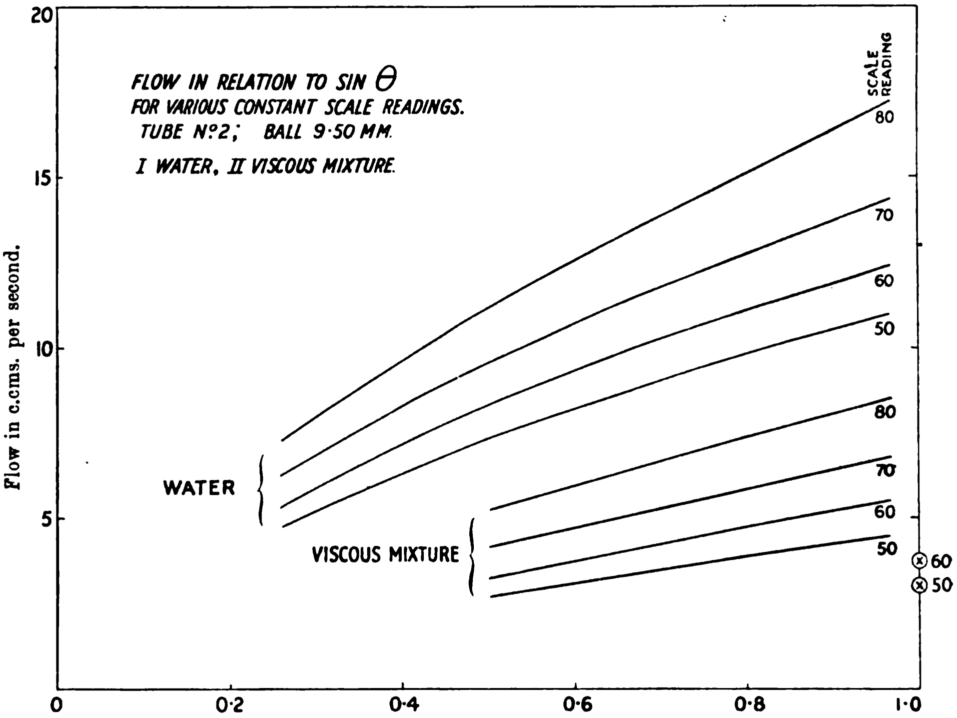

Fig. 10.—Sine of angle of slope.

$$\frac{A\rho}{B\mu} \quad \text{or} \quad \frac{r\frac{\mu'}{\mu}-1}{1-r} = 4{\cdot}8 \text{ nearly,}$$

and in the mixture

$$\frac{A\rho}{B\mu'} = 0{\cdot}4.$$

Thus under these conditions the viscosity effect contributes only 17 per cent. of the whole sustaining force in water, but fully 70 per cent. in the mixture.

When the comparison is made for smaller rates of flow, the relative effects of viscosity are greater, for r becomes somewhat less. The curves of fig. 10 give the following values of r:

At Position 80 cms.		At Position 70 cms.		At Position 60 cms.	
Flow.	r.	Flow.	r.	Flow.	r.
8·5	0·344	6·8	0·309	5·5	0·286
7·8	0·337	6·2	0·299		
7·32	0·331				

Taking the lowest of these values (0·286) one finds

$$\frac{A\rho}{B\mu} = 3{\cdot}4 \quad \text{and} \quad \frac{A\rho}{B\mu'} = 0{\cdot}28,$$

showing that with this slower flow (and slightly smaller area of clearance over the ball) the viscous effect is about 23 per cent. of the whole in water, and 78 per cent. in the mixture.

On the other hand, the relatively greater effects of inertia are apparent when one deals with larger flows in wider tubes. Thus in a similar series of comparative experiments made at the National Physical Laboratory by Mr Awbery with a tube tapering from 13·1 to 15·7 mm., a ball 13·1 mm. in diameter indicating near the top of the tube a flow of about 40 c.cm. per sec. has given a value of r such that the ratio $A\rho/B\mu$ is as high as 7. A and B, which have been treated in the above analysis as constant in a given tube, so long as the flow and the position of the ball are unchanged, are not general constants; the ratio of A to B tends to rise under conditions that increase the flow.

It results from this that, if one expresses the sustaining force, for a given fluid, in the form

$$F = \text{const.} \times (\text{flow})^n,$$

the index n is not constant, even for the same position of a given ball in a given tube. When log sin θ is plotted against log (flow) for tests made at various slopes but at a constant scale-reading, n is found to increase slightly as the flow becomes stronger. Thus the 9·50 mm. ball in Tube No. 2 gave values of n rising from about 1·4 at small flows to about 1·62. With smaller balls in the same tube (and consequently larger flows) the index was more nearly constant, ranging only from 1·63 to 1·65. All the observed values are consistent with the idea that the sustaining force is due to two terms, one of which depends on the square of the flow and the other on the first power.

When the device is used to indicate the flow of a liquid such as oil, where the viscosity is large and is liable to change much with temperature, it may be necessary to apply a table of temperature corrections in interpreting the readings of the scale.

To illustrate the action when the size of the ball is varied, fig. 11 gives representative curves, all taken at scale-reading 80 in Tube No. 2, for the flow of water in that tube at various inclinations, with four different balls of diameters 9·91, 9·50, 8·32, and 7·15 mm. The diameter of the bore at that place was 10·74 mm. The crescent-shaped clearance is shown for the four balls, drawn to double the actual size, in fig. 12. Readings of the flow were observed at slopes ranging from 15° upwards, and the curves of fig. 11 give the results by showing flow in relation to sin θ. Readings were also taken with the tube vertical, and these are separately indicated thus, ⊙, on the line sin $\theta = 1$. It will be observed that for the two larger balls the points so marked conform very nearly to the curves, but for the two smaller balls they lie notably higher. Thus for the 7·15 mm. ball the observed flow (at this constant scale-reading) when the tube was vertical was 47·4 c.cms. per sec., whereas the curve would lead one to expect a reading of about 43·7. Similarly with the 8·32 mm. ball the actual flow is 34·5 instead of, say, 32·4.

Thus, with a ball considerably smaller than the tube, the scale-reading in the vertical position is abnormally low for a given flow of water, when compared with readings taken at various angles of slope. Vibration has nothing to do with this abnormality: the readings in question were all taken with the ball in a steady state.

It is to be ascribed to the difference in the shape of the passage—annulus as compared with crescent—and the consequent change in the sustaining force which the same current exerts on the ball. In these experiments the conditions and the nature of the fluid are such that what has been called the inertia effect contributes the greater part of the sustaining force. It appears that when this is the case, the crescent is more effective than the annulus, for the same area and the same total flow, in holding up the ball. This is probably because in the crescent there is a wider range of velocities throughout various parts of the minimum cross-section at the narrows, and consequently a greater aggregate of elements that depend individually on V^2.

With the much more viscous mixture (of glycerine and spirit) an even more striking abnormality shows itself when the tube is set in the vertical position, but in that case the difference is opposite in kind. The observed point for sin $\theta=1$ then lies notably *below* a prolongation of the curve for smaller values of sin θ. This is shown in the lower part of fig. 10, where the relation of flow to sine of slope is shown by curves for the same ball and for stated scale-readings.

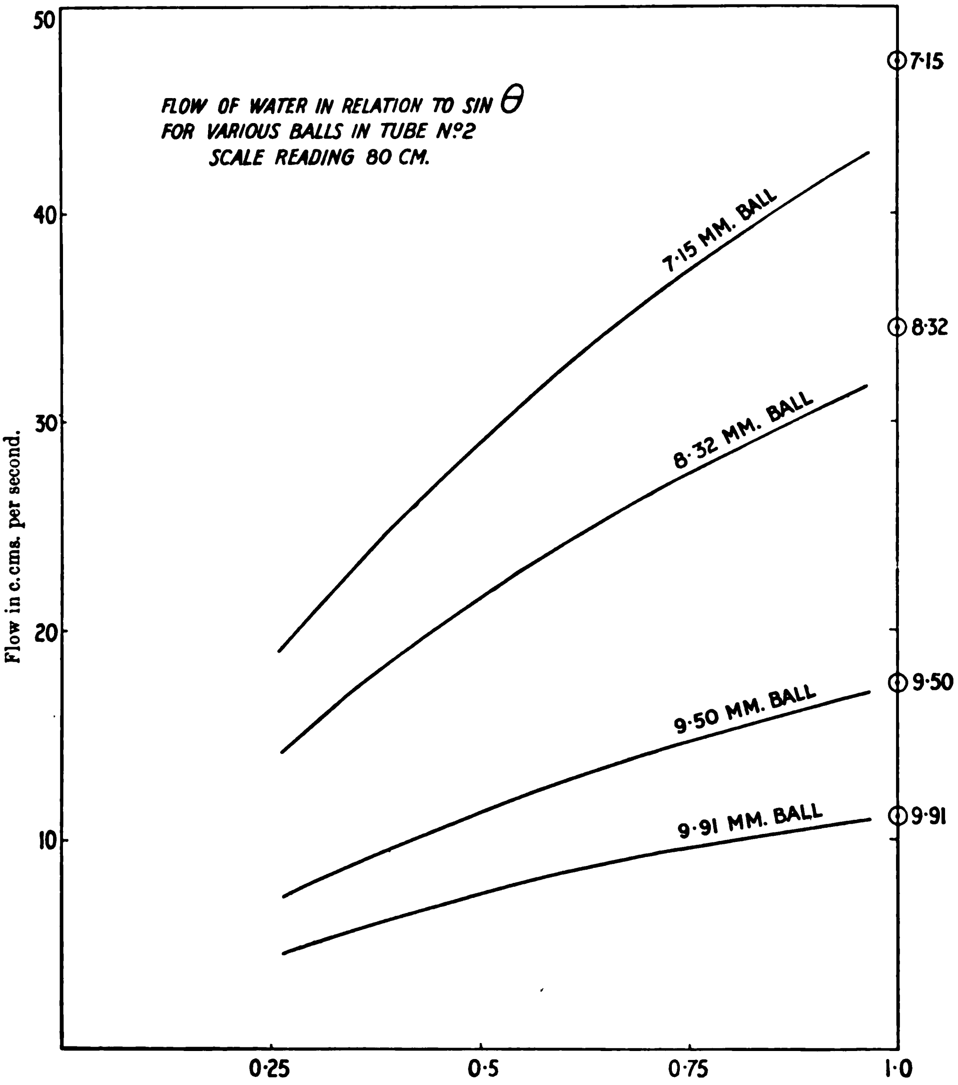

Fig. 11.—Sine of angle of slope.

On the line sin $\theta=1$ two points marked thus ⊗ will be seen, which exhibit the observed flow, of the viscous mixture, for scale-readings of 60 and 50 cms. with the tube set vertical. Each of these points will be seen to lie much below the corresponding curve, showing that with a very viscous liquid the abnormal feature found with water is reversed. The ball is now held at a given scale-reading, in the vertical tube, by a much smaller flow than is required when the tube slopes. In other words, the annulus gives a much greater sustaining force than the crescent for the same rate of flow. This anomaly will be seen perhaps more clearly by reference to fig. 9, where the broken line marked 90° is drawn from observed readings of flow for the vertical tube. Instead of lying to the right of the curve for a slope of 75°, as might be expected, it lies between the curves for 30° and 45°. Here we are dealing with conditions in which the viscosity effect is far more important than the inertia effect. It appears that the drag due to viscosity is less in the crescent, for the same total flow, because it depends on the velocity of those portions of the stream that are contiguous to the ball, and there is a greater excess of velocity in the more central portions of the crescent-shaped area than is found in the annular area. The variations of velocity that occur from point to point in the plane of least area are greater in the crescent than in the annulus, and this operates to make that part of the sustaining force which is due to viscosity relatively less in the crescent, and that part which is due to inertia relatively greater. This consideration also explains why, with water, the anomaly becomes conspicuous only when the ball is considerably smaller than the cross-section of the tube. It is then that the effect of viscosity is relatively unimportant: with a ball that nearly fills the tube the effect of viscosity is so much more considerable that its influence in causing abnormality when the tube is vertical cancels more or less completely the opposite influence (due to inertia) which is dominant with smaller balls. Thus in fig. 11 this approximate cancelling occurs with the 9·91 and 9·50 mm. balls. With the 8·32 mm. ball there is a marked excess in the influence due to inertia, and this is still greater with the 7·15 mm. ball.

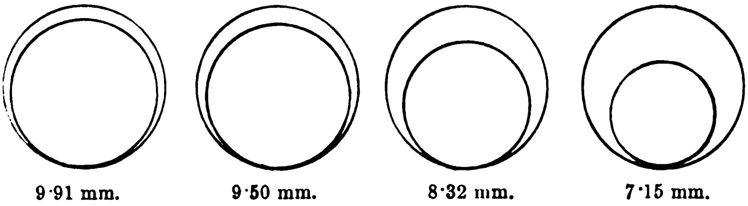

Fig. 12.—Sections of tube with various balls, double full size.

Incidentally these results show that the vertical position is to be avoided, not only because of its liability to vibration but because the action in the annulus may differ markedly from that in the crescent. By giving the tube a sufficient slope one secures not only stability but also regularity in the relation of flow to sin θ.

Note added July 3, 1925.

Since these experiments were communicated to the Royal Society, I have heard from Dr Griffiths that a ball-and-tube device for indicating the flow of petrol to the carburetters of aeroplanes was submitted to the National Physical Laboratory in 1916. The petrol flowed through a vertical glass tube ground internally so that the inner surface was conical, and the flow was indicated by a phosphor-bronze ball, the position of which was read against a scale graduated experimentally to show gallons per hour. The account continues: " If a smooth glass tube is used, it is often possible, by shaking the glass tube, to cause the ball to start a rapid rotary vibration which, when once started, persists. When thus vibrating, the readings are incorrect, as the ball rises too high in the tube. This rotary vibration can be prevented by placing a thin wire stretched tight along the inner surface of the glass tube." Nothing is said of avoidance of vibration by giving the tube a slope.

(*Issued separately September* 24, 1925.)

XXIX.—The Precipitation of Sols by Polyvalent Ions. By W. W. Taylor, M.A., D.Sc.

(MS. received July 3, 1925. Read July 6, 1925.)

Part I.

The precipitation of sols by electrolytes naturally attracted the attention of the early investigators, and certain general relations were established. As these are to be found in any standard textbook on colloids, it is unnecessary to restate them here at length. They may be summarised as follows:—(i) A sol is precipitated by the ion of opposite electric charge, *i.e.* negative sols by kations, positive sols by anions; (ii) the precipitating power increases with the valency of the ion, the ratio for univalent, bivalent, trivalent being $I : X : X^2$, the so-called valency rule; (iii) the other ion of the electrolyte tends to stabilisation of the sol; (iv) excess of the precipitating ion reverses the electric charge of the sol, which is then only precipitated by the ion of opposite electric charge; this occurs more readily with polyvalent ions.

Consequently, the complete sequence of events with increasing amounts of electrolyte is: (*a*) no precipitation; (*b*) partial precipitation; (*c*) complete precipitation; (*d*) partial precipitation; (*e*) no precipitation (reversal); (*f*) partial passing to complete precipitation of the reversed sol by the other ion of the electrolyte.

As a reference to the textbooks will show, there is no little doubt as to whether the effect of polyvalent ions, uncomplicated by other disturbing effects, has ever been investigated. Since the majority of sols are negative sols, especially those which are most convenient to prepare in a comparatively pure state, and are easily observed, *e.g.* gold, arsenious sulphide, etc., a great part of the work is concerned with the action of kations on these. The difficulty is that already the salts of trivalent kations, as $Fe^{\cdots}$ $Al^{\cdots}$ $Cr^{\cdots}$ etc., are not normal electrolytes, their solutions being hydrolysed with production of free acid and basic salt, or positive sols of the hydroxide, if the latter is insoluble. Those which are not hydrolysed are otherwise abnornal, the metal forming part of a complex anion; in some, *e.g.* chromic salts, both complications may occur at the same time. The case of quadrivalent ions is still worse.

Although salts of Ce, La, and the complex cobaltammine salts, which have also been used, are less open to this objection, it appeared very desirable to work with a positive sol and polyvalent anions.

So far as I am aware, no systematic investigation in this direction has been made, for the alkali orthophosphates, which have been frequently used, are hydrolysed in solution, and besides, contain little or no PO_4''', the main anion in the alkaline solution of Na_2HPO_4 being HPO_4''.

By good fortune, I was in possession of a number of sodium and potassium salts of normal (*i.e.* unhydrolysed) polybasic acids, which had been prepared in a very pure state, and had been analysed. The present communication contains an account of a systematic study of the precipitation of a ferric hydroxide sol by the trivalent anions of

Potassium methane-trisulphonate, $CH(SO_3K)_3$, H_2O,
Sodium naphthalene-trisulphonate, $C_{10}H_5(SO_3Na)_3$, $5H_2O$,

and, for purposes of comparison, by sodium chloride, sodium sulphate and di-sodium hydrogen orthophosphate.

MATERIAL AND METHOD.

The ferric hydroxide sol was made from dialysed iron (Merck); this was dialysed in collodion tubes in a current of distilled water for about six weeks, long after no trace of chloride could be detected in the dialysate. The stock solution was kept in Jena glass bottles. It contained 0·920 gm. of $Fe(OH)_3$ in 100 c.c.; for use 15 c.c. of this sol was made up with pure water to 200 c.c.

The procedure finally adopted was as follows:—The desired volumes of salt solution were run from a microburette into glass tubes of uniform bore, pure water added to make a total volume of 1 c.c., and, after mixing, 2 c.c. of the sol were run in from a pipette with constant stirring by means of a platinum stirrer. The times of mixing were noted, and the rate of sedimentation measured by observing the depth of clear liquid at stated intervals of time. As is well known, only by scrupulous care in cleaning all apparatus and by rigid adherence to a standardised procedure can consistent and comparable results be obtained.

A brief description of the systematic procedure may be given. Commencing with, say, an exact 0·1 n solution of the salt, a series of eight tubes was set up with amounts of the solution varying from 0·02–1·0 c.c.; and the rate of clearing noted at 10′, 20′ 30′, . . . until sedimentation was nearing completion. Intermediate series were then set up with a narrower range of concentration, wherever the results indicated the desirability of doing so. And similarly with 0·01 n., 0·001 n., 0·0001 n., and 0·00001 n. solutions; and where possible with 1 n. and saturated solutions, care being taken that in passing from one dilution to another, there was overlap between the last tube of one series and the first of the next.

In order to compare the different salts, it was my original intention to obtain by interpolation the exact concentration of each which produced a fixed amount of clearing in a fixed interval of time. It was soon evident that this could not be done, for with the trivalent ions, which were first investigated, the rate did not increase regularly with increasing concentration, but varied in a more or less rhythmic manner, giving the impression of a succession of maxima and minima. This is not to be put down to errors in working, for the irregularities appeared at the same points, both on repetition and in the intermediate series; further, consistent results were obtained in series of four to eight tubes, each of which contained the same amount of salt.

RESULTS.

It is impossible to give the data of all the experiments which have been performed; it must be stated, however, that the selection is not arbitrary, since what purports to be a series is actually the results of a series, and is not a composite series, suitably selected.

SODIUM CHLORIDE.

Solution 1 n. (5·846 gm. in 100 c.c.).

I.		II.		III.		IV.	
c.c.	30′	c.c.	30′	c.c.	30′	c.c.	30′
0·1	opal.	0·15	?	0·50	6	0·80	10
·2	0·5	·20	0·5 ?	·53	4	0·85	5·5
·3	5	·25	1·5	·56	6	0·90	9
·4	4	·30	5·0	·60	6	0·95	6
·5	7	·35	11·0	·63	3·5	1·00	9
·6	6	·40	4·0	·66	5		
·8	10	·45	2·5	·70	4		
1·0	8	·50	7·0	·75	3		

Solution 0·1 n.

I.				II.			
c.c.	30′	7 hrs.	24 hrs.	c.c.	30′	7 hrs.	24 hrs.
0·2 to 0·4	0	0	0	0·5	0	0	0
0·6	opal	partial pp.	complete pp.	·6	0	opal.	partial pp.
0·8	”			·65	0	partial pp.	”
1·0	”			·7	0	”	complete pp. ?
				·75	0	14	complete pp.
				·8	0	16	”
				·85	0	11	”
				·9	0	10	”

Solutions 0·01 n. and 0·001 n., 0·02 to 1·0 c.c., 0.

Solution 5 n., 0·2 to 2·0 c.c., complete precipitation (flocculent).

SODIUM SULPHATE.

Solution 4 n. (saturated solution at 30° C.),
0·2 to 2 c.c., complete precipitation (at equal rates).

Solution 1 n.		Solution 0·1 n. (1·6111 gm. in 100 c.c.).	
c.c.	20′	c.c.	30′
0·1	16	0·05	11
·2	12	·10	11
·4	10	·15	12
·6	5	·2	8
·8	10	·4	13
1·0	7	·6	14
		·8	10
		1·0	13

Solution 0·01 n.

I.		II.		III.		
c.c.	30′	c.c.	30′	c.c.	30′	24 hrs.
0·1	0·5 ?	0·25	10	0·02	0	0
·2	2	·30	7	·05	0	0
·3	7	·35	3	·07	0 ?	partial pp.
·4	5	·40	8	·10	0 ?	complete pp.
·5	10·5	·45	12·5	·15	2	,,
·6	9	·50	9·5			
·8	11	·55	8			
1·0	13	·60	8			

SODIUM NAPHTHALENE-TRISULPHONATE.

Solution 0·01 n.		Solution 0·002 n.		
c.c.	30′	c.c.	30′	24 hrs.
0·05	opal.	·20 to ·22	0	0
·1	14	·25	opal.	partial pp.
·2	16	·30	?	complete pp.
·3	19	·32	0·5 ?	,,
·4	19	·35	4	,,
·6	18	·37	14	,,
·8	15	·40	16	,,
1·0	19	·50	17	,,

SODIUM NAPHTHALENE-TRISULPHONATE—*continued.*

Solution 0·001 n.			Solution 0·1 n. (1·749 gm. in 100 c.c.).		Solution 1 n.	
c.c.	30′	24 hrs.	c.c.	30′	c.c.	30′
·02 to ·2	0	0	0·05	18	0·1	13
·4	opal.	trace	·1	15	0·2	8
·6	1·5	complete pp.	·2	12	0·4	5
·8	8·5	,,	·3	14	0·6	9
1·0	11·0	,,	·4	10	0·8	3
			·5	12	1·0	8
			·6	10	2·0	11
			·8	12		
			1·0	14		

Solution *ca.* 1·5 n (25 per cent.).
2 c.c., 10 in 30′ (precipitate at once).

Solutions 0·0001 n., and 0·00001 n.
0·02 to 1·0 c.c., 0 in 24 hours (nor in many days).

POTASSIUM METHANE-TRISULPHONATE.*

Solution 0·05 n. (0·6475 gm. in 100 c.c.).		Solution 0·01 n.		
c.c.	30′	c.c.	20′	30′
0·2	14	0·16	10	14
·3	12	·20	5	8
·4	8	·25	12	16
·5	11	·3	14	17
		·4	15	17
	20′	·5	16	18
·6	14	·6	18	21
·7	16	·8	12	15
·8	19	1·0	16	18
1·0	18			

Solution 0·002 n.

I.			II.		III.	
c.c.	30′	24 hrs.	c.c.	30′	c.c.	30′
0·20	opal.	partial pp.	0·34	10	0·50	17
·22	,,	,,	·35	14	·55	20
·24	partial pp.	,,	·36	17	·60	10
·25	,,	,,	·37	8	·65	15
·26	4	complete pp.	·40	14	·70	12
·28	6	,,	·43	18	·75	12
·30	8	,,	·45	20	·80	13
					1·00	10

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* The saturated solution is less than 0·1 n.

DI-SODIUM HYDROGEN PHOSPHATE.

Solutions 0·00001 n. and 0·0001 n.
0·02 to 1·0 c.c., 0 in 24 hours (nor in many days).

Solution 0·001 n.

c.c.	30′	24 hrs.	72 hrs.
0·1 to 0·4	0	0	0
1·0	0·5	18	19

Solution 0·002 n.

I.			II.			III.		
c.c.	30′	24 hrs.	c.c.	30′	24 hrs.	c.c.	30′	24 hrs.
0·05 to 0·4	0	0	0·40	0 ?	0	0·60	4·0	18
·5	7	19	·43	0	0 ?	·65	1·5	18
·6	0·5	18	·45	0	partial pp.	·70	1·0	17
·7	0 ?	17	·47	0	„	·75	0·5	14
·8	0 ??	10	·50	partial pp.	0·5	·80	opal.	9
			·53	„	18	·90	0 ?	12
			·56	3	17	1·00	0	0 ?

Solution 0·01 n.

c.c.	30′	24 hrs.	c.c.	30′	24 hrs.
0·05	0	0	0·15	0·5	2
0·1	partial pp.	17	0·2 to 2·0	0	0

Solution 0·1 n. ($\frac{1}{3}$ Na_2HPO_4, $12H_2O$; 1·194 gm. in 100 c.c.).
0·2 to 1·0 c.c., 0 in 24 hours (nor in many days).

Solution *ca.* 1 n. (saturated solution).

c.c.	30′	c.c.	30′
0·2 to 0·4	opal.	0·8	4
0·5	0·5 ?	0·9	7
0·6	1	1·0	7
0·7	3		

All complete in 24 hours.

Discussion.

When these results are considered, it will be seen that there are three main points of interest, which will be taken in order of their importance.

I. When the positive sol is added to solutions containing increasing concentrations of trivalent anions, the sequence of events is:—(1) No precipitation followed very abruptly by (2) partial precipitation and (3) complete precipitation, which persists throughout the whole range of concentrations beyond that at which it once begins. That is, in the absence of an excess of OH′, due to hydrolysis, there is no second zone of non-precipitation, which in turn may itself be succeeded by a second zone of precipitation at high concentrations.

To make certain on this point, observations were extended in both directions, up to saturation and well below the concentration of initial complete precipitation, as calculated from the data for NaCl and Na_2SO_4 on the assumption that the valency rule held, at least approximately (see III. below). The concentration indicated was 0·0000043 n. corresponding in my procedure to the addition of 1·28 c.c. of 0·00001 n. solution or of 0·28 c.c. of 0·0001 n. solution. The two series examined contained 0·02, 0·05, 0·1, 0·2, 0·4, 0·6, 0·8, and 1 c.c. of these two solutions, the lowest concentration being 7×10^{-8} n. No change of any kind was observed, either immediately or at any time during a period of many days. If there were any zone of precipitation in this region, it is improbable that some slight indication of it would not have been noticed in the above series.

The results given by sodium phosphate are in striking contrast with those of the salts of the two trisulphonic acids. The first zone of precipitation, which is narrow, is followed by a wide zone of no precipitation, which finally passes to a second zone of complete precipitation at high concentrations, and yet the solutions contain little trivalent PO_4''', if any at all, the chief anion being the bivalent HPO_4''. Nevertheless, it is well known that this sequence is not observed with normal bivalent ions, *e.g.* with SO_4'' and positive sols, or with $Ca^{\cdot\cdot}$, $Ba^{\cdot\cdot}$, and negative sols such as gold sol; with the latter, "reversal" followed by a second zone of precipitation, has only been observed with solutions containing $H^{\cdot}$ in addition to the trivalent kation.

The conclusion is that a normal trivalent anion behaves towards a positive sol in exactly the same manner as do univalent and bivalent anions, and that the phenomena usually attributed to it are really due to the presence of the OH′. Whether the presence of a polyvalent ion, in addition to that of the OH′, is essential, I am at present unable to say;

but the sodium phosphate results would certainly lead to the conclusion that, if it is essential, its valency need not be higher than two. This is a matter for further investigation.

That the ions H· and OH′ occupy unique positions with respect to negative and positive sols respectively has long been recognised. The phenomena described above could be readily explained if the positive and negative hydrosols owed their electric charges to the attachment, in some way, of OH′ and H· respectively to the particles of the sol, a suggestion which is by no means new.

II. It was very noticeable in the early experiments with the trisulphonates that the rate of clearing did not increase steadily with increasing concentration after the stage of first precipitation was passed. These irregularities were at first attributed to slight differences in procedure; they appeared, however, at or near the same points in series of intermediate concentration. Further, they still occurred in similar series in which each concentration was represented by two, three, or four tubes. As a rule the results were consistent, although occasionally one would be quite out of line with the others.

A difficulty in investigating this behaviour was that, when a particular series was repeated, although the figures had the same general trend, they might be correspondingly higher or lower throughout than the previous ones.

These variations persist until precipitation is approximately complete, so that they disappear after an interval of thirty to sixty minutes. In no case was it found that precipitation was not complete in a comparatively short period of time. Subsequent observations showed that this is not a peculiarity of the trivalent anion, as it occurs perhaps to a smaller degree with both sulphate and chloride; on the other hand it is absent or very slight in the case of phosphate.

As the data given above have been reduced to a minimum in order to give a clear view of the whole phenomena, a few examples may be added which are representative and which fairly illustrate the points under discussion.

POTASSIUM METHANE-TRISULPHONATE.

Solution 0·002 n.

c.c.	15′	30′	50′	c.c.	15′	30′	40′
0·26	1	2	2	0·36	8	16	17
	1·5	2·5	4		10	17	18
	1·5	2·5	4		9	18	20
0·28	3	7	15				
	2	4·5	13				

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Potassium Methane-trisulphonate—*continued.*

Solution 0·01 n.

c.c.	10′	20′	30′	c.c.	10′	15′	20′
0·16	4	11	15	0·3	6	10	13
	4	10	13		6	9	13·5
·20	2·5	5	9	·4	8	12	14
	2·5	5	8		8	12·5	15
·25	7	14	16·5	·5	4·5	9	13
	5	11	15		12	16	18
·30	8	15	17·5	·6	13	17	19
	7	14	16		12	16·5	18
				·7	5	9	13
					6	11	14
				·8	5	9	12
					5	8·5	11
				·9	8	12	15
					9	13	16

As the figures show, the more concentrated solution gives better agreement than the more dilute one.

When due consideration is given to these points, it must be admitted that these more or less periodic variations of the rate of precipitation are real. How they are to be accounted for, and what bearing they may have on the mechanism of the precipitation of sols by ions, is not at all clear.

III. The remaining point concerns the validity of the "valency rule." The concentrations of chloride and of sulphate at which complete precipitation first occurred under strictly comparable conditions were determined with considerable accuracy, with the primary object of obtaining a clue to the corresponding points for the trivalent anions. But, as has been stated already, no trace of precipitation was to be found anywhere in the region indicated.

The actual data may be summarised briefly.

A. Taking the concentrations at first complete precipitation in twenty-four hours:—

	Concentration.	Ratio.	Ratio. Calculated.
	n.		
1. Cl' 0·75 c.c. of 0·1 n.	0·025	1	...
2. SO_4'' 0·10 c.c. of 0·01 n. . . .	0·00033	0·013	...
3. $CH(SO_3)_3'''$ 0·26 c.c. of 0·002 n. . .	0·00017	0·007	0·00017
4. $C_{10}H_8(SO_3)_3'''$ 0·30 c.c. of 0·002 n. .	0·00020	0·008	0·00017
5. HPO_4'' 0·53 c.c. of 0·00134 n. . .	0·00024	0·0095	0·013
or PO_4''' 0·53 c.c. of 0·002 n. . .	0·00035	0·014	0·00017

From the calculated ratio for the trivalent anion, the initial precipitating concentration should be 0·0000043 n. The actual range of experiment was from 7×10^{-8} n. to 1 n.

B. Taking the concentrations which clear 10 mm. in thirty minutes:—

	Concentration.	Ratio.	Ratio. Calculated.
	n.		
1. 0·35 c.c. of 1 n.	0·12	1	...
2. 0·5 c.c. of 0·01 n.	0·0017	0·014	...
3. 0·34 c.c. of 0·002 n.	0·00022	0·0018	0·0002
4. 0·36 c.c. of 0·002 n.	0·00024	0·0020	0·0002
5. At no concentration examined did phosphate cause 10 mm. clearing within 30 minutes.			

The valency rule does not seem to apply in the case at least of the trivalent anions and ferric hydroxide sol.

An interesting feature of these results is that while the ratio for chloride : sulphate is the same in A and B, for the two trisulphonates it is very much less in B than in A, due to the fact that the concentration of these salts which produces B is only very slightly greater than that which produces A. It is also worth noting that there is practically no difference between the sodium salt and the potassium salt of the two tribasic acids.

CONCLUSIONS.

I. When the positive sol ferric hydroxide is added to solutions of the *normal* trivalent anions of potassium methane-trisulphonate and sodium naphthalene-trisulphonate, there is *only one zone of precipitation*, which commences abruptly at a concentration of about 0·0002 n. and then persists up to high concentrations (at least to 1 n. or 25 per cent. solution).

II. The occurrence of two zones of precipitation separated by a zone of no precipitation, due to "reversal" of the sol in the case of sodium phosphate, appears to be due to the presence of OH', produced by the hydrolysis of the salt. If this be so, the similar phenomena observed with negative sols and trivalent kations, as $Al^{\cdots}$, $Fe^{\cdots}$, would also appear to be due to the presence of $H^{\cdot}$, produced by their hydrolysis.

III. Well-marked variations, of a more or less periodic nature, in the rate of precipitation occur when the concentration of the salt is steadily increased. These disappear with time as precipitation approaches completion. They are more marked at low concentrations.

IV. The valency rule, which connects the precipitating power of an ion with its valency, does not hold in the case of the normal trivalent anions of the two trisulphonic acids examined.

There are many directions in which this work promises interesting developments, some sufficiently obvious. Work in several directions is already on the way, including the behaviour of tetrasulphonates and of penta- and hexa-carboxylates, some of which are hydrolysed.

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(*Issued separately January* 11, 1926.)

XXX.—Salmon (*Salmo salar*) of the River Moisie (Eastern Canada). By **Mr W. J. M. Menzies.** *Communicated by* W. L. CALDERWOOD, I.S.O. (With Two Plates.)

(MS. received May 15, 1925. Read December 7, 1925.)

INCLUDED in the area of distribution of *Salmo salar* are the western coasts of Europe as far south as the Franco-Spanish border as well as the British Isles and Iceland, and, in addition, the eastern coast of Canada and the United States down to the State of Maine. A very large number of investigations have been made in Great Britain and various European countries, both by marking the fish in order to trace their subsequent growth and movements, and by reading their age and history from the scales. Length calculations from scale measurements have also been made in Scotland, Norway, and Sweden.

Certain marking experiments have been made with *Salmo salar* as a subject both in Canada and the United States, but, so far as I am aware, no effort has been made to deal with their scales, or if so, no results of such investigations have been published.

A. MATERIAL.

I therefore approached with some care a small collection of scales of salmon from the river Moisie which flows into the north side of the Gulf of St Lawrence. Unfortunately the collection could not by any means be called representative, since the sample I have to deal with consists of particulars of only 120 fish caught in 1922 and 257 taken in 1923, while other details of the catch which were supplied to me showed that some seven or eight thousand fish may be caught by nets and rods in the course of a short season of about two months.

The period of collection also of these small samples did not extend throughout the season, but was confined to about a month. One was therefore at a grave disadvantage in the examination of scales, not only from a strange river but also from another country. As a preliminary measure it was only possible to use accustomed methods, but after trial the formation of the ridges and zones of growth were seen to resemble exactly that of the scales of salmon of this country, and no special difficulties arose in reading the apparent age.

The parr scale (fig. 1), that is, the portion of the adult scale indicating the early river-life of the fish, is usually well defined, and the number of scales difficult of interpretation is no greater than in most rivers of this country: the indistinct scales are fewer than in, for instance, the Aberdeenshire Dee where the line of demarcation between summers and winters spent in fresh water is by no means always clear.

The indication of the period of slow growth in the sea, which in Scotland has been found to coincide in a general way with the winter months, is always well marked on the Moisie scales, and is often much clearer than the evidence of winter growth in this country. From the material available it was quite impossible to determine the time of formation of the zones of widely spaced and narrowly spaced ridges in the case of growth in either river or sea, and while in this paper I shall designate these by the conventional terms of summer and winter growth, it is quite impossible for me to say whether the seasonal limits coincide, as in this country, with the growth zones, or if there be some overlap between slow growth and the summer period.

B. AGE GROUPS.

The salmon from which the scales were taken were all caught in June or the early part of July, and if from this country, one would have expected the sample to consist almost exclusively of "summer" fish, that is, salmon leaving the sea during the period of rapid growth indicated by widely spaced ridges on the scales. Our experience in Scotland is that such fish may come into the rivers as early as the second week in April, and that by the end of May it is exceptional to find clean-run fish of the "spring" type, *i.e.* those which have not recommenced rapid growth. In the Moisie, however, among these June and July fish the reverse is the case: 81·6 per cent. are spring fish and only 18·4 per cent. belong to the summer groups. As in this country the majority are "maiden" salmon which are on their first return to fresh water: they have spent from two to four years in the sea, and it is noteworthy that grilse (1+ winters group) are unrepresented. The numbers and percentages in each age group are:—

Winters in Sea.	2.	2+.	3.	3+.	4.
Numbers	1	47	256	10	1
Percentages of total . . .	·3	12·5	68	2·6	·3

In this table, and in others concerned with age only, I have treated the 1922 and 1923 collections as one.

Just over two-thirds of all the fish belong to the important 3-winters group, while the main summer group, consisting of fish which have spent 2+ winters in the sea, form only 12·5 per cent. of the whole. The sea-life of one fish was estimated as only two winters, but, while the scales indicate this age, their formation is not so definite as to put the interpretation beyond doubt. No such dubiety attends the interpretation of the scales of the single fish which had spent four winters away from fresh water (fig. 2).

In such rivers as the Moisie, subject to extreme climatic conditions and liable to prolonged and severe cold during the winter months, one does not know in what manner kelts may behave after spawning. I approached the subject of the spawning mark on the scales with the possibility in my mind of a rapid migration from the river by the kelts immediately after the completion of the redds. I was prepared to find a somewhat less definite scar than is usual on the scales of previously spawned fish caught in this country. Fortunately, among the collection sent were the scales of three kelts and a "baggot" (*i.e.* an unspawned female), all caught in late spring or early summer, and these seemed to suggest that some kelts at least spend the winter in the river. Scales of the kelts showed absorption extending completely round the periphery, while two-thirds of the circumference of those of the baggot were similarly affected. Absorption of the scales of a few of the clean fish also had commenced, and appeared to be proceeding normally. One kelt was described as a male, and while in this instance the absorption is rather greater than in the other cases, the extreme degree common in many British male fish has not been reached. The amount of absorption exhibited by the scales of the kelts and the baggot does not suggest the possibility of Moisie salmon spawning without the fact being placed on record on their scales, and none of the clean fish showed what might be termed a "doubtful" spawning mark: in every case the scar was either without doubt absent or undeniably noticeable.

Acting on the assumption that a spawning mark is shown by all fish which have spawned and have again returned to the river as clean fish, I find that of the total of 377 salmon, 63, or 16·7 per cent., fall into this category. This proportion is very much higher than we are accustomed to find in the rivers of the east coast of Scotland, where it is very unusual for more than 5 per cent. of the stock to be on other than their first return to fresh water.

Of these 63 Moisie salmon, 35, or 9·3 per cent. of the total sample, had spawned once; 11, or 2·9 per cent., had spawned twice; 16, or 4·2 per cent., had spawned three times; 1, or ·3 per cent., had spawned four times.

Of the thirty-five fish which had spawned once in their lives and were again in the river, four had been absent a very short time, not more than a few months at most, in the sea, after descending as a kelt and before reascending as a clean fish. Twenty-nine had been away a complete year, and two had started a second summer's feeding before returning to fresh water. Both of the last were males, and I have noticed in other cases that the few fish which adopt this habit are commonly of this sex. The four which had been absent only a few months call for special remark, since the amount of scale growth made after the formation of the actual scar is, as far as British experience goes, most exceptionally small, and consists of only one or two ridges. One of the fish was described as a kelt, although it was also noted that sea-lice were adhering to it. It was by far the smallest fish in the collection; and the dimensions, 28 inches long and 4½ lb. in weight, lent colour to its suggested condition, since a normal fish of 28 inches should weigh about 7½ lb. The other three fish of this same history exhibited lack of complete recovery from the hardships and emaciation incident to the spawning journey, and were noticeably thin.

The scale of the very small fish exhibits less growth after spawning than any Scottish scale I have ever seen, with one exception, and that was a salmon, described as a kelt, taken by a trawler in the North Sea thirty miles north-east of Buchan Ness. In the Moisie scale we can see the growth of new material in the posterior, or exposed, part, and in the projection of the fibrous layer beyond the ridges. In the North Sea scale, as far as one can see, there is no increase in the scale area, but new ridges are in process of formation on the part of the fibrous layer denuded of hyalodentine by absorption during the freshwater residence of the spawning period.

The numbers of salmon which survive to return to the Moisie for a third and fourth time are noteworthy, and more especially is this so in the case of the single fish which had spawned four times (fig. 3), and was on its fifth journey to the river when netted and killed. Of all the scales examined in Scotland, we have records of only three fish which had helped so materially in the reproduction of their species. Two were clean fish, one of 19 lb. from the Conon, and another of 29½ lb. from Loch Maree; the third was a kelt of 12 lb. caught in the Add in Argyllshire, which, since it was again in the kelt stage, had spawned on five occasions. The Moisie fish weighed 38½ lb., and, like the Scottish fish, was a female. Its scales show two years of river-life, followed by apparently three years in the sea, and then a full year's feeding between each spawning period—that is, it adopted the long-absence habit and spawned biennially, descending as a kelt in the spring of one year, and remaining in the sea until the spring or early summer of the following year. Its total age was consequently thirteen years.

As absorption of scale material during the freshwater fast may remove evidence of a complete winter's growth, the age of any of these spawned fish at the time of their first ascent of the river cannot be estimated with absolute certainty, but, as far as can be judged, 58 of the 63 fish spawned for the first time after spending three winters continuously in the sea.

C. Smolt Ages.

The average age at migration of the Moisie smolts is slightly higher than is usual in the rivers of the north-east of Scotland, but is lower than is found in Norway. In Scotland the majority of smolts are two years of age, but in certain districts as many as 25 per cent. of the total may be three years old. In Norway, the proportion of three-year-old smolts may be as high as 75 per cent. In the Moisie, the percentages of two- (fig. 2) and three-year-old migrants were 56 and 39 respectively; the remainder, consisting of those four years old (fig. 1), formed 5 per cent. of the whole, while one fish had spent five years as a parr in the river. To what extent these conditions may be considered typical one is unable to judge, since the figures from the two samples separately (*i.e.* those taken in 1922 and 1923 respectively) vary considerably; in 1922, of 120 salmon, 76 per cent. were two-year-old smolts, and in 1923, of 257 salmon, only 46 per cent. were of that age.

The proportion of each smolt age varies in the different adult age-groups, and, as in Great Britain, the older smolts show a slight tendency to remain a shorter time in the sea than the younger smolts. For instance, of the 2+ winters sea-group, 51 per cent. were three-year-old smolts, but of the 3-winters sea-group only 41 per cent. were of the same smolt age. However, when divided into separate age-groups, the number of specimens is extremely small, and the figures consequently are not conclusive.

D. Average Sizes.

In dealing with the average sizes of the age-groups I shall only use the 1923 sample, since in 1922 the weights were recorded without the length of the fish.

The one fish of the 2-winters group weighed 13 lb. and was 32 inches in length.

In the 2+ winters group the average sizes are 9·15 lb. and 29·4 inches, with a minimum of 6 lb., 26 inches, and a maximum of 14 lb., 33 inches. The smallest fish corresponds in size very closely to the smaller specimens of the same age-group in Scotland, but both the average and the maximum are considerably less than the usual Scottish fish. Perhaps this is scarcely surprising, since our Scottish averages include salmon caught in August, by which date growth in the sea, as far as they are concerned, has reached its maximum rate.

It seems more possible that some factor of the time of growth and date of catch is involved, since in the next older group of fish, which have spent three winters in the sea, the same variation does not exist, and all the Moisie figures might well represent fish from a Scottish river such, for instance, as the Spey. It is also probable that in this group the larger number of samples has given a more accurate picture of the true conditions. The actual details are :—

	Lb.	Inches.
Minimum . . .	12	32
Average	20·7	37·3
Maximum . . .	32·5	42·5

The nine fish of the 3+ winters group varied from 15 lb., 34·5 inches to 26 lb., 40 inches, while the average of the group is 22·3 lb., 38·1 inches.

The average sizes of the fish which have spawned are of little value owing to the varying and, to some extent, uncertain history. One fish perhaps calls for special comment and has already been mentioned: although it had spawned and again returned to the river, it was one of the smallest fish in the collection—at 4·5 lb. it was actually the lightest specimen, but its length of 28 inches was 2 inches longer than the shortest of the 2+ winters group. Among these spawned fish some of notable size were to be found: among those which had spawned once are five over 30 lb. each, the largest being 35·5 lb., two of 33 and 36 lb. respectively had spawned twice, and the solitary fish which survived after spawning four times weighed 38·5 lb.

E. CONDITION.

A valuable means of comparison of the salmon of different rivers and of the salmon of different runs or age-groups in any one river is to be found in the shape, and state of fatness, or the "condition," of the fish. An index, known as the "coefficient of condition," is obtained by division of the weight by the cube of the length: when the weight is in pounds and the length in inches, the resulting fraction is somewhere between ·0003 and ·0005, and in Scotland we have carried the calculation one stage further by the division of the result by ·00036. We then finally have an index which for fish of normal shape and fatness is in the neighbourhood of 1; poor, thin fish give an index of between ·8 and ·9, and for those exceptionally fat the figure may rise to 1·4, or in extreme cases 1·5.

The indices of condition obtained from the average weights and lengths of the 2+, 3, and 3+ winters groups of Moisie salmon are 1, 1·23, and 1·12 respectively, and show a somewhat surprising variation. The trend is the same as Mr J. Arthur Hutton has found in the Wye in England, in that the spring fish are considerably fatter than the summer fish, but while the 2+ and 3+ winters groups have values closely akin to those of the salmon of the Aberdeenshire Dee, which are somewhat thin, the 3-winters group are even fatter than fish of that age from the Spey, and they are among the best shaped fish in Scotland. From the available material it is impossible to suggest any reason for these wide differences.

F. SEX.

In the particulars sent with the Moisie scales, the method of determining sex, whether by dissection or only by external examination, was not stated, and in the absence of this information two causes for disquiet arise as to the accuracy of the details. One is the very considerable variation which exists between the figures for 1922 and 1923—in the former year the quite exceptional proportion of 53·4 per cent. were recorded as males, and in the latter year 39·5 per cent. (a much more usual proportion for Scottish rivers) were stated to belong to that sex. Unless conditions in the Moisie differ a great deal from those which obtain in this country, another cause for venturing to doubt the accuracy of the sex-determination is the large number of male fish among those which have spawned in previous years. In this group of salmon of diverse age and history, the total of each sex differs by only one fish: in Great Britain a male fish which, after spawning, has survived to return to the river a second time is a rarity; and there is no record of a male fish spawning more than twice. Of the sixteen Moisie fish which had spawned three times and were on their fourth return, six are said to have been males, although the type of the spawning marks on the scales does not suggest that this was the case. If the sex was determined by external examination only, one is not surprised if an observer be in error, since the spawning journey has a very definite effect on the appearance of the fish, such as might easily lead to an incorrect estimation by one without considerable experience.

G. LENGTH CALCULATIONS.

The scales of every unspawned fish in the 1923 sample were measured, and therefrom the lengths at the end of each winter of life were calculated in the ordinary way, no factor being allowed for the difference in time between hatching and scale formation. The averages of the calculated lengths accord closely with similar averages obtained from salmon of the east coast of Scotland, and I may perhaps be allowed to quote the more representative figures.

When the smolts migrate to the sea, those two years of age average 9·8 cm. in length, and those three years old 11·9 cm. Both ages are somewhat smaller than the corresponding smolts of the east Scottish coast, where the average lengths of the two-year-olds vary between 10 and 13 cm., and the three-year-old smolts between 12·5 and 14 cm. In Scotland we have not found any differences of importance in the average lengths of the smolts, which subsequently remain for one year, or for several years, in the sea respectively, but in the Moisie salmon there is a small difference: the average length of the two-year-old smolts which remain 2+ winters in the sea is 9·9 cm., and of those which remain 3 winters in the sea, 9·7 cm.; in the case of the three-year-old smolts the figures are 12·1 and 11·8 cm. respectively. These small differences in length correspond with the phenomenon of the change in growth rate observed in other fishes, and the significance is therefore uncertain. The same condition obtains in the case of the two-year-old smolts at the end of the first parr year, when the average length is 4·1 cm., since the parr which become 2+ winters fish average 4·3 cm., and those remaining 3 winters away from the river 4 cm.

As is always the case at the end of both the first and second year, the average lengths of the parr which become smolts at two years of age are greater than the lengths of those which remain three or more years in the river. For instance, the first-year length of the two-year smolts is 4·1 cm., while at the same age the three-year-old smolts average only 3·4 cm. I have stated that the two-year-old smolts are 9·8 cm. at the time of migration; the three-year-old smolts at the end of their second year are nearly two and a half centimetres shorter, and average 7·4 cm. The four- and five-year-old smolts are too few in number to give averages of any value, and it suffices to mention that the total lengths of these two ages at migration are 15·2 and 15·9 cm. respectively.

In the average lengths at the end of the first year in the sea there is exhibited a somewhat surprising condition shown by certain samples of Scottish salmon: it is that the average length of the fish staying the shortest time in the sea is *shorter* than the average length of those remaining in the sea for a longer period. The average first sea-year lengths of the fish at present under consideration are:—

	Winters in Sea.	
	2+.	3.
	cm.	cm.
Two-year smolts . .	40·9	41·4
Three-year smolts . .	42·1	42·8

It is not a case of the longer smolts developing into longer fish in the sea, since both the two- and three-year smolts are longer in the 2+ than in the 3-winters group, nor can the difference be ascribed to the condition known as the phenomenon of the change in growth-rate, since the present is the reverse of that state. It is a surprising fact, the explanation of which is obscure. A fact, moreover, which is exhibited in a still more marked degree at the end of the second year in the sea, when the difference in lengths is even greater, as is shown by this table.

	Winters in Sea.	
	2+.	3.
	cm.	cm.
Two-year smolts . .	67·9	71·4
Three-year smolts . .	68·5	71·3

The relationship between the amount of growth made by the parr and the length attained in the sea cannot be shown owing to the small number of fish, although one may note that the older, and therefore longer, smolts attain a greater length in the first sea-year than do the younger, and therefore shorter, smolts.

I now come to the increment of growth made in each year of sea-life, and in this table of the 3-winters group the figures showing the increase in length in each year are not at all dissimilar from the corresponding figures for salmon which have fed in British waters.

Smolt Age.	Year in Sea.		
	First.	Second.	Third.
	cm.	cm.	cm.
Two years	31·7	30	23·6
Three years	31	28·4	23·4
Four years	30·3	26·9	30·7

The only difference to be noted is that the Moisie fish seem to make a rather greater growth in the third year than our fish.

H. FIN DISTANCE MEASUREMENTS.

In recent years in Scotland a great deal of material in the form of certain external body-measurements relating to the position of the fins has been accumulated with a view to investigating the possibilities of the existence of local races. So far this material has been productive only of negative results, and a series of measurements taken in the same manner from the salmon at present under consideration is no more informative. The points of measurement are from the extremity of the snout to the posterior edge of the gill cover, and to points on the lateral line immediately below the posterior insertion of the dorsal fin and immediately above the anterior insertion of the anal fin, as well as to the extremity of the central rays of the caudal fin. The head-length is then deducted from the other measurements, and the fin positions are described in terms of distance from the edge of the gill cover expressed as a percentage of the total trunk-length. Of the Moisie details (in the table below) the dorsal fin figures are very similar to the percentages obtained from salmon of the Aberdeenshire Dee.

	Winters in Sea.	
	2+.	3.
	Per cent.	Per cent.
Dorsal fin	39·9	40·7
Anal fin	61·5	61·6

SUMMARY.

The analysis is based on a rather small collection of scales and measurements which may possibly be not quite representative of the salmon (*Salmo salar*) of the river Moisie (E. Canada).

The portions of the scales representing both river- and sea-life are easily read, and closely resemble the scales of European salmon in formation and growth, but it is not possible to decide, from the material available, if the periods of slow and rapid growth correspond with the winter and summer seasons respectively.

The majority of Moisie salmon return to the river after spending three winters in the sea; some may do so after an absence of 2+ winters, but those which spend 2, 3+, and 4 winters away from the river are rare, and apparently grilse (1+ winters group) are entirely absent.

Erosion of the scales during the period of abstinence prior to spawning takes place normally, and a spawning mark is subsequently formed on the scales in the usual manner. The number of fish on their second or subsequent return to the river (16·7 per cent.) is high, and the number which have spawned more than once (11 twice, 16 three times, 1 four times) is relatively considerable.

The average age of the Moisie smolts is intermediate between that of the smolts of N.E. Scotland and of Norway; 56 per cent. were two years, and 39 per cent. three years, old when they first entered salt water. The balance were four years of age, with the exception of one, which had spent five years in the river.

The average sizes of the 2+ winters group, 9·15 lb. and 29·4 inches, are considerably less than the averages of fish of the same history in Scotland, but the result may be affected by the period of the season at which the samples were taken. In the older group of 3-winters fish the averages, 20·7 lb. 37·3 inches, correspond closely with the Scottish figures. These latter fish are fat and well nourished, as their coefficient of condition, 1·23, shows; they compare favourably with the most finely shaped Scottish fish. The younger group, however, besides having a small average size, have for their length a rather low coefficient of condition—1.

The sex of each fish was given with the other details, but presumably was judged from external examination only. The proportion of males seems unduly high, and the erosion of the scales also in many cases does not suggest that the sex has been accurately determined.

Length calculations from scale measurements give the following as the average lengths attained during the preliminary river-life:—

	Two-year Smolts.	Three-year Smolts.
	cm.	cm.
First year	4·1	3·4
Second year . . .	9·8	7·4
Third year	...	11·9

At the end of the first year in the sea, the average lengths of the fish of each group are:—

	Winters in Sea.	
	2+.	3.
	cm.	cm.
Two-year smolts . .	40·9	41·4
Three-year smolts . .	42·1	42·8

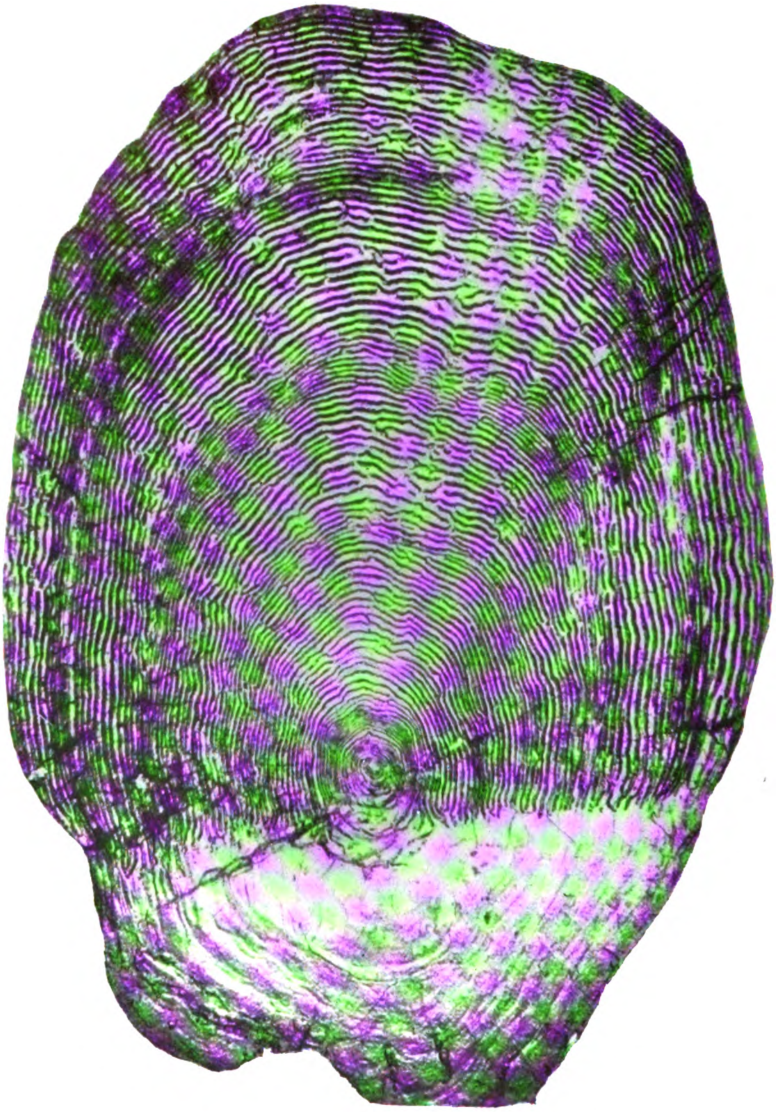


Fig. 2.

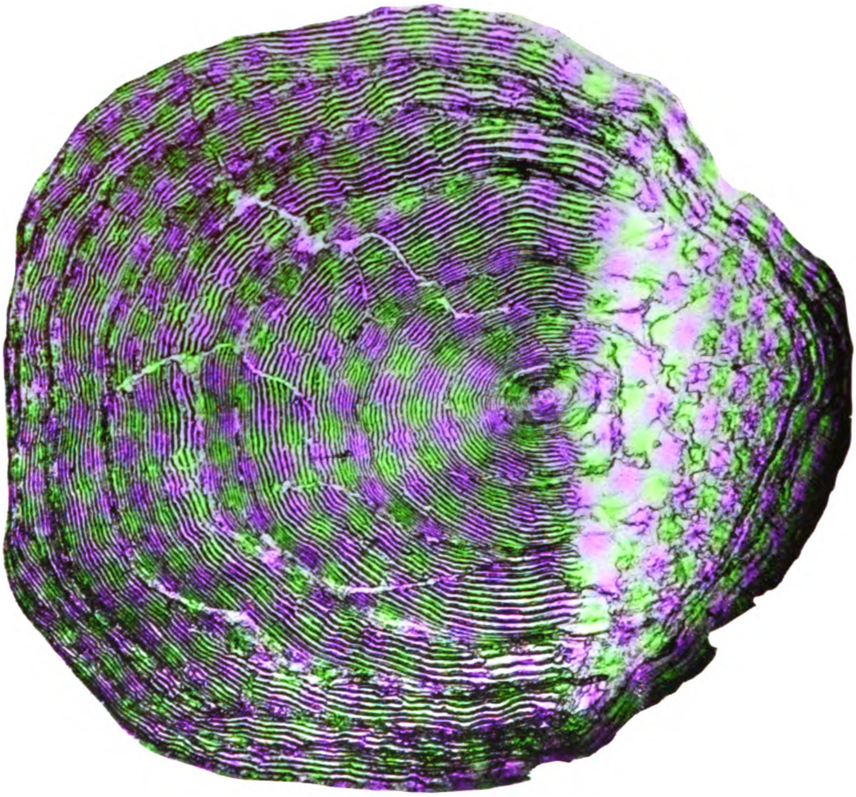


Fig. 3.

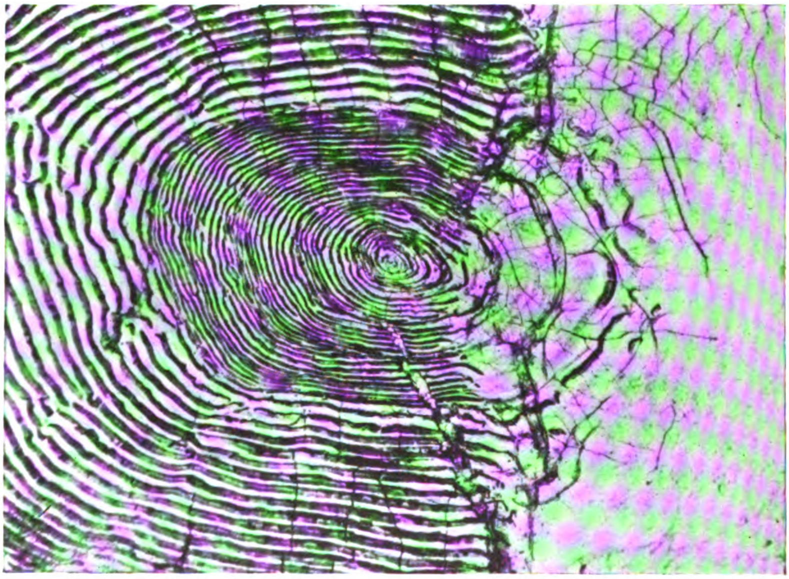


Fig. 1.

W. J. M. Menzies. [Plate I.

At the end of the second sea winter the figures are:—

	Winters in Sea.	
	2+.	3.
	cm.	cm.
Two-year smolts . .	67·9	71·4
Three-year smolts . .	68·5	71·3

The annual increment of growth in the sea is:—

Smolt Age.	Year in Sea.		
	First.	Second.	Third.
	cm.	cm.	cm.
Two years	31·7	30	23·6
Three years . . .	31	28·4	23·4
Four years	30·3	26·9	30·7

All these calculated lengths do not differ materially from those of Scottish salmon; the position of the dorsal and anal fins, as shown by their distances from the posterior edge of the gill cover, is also similar in Moisie and Scottish fish.

LIST OF SCALE ILLUSTRATIONS. (Plates I, II.)

Fig. 1. Four-year smolt.
„ 2. 36 lb. Male. Two years in river, four years in sea.
„ 3. 38 lb. Female. Four spawning marks.

(*Issued separately January* 11, 1926.)

OBITUARY NOTICES.

The Scientific Career of Sir Archibald Geikie, O.M., K.C.B., F.R.S.

By **B. N. Peach,** LL.D., F.R.S., and **John Horne,** LL.D., F.R.S.

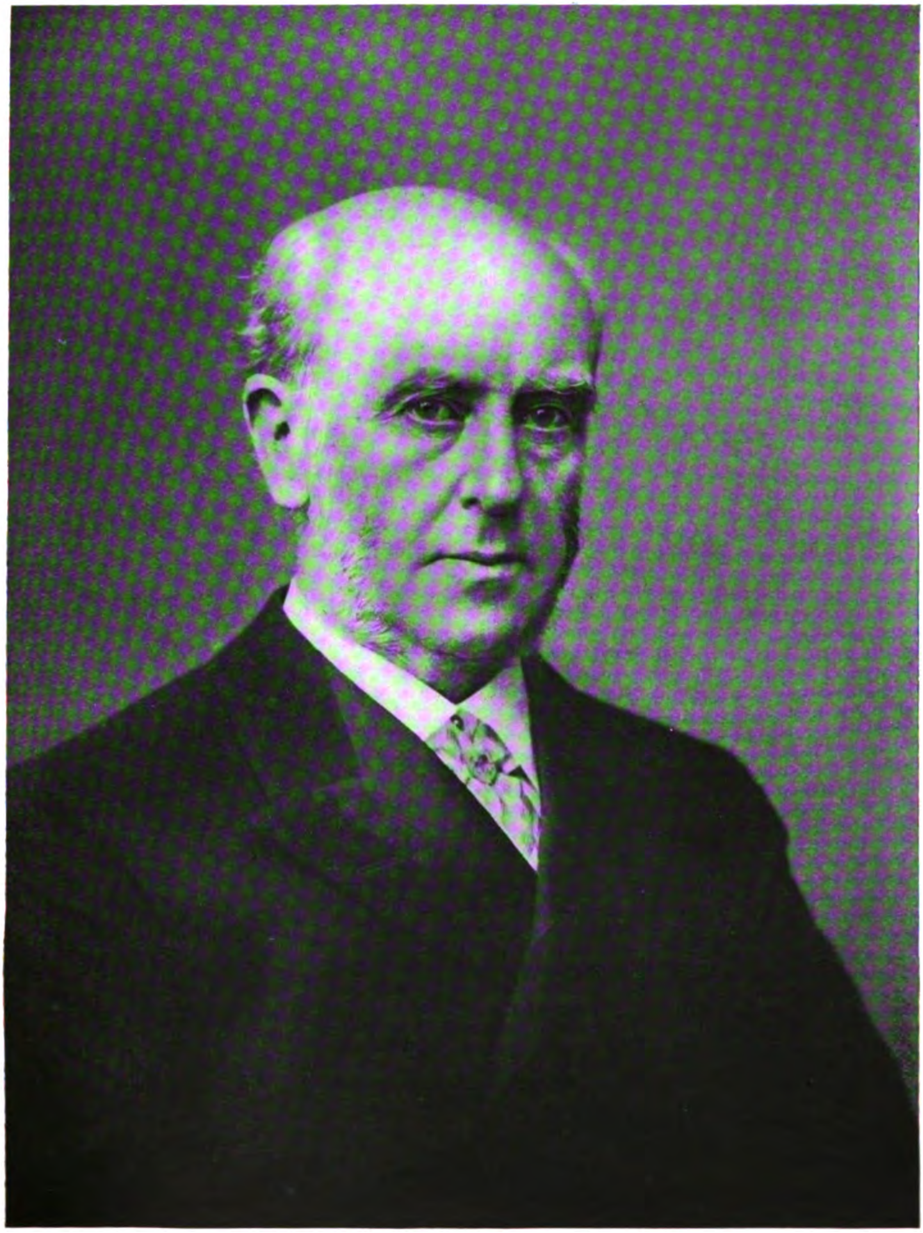
(Read February 23, 1925.)

The scientific career of Sir Archibald Geikie was one of great distinction. Joining the Geological Survey in 1855, he became Director-General of the Geological Survey of the United Kingdom in 1882; President of the British Association at the Edinburgh Meeting in 1892; President of the Geological Society, London, 1890-92, and specially selected by the Council for a second term of office, to preside at the centenary celebrations in 1907; President of the Royal Society, London, 1908–13.

Honours were showered upon him by scientific societies at home and abroad, and by Universities, which were crowned by the Order of Merit in 1914.

The clue to this successful career is to be found in certain traits of his personality, displayed in his early years and developed as the time slipped past. Prominent among these traits were—his enthusiasm for certain lines of geological research; his firm determination to increase his knowledge of the geological record in his own and other lands; his keen desire to communicate that knowledge to others in artistic form with pen and pencil; his vivid imagination which led him to visualise his interpretation of geological evidence and to present it in clear and convincing terms; his love of literature, ancient and modern, which gave him intense pleasure in the midst of a strenuous life; his unwearied industry and great capacity for work. These qualities combined to make a strong personality which left its mark in every sphere of labour that he undertook. But some of these sources of strength proved to be sources of weakness when he dealt with complicated geological problems in the field.

Born in Edinburgh in 1835, and educated at the High School and University, he derived inspiration from the classic surroundings—the scene of the fierce controversy between the Huttonians and Wernerians at the beginning of last century. His finding fossils in the Burdiehouse limestone in his boyhood marked the first stage in his field experience. He read eagerly any geological works on which he could lay hands. Hugh Miller's volume on *The Old Red Sandstone* fired his imagination by its vivid descriptions of the members of the system, its included fish-remains, and



Arch Geikie

conditions of deposition. He resolved to follow that descriptive style of writing, as it appealed so effectively to the general reader. Later in life it was superseded by Playfair's *Illustrations of the Huttonian Theory*, which he regarded as a model of lucidity, brevity, and elegance of style. He roamed over Arthur's Seat and the Pentland Hills with Charles Maclaren's volume on *The Geology of Fife and the Lothians* as his guide.

Not content with excursions in the neighbourhood of Edinburgh, he went to Arran with Ramsay's account of the geology of that island in his hand, to examine the metamorphic rocks and palæozoic strata, pierced by the granite masses in the north of the island. This was followed, in his eighteenth year, by a visit to Skye, where he began the mapping of the Lias, in the district of Strath, after a careful study of Macculloch's volume on *The Geology of the Western Islands*.

All these excursions sprang from his own initiative, which showed that he was clearly destined for a geological career, if such could be obtained. Two articles which he wrote on his work in Arran secured an introduction to Hugh Miller, then living in Edinburgh, which led to a close friendship that lasted till the tragic end of Hugh Miller's career. The young geologist paid frequent visits to that home, with his specimens, his maps and notes, receiving in return great encouragement to continue the work. Soon after this introduction, Hugh Miller was asked by Sir Roderick Murchison, who had been appointed Director-General of the Geological Survey, to recommend a young geologist to carry on the mapping in East Lothian, begun by Professor Ramsay in 1854. Archibald Geikie was recommended, and he joined the staff in his twentieth year.

He made the acquaintance of other scientific men in Edinburgh at that time, who were impressed with his enthusiasm for geological research. Dr George Wilson, lecturer on Chemistry in the extra-mural school and Director of the Science and Art Museum, in whose laboratory he did chemical analyses, followed his early progress with great interest, and spurred him on with the suggestion that his goal ought to be a Chair of Geology in Edinburgh University which would certainly be established in the future. Robert Chambers, Professor Fleming, one of the able naturalists of that period, and Professor James David Forbes, whose work among the glaciers of the Alps and Norway is familiar to geologists, befriended him. Forbes, who was Secretary to the Royal Society of Edinburgh for twenty years, appealed to him to contribute papers to the Society, for he regretted the decline of geological research in Scotland since the days of Hutton, Hall, and Playfair. The last time that Forbes addressed the Royal Society of Edinburgh he uttered these impressive words: "Of all the changes which have befallen Scottish science during the last half century, that which I most deeply deplore, and, at the same time, wonder at, is the progressive decay of our once illustrious Geological School." Archibald Geikie responded cordially to this appeal in later years by communicating his best papers to this Society.

From the point of view of worldly advancement, Murchison was his greatest benefactor. In 1860, after a few years' experience of field work in the Lowlands, the young geologist was requested by Murchison to accompany him in the North-west and Central Highlands to determine the order of succession of the rocks in those regions. From these traverses arose an intimate friendship which lasted till the close of Murchison's life in 1871. During these years Murchison realised Archibald Geikie's great ability, his enthusiasm for geological work, his graphic descriptive power, and showed his appreciation in tangible forms.

Mainly through Murchison's influence he was admitted to the Fellowship of the Royal Society, London, at the early age of twenty-nine, and appointed first Director of the Geological Survey in Scotland in 1867. When Murchison offered to share the endowment of the new Chair of Geology in Edinburgh University, he stipulated with the Government that Archibald Geikie should be the first Professor. In his will Murchison made him literary executor and left him a legacy of £1000. These are striking proofs of the impressions produced by the young geologist on the mind of the veteran, who, before the close of his career, dominated British geology.

The areas fixed by Professor Ramsay, then Director of the Geological Survey of Great Britain, to be mapped by Archibald Geikie in his early official career gave him great gratification. His colleague was H. H. Howell, a coalfield expert, and after completing the survey of the low ground in the county of Haddington, they moved westward to Midlothian. To Geikie was assigned the area west of the coalfield, ranging from Arthur's Seat and the Pentland Hills to the Bathgate Hills, and northwards into Fife. His experience in those areas determined the lines of research in which he did his best work. The evidence bearing on the glaciation of the region, the development of the Old Red Sandstone formation in the Pentland Hills, and the occurrence of volcanic rocks on different geological horizons, suggested fields of investigation that might bear rich fruit.

In his early Survey days, in common with the prevalent opinion of that time, he attributed the polishing and grooving of the rocks to the action of currents and icebergs, but under the influence of Professor Ramsay, who had already described the vanished glaciers of North Wales, he adopted the view, propounded by Agassiz during his visit to Scotland in 1840, that these phenomena were due to the action of land ice. In 1862 he contributed to the Glasgow Geological Society a valuable paper on "The Glacial Drift of Scotland" (published in 1863), which helped to place the investigation of these deposits on right lines. He showed how the rock striations radiated from the main mountain ranges; he described the local character of the boulder-clay, its relation to the underlying rocks, the direction of transport of the materials, its striated pavements, the intercalated deposits of sand and gravel—observations which have stood the test of time. From these phenomena he inferred that during the earlier part of the glacial period Scotland was covered with an ice-sheet after the manner of Greenland, that the boulder-clay was a product of this ice-sheet, and that the stratified beds in the Till marked periods of lessened severity of climate when the ice retired for some distance. He described the inland stratified drift and the fossiliferous marine shelly clays of the Clyde basin, so admirably worked out by Smith of Jordanhill. He suggested that the moraines in the upland valleys indicated the gradual shrinkage of the ice-sheet into local glaciers, and the final disappearance of the ice.

Availing himself of the researches of Smith of Jordanhill, Edward Forbes, T. F. Jamieson, and others, he appended to this paper a list of organic remains from the glacial deposits of Scotland.

This paper proved of great service to British geologists at that time, as it gave an outline of the conditions which probably prevailed during the glacial period in this country, and suggested the lines along which investigation should proceed. It held the field till the publication in 1874 of the well-known volume, *The Great Ice Age*, by his younger brother, Professor James Geikie.

As the Scottish Survey work advanced, it fell to the lot of Archibald Geikie to map in detail large areas of the Old Red Sandstone in Midlothian, Lanarkshire, Ayrshire, and in the counties of Fife, Perth, and Kinross. He also examined the districts surveyed by other members of the staff south of the Grampians. Throughout this region it had been proved that this formation consisted of two divisions, an upper and lower, separated by a marked unconformability, and characterised by different fish-faunas.

North of the Grampians, however, Murchison adopted a triple classification by introducing a middle division, composed of the Caithness flagstones. From their lithological characters and fossil contents he maintained that this flagstone series belonged to a different division of the formation. He argued that the fish-fauna found in this series differed in important points from that occurring in the flagstones, sandstones, and shales of Lower Old Red Sandstone age in Forfarshire. It was therefore of younger date. He also suggested that the Caithness flagstones might have been laid down during the long interval separating the lower and upper divisions south of the Grampians.

Hugh Miller and Dr Malcolmson had proved the occurrence of the upper division with its characteristic fish-remains in the basin of the Moray Firth, and they further showed that the ichthyolites found in the lower beds in that region resembled those met with in the flagstones of Caithness and Orkney.

Such was the position of enquiry when Geikie began a series of traverses in the basin of the Moray Firth, Caithness, Orkney, and the Shetland Isles. In 1878 he communicated to this Society an elaborate paper on the "Old Red Sandstone of Western Europe" which embodied the results of these traverses. It was intended to be the first of a series, descriptive of the other areas occupied by this formation in the British Isles, but his removal to London in 1882 prevented him from carrying out this intention. The other areas were dealt with at a later date in his treatise on the *Ancient Volcanoes of Great Britain*, with special reference to the records of volcanic action associated with them.

One of the special features of this communication is the vivid description of the geographical changes in Western Europe that followed the marine conditions of Silurian time. Adopting the theory of the lacustrine origin of the Old Red Sandstone suggested by Fleming and Godwin Austin, and supported by Rupert Jones on palæontological evidence, and by Professor Ramsay on lithological grounds, he held that in North-west Europe the Silurian sea gave place to continental conditions with large inland lakes. He defined the areas of the various basins of deposit of the Old Red Sandstone in the British Isles and gave them the following names: (1) Lake Orcadie including the extensive region north of the Grampian range and stretching north to the Shetland Isles; (2) Lake Caledonia or the Mid-Scottish Basin between the Grampian Highlands and the Southern Uplands; (3) the Lake of Lorne extending from the south-east of Mull to Loch Awe; (4) Lake Cheviot, including a part of the south-east of Scotland and north of England; (5) the Welsh Lake, bounded on the north by the Cambrian and Silurian high grounds, its southern and eastern extension being obscured by later formations.

The deposits in Lake Orcadie were grouped by him in two divisions; a lower, comprising the Caithness flagstone series, and an upper, composed of false-bedded sandstones and marls with its characteristic fish-fauna, resting unconformably on the lower as in the midland valley of Scotland.

He pointed out that Murchison's lower division in Caithness consisted merely of the thick accumulation of sandstones, breccias, and conglomerates that underlie the Caithness flagstone series.

The distinctive feature of the paper was his correlation of the Caithness flagstone series (Murchison's middle division) with the true Lower Old Red Sandstone south of the Grampians. He admitted the marked differences in the lithological characters and fossil contents of the deposits in the two areas, but he contended that the palæontological distinctions are probably not greater "than the contrast between the ichthyic faunas of adjacent but disconnected water basins at the present time."

Much new information bearing on the distribution and field relations of these deposits was embodied in this communication. The main features in the geological structure of Caithness were indicated; the great overlap at Reay in the north of that county, where the higher members of the flagstones rest unconformably on the crystalline schists, was clearly established; his suggestion that the fish-bearing bands and associated strata in the Moray Firth basin are the equivalents of part of the higher portion of the Caithness flagstone series has been confirmed by the later detailed mapping of the Geological Survey. He also recorded for the first time the unconformability between the Upper Old Red Sandstone and the Caithness flagstones visible on the west coast of the island of Hoy, Orkney.

In accordance with his classification, the Caithness flagstone series and associated strata were grouped with the lower division of the system in the official publications of the Geological Survey. But after his retirement from the Survey, the palæobotanical evidence threw new light upon the problem. The researches of Dr Kidston and Mr P. Macnair showed that the assemblage of plants found in the Caithness flagstones differed from those met with in the Lower Old Red Sandstone south of the Grampian Chain, and ought to be referred to a middle division. The triple classification based upon the plants harmonised with that advanced by Dr Traquair based upon the fishes. The value of this evidence was appreciated by the Geological Survey in 1902, and the threefold grouping of the system has been adopted in the official publications since that date.

Turning now to volcanic geology, we enter a sphere of research in which Archibald Geikie laboured with great success. This branch of enquiry roused his enthusiasm and led him to study volcanic phenomena in the British Isles and other lands. As his knowledge of volcanic activity in different geological periods increased, he showed great aptitude in modifying his opinions in accordance with fresh evidence.

In mapping the areas assigned to him in Midlothian, West Lothian, and Fife in his early official life, he frankly acknowledged his obligations to the clear descriptions of Charles Maclaren, whose work was far in advance of its time. These descriptions proved the occurrence of volcanic action in Old Red Sandstone and Calciferous Sandstone time. His own field work in the Bathgate Hills showed that volcanoes were active at intervals during the Carboniferous Limestone period on higher horizons than those of the Arthur's Seat volcano. He recorded the prevalence of later intrusive dykes traversing the Carboniferous strata. From his own researches and the work of other investigators, he prepared a paper on the "Chronology of the Trap Rocks of Scotland," published in the *Transactions* of this Society in 1861. In the map illustrating this communication the Trap Rocks are referred to the Old Red Sandstone, Carboniferous, Oolitic, and Tertiary periods. In accordance with the classification of Edward Forbes, the extensive basaltic plateaux of the West Highlands were assigned to the Jurassic period, except the lavas at Ardtun, in Mull, which are associated with the well-known leaf beds described by the Duke of Argyll. This grouping was corrected in an important paper that appeared in the *Proceedings* of this Society in 1867, in which he suggested that the basaltic plateaux, extending from the North of Ireland along the West Coast of Scotland to the Faroe Islands and Iceland, were all erupted probably during the Tertiary period. Even at that date he emphasised the importance of the system of intrusive dykes, which in his opinion was possibly the most striking manifestation of Tertiary volcanic activity.

While mapping the volcanic area in Fife between Burntisland and the Saline Hills, he visited the Auvergne in Central France, with its extinct but recent volcanoes, to increase his knowledge of volcanic phenomena. The evidence, which appealed strongly to his imagination, is described by him in the following artistic style:—

"To one who had been at work for some years among a set of old and fragmentary volcanic rocks, trying to piece together porphyrites, dolerites, basalts, and tuffs, the sight of those Puys, with their fresh cones and craters of ashes and scoriæ, and their still perfect floods of lava, was inexpressibly instructive. Merely to cast the eye over the landscape was of itself a memorable lesson. The scene was exactly what was needed to enable one to realise the character of those old British Carboniferous Volcanoes of which such mere fragments now remain. High up among the uplands of Central France my eye was ever instinctively recalling the hills and valleys of Central Scotland, and picturing their original scenery by transferring to them some of the main features in the landscapes of Auvergne. The imagination easily filled again with a sheet of deep blue water the broad expanse of yonder Limagne. Vines, and acacias, and mulberry-trees seemed to melt of their own accord into stately *sigillariæ*, *lepidodendra*, and *calamites*; the orchards and corn-fields along the slopes began to wave a dense underwood of ferns and shrubby vegetation; some of the cones rose fresh and bare, others were dark with a growth of araucarian conifers, and there, with but little further change, lay a landscape in the central valley of Scotland during an early part of the great Carboniferous period."

In 1868 he visited the volcanic district of the Eifel in Germany, and in 1870, at the request of Poulett-Scrope—author of the well-known volume on *Geology and Extinct Volcanoes of Central France*—he undertook an examination of the volcanic districts of Southern Italy and the Lipari Islands. He was impressed by the contrast between the comparatively low craters of the Phlegrean Fields, where the eruptive materials consist mainly of tuffs with few lava flows, and the great crater wall of Monte Somma, whose lavas are piled on each other to a great height and pierced by innumerable vertical dykes filling fissures made at successive eruptions. An attack of malarial fever prevented him from carrying out his intention of examining the Lipari Islands.

Another stage in his pursuit of the study of volcanic phenomena in Scotland is marked by an important paper contributed to the Royal Society of Edinburgh in 1879, on the "Carboniferous Volcanic Rocks of the Firth of Forth Basin—their Structure in the Field and under the Microscope."

As his field work proceeded in the Lothians and Fife, he felt the necessity of applying the microscope to the study of the igneous rocks in order to gain definite knowledge regarding their internal composition and structure. He realised that much new light might be thrown on the history of volcanic action in this region by this field of enquiry.

In his historical sketch he notes with pleasure that the igneous rocks of the Edinburgh district furnished Hutton with the evidence whereby he established the igneous origin of "whinstone" (basalt) and led to the famous experiments of Hall which laid the foundations of experimental geology.

The volcanic masses were grouped by him in four subdivisions: (1) Necks or Vents; (2) Intrusive Sheets and Dykes; (3) Contemporaneous Lavas; (4) Tuffs. A notable feature of this communication was the prominence given to the numerous necks or vents occurring in the Lothians, Fife, and Stirlingshire, from which proceeded showers of ashes and sheets of lava. Most of them were regarded as belonging to different stages of Lower Carboniferous time, while others were supposed to be later than the folding and faulting of the Carboniferous sediments, and were referred to the Permian period. The occurrence of these vents and their wide distribution is one of the valuable contributions made by Archibald Geikie to Scottish volcanic geology.

In the petrographical part of this paper he acknowledges Allport's researches on the Carboniferous dolerites round Edinburgh and gives an outline of his own investigations. Subsequent petrographical work has shown that he did valuable pioneer work in this branch of enquiry, for he was the first to describe many important features of the microscopic characters of these rocks.

Archibald Geikie's discovery of proofs of volcanic action during Permian time in Scotland is of special interest. While mapping the Mauchline district in Ayrshire, he recorded a series of contemporaneous lavas and tuffs underlying the Permian sandstones of that region, and forming a ring of higher ground between the Carboniferous and Permian sediments. This discovery was announced in the *Geological Magazine* in 1866. Similar types of lava were mapped by him in the Thornhill basin in 1868, where they are also associated with Permian sandstones. He laid special emphasis on the occurrence of volcanic necks in excellent preservation in Ayrshire which he referred to this period. Some of the smaller ones rise through the ring of Permian lavas, while others occur in the upper division of the Coal Measures. They are filled with agglomerate, pierced in some cases by igneous intrusions.

In describing the volcanic phenomena of Permian age, he suggested that the great series of volcanic vents in East Fife probably belonged to this period. He also suggested that the coarse agglomerates of the Arthur's Seat volcano, and the associated igneous rocks, which, in his opinion, marked a second period of volcanic activity, were probably erupted during Permian time. But the subsequent detailed mapping of the Geological Survey led him to accept Maclaren's later interpretation, confirmed by Professor Judd, that these coarse agglomerates occurred within the vent from which the lavas and tuffs of the hill had been discharged during one period of volcanic action in Lower Carboniferous time.

Another great opportunity of extending his knowledge of volcanic phenomena occurred to Archibald Geikie in 1879. He had arranged to give a course of lectures at the Lowell Institute, Boston, in the autumn of that year, and he resolved to spend the summer in traversing the extensive lava-fields, drained by the Snake River, in Idaho, on the Pacific slope of the United States. The vast floods of lava in that region, with no visible cones and craters, were explained by Richthofen as being due to fissure eruptions. Hitherto Geikie had regarded the Scottish volcanic plateaux as having issued from local vents, but this visit widened his conceptions. The great lava-plain looked as if it "had been filled with molten rock which had kept its level and wound in and out along the bays and promontories of the mountain slopes as a sheet of water would have done." The Snake River has cut a gorge through this plain, which exposes a succession of sheets of basalt to a depth of several hundred feet. No central cone from which these lavas might have been erupted was visible; only a few cinder cones of secondary origin appeared at wide intervals on the basaltic plain. The suggestion then occurred to him that the Tertiary volcanic plateaux of Western Europe might have had a similar origin.

Inspired by this conception, he revisited the West Highlands at intervals for several years to continue his investigations among the Tertiary Igneous Rocks. He was also impelled to do so by the remarkable interpretation of the volcanic history of that region advanced by Professor Judd. In a paper communicated to the Geological Society on "The Ancient Volcanoes of the Highlands" in 1874, Professor Judd described the basal wrecks of five great extinct Tertiary volcanoes (Skye, Mull, Rum, Ardnamurchan, and St Kilda); the one in Mull was estimated by him to have reached a height of 14,500 feet. They indicated three periods of volcanic activity. The first was marked by the extrusion of acid lavas and tuffs connected with plutonic masses of granite, the second by basaltic lavas and tuffs related to deep-seated masses of gabbro, and the third by the discharge of lavas from small sporadic cones after the great central volcanoes had become extinct.

At last, after a quarter of a century of intermittent labour in this subject, Archibald Geikie presented in 1888 to the Royal Society of Edinburgh his great memoir on "The History of Volcanic Action during the Tertiary Period in the British Isles." In the preparation of this monograph he acknowledges the assistance he received from several of his colleagues. A prominent feature of it is the elaborate description of the system of basic dykes, the importance of which he recognised early in his career. His main conclusions may thus be briefly summarised.

Owing to enormous horizontal tension, a series of more or less parallel fissures arose in Tertiary time in a tract of country including the North of England and Ireland, the southern half and the West Coast of Scotland—a total area of about 40,000 square miles. Molten material rose up these fissures, thereby giving rise to the numerous basic dykes which are the distinctive feature of the volcanic region. The basalt plateaux are supposed to be due to streams of lava issuing from these fissures and from vents occurring along these lines of weakness. After these sheets of lava had accumulated to a great thickness, they were injected by laccolitic masses of gabbro, and sills and veins of dolerite. At a later period the gabbros and basalt lavas were alike disrupted and pierced by acid igneous rocks, ranging from granites and granophyres to porphyries and felsites. Crustal movements again ensued, whereby another series of fissures was established, now filled with basic dykes that traverse alike the basalt plateaux, the later gabbros, granophyres, and granites.

It is interesting to observe that these main conclusions were confirmed by Dr Harker in the course of his detailed mapping of the central mountain group of Skye for the Geological Survey. But from the recent exhaustive memoir issued by the Geological Survey on the *Tertiary and Post-Tertiary Geology of Mull, Loch Aline, and Oban*, it is evident that the volcanic history of Mull is much more complicated than Sir A. Geikie or Professor Judd imagined. Mull is regarded as a volcanic centre of extreme complexity which "has repeatedly served as a focus of fissure eruptions; but it is doubtful whether the lavas still spared by erosion are not, in the main, the products of a central volcano, an idea always linked with the name of Professor Judd."

His last contribution to this branch of geology was his comprehensive treatise on the *Ancient Volcanoes of Great Britain*, which appeared in 1897. It presented a summary of the knowledge then ascertained regarding these volcanoes, and embodied the results of his own researches and of others who had worked at volcanic problems in the field and in the laboratory. It traversed a wide field of enquiry, for the opening chapters of the treatise are devoted to the discussion of the general principles and methods of investigation of volcanic phenomena, which are followed by detailed descriptions of the proofs of volcanic activity in Britain ranging from pre-Cambrian to Tertiary time.

In the traverses with Murchison in the North-west and Central Highlands in 1860, Archibald Geikie had to deal with problems of fundamental importance connected with the geology of the Highlands. The researches of Macculloch, Murchison and Sedgwick, Hay Cunningham and Hugh Miller, showed that the belt of quartzites and limestones in the West of Sutherland and Ross are succeeded eastwards by metamorphic rocks that stretch across the Great Glen to the eastern border of the Highlands. Accepting Salter's determination of the fossils found in these limestones by Mr C. W. Peach, Murchison regarded these strata as Silurian (now known to be Cambrian by Mr A. Macconochie's discovery of the *Olenellus* fauna in the Fucoid beds). He contended that these Silurian strata pass *conformably* below, and are overlain by, the metamorphic rocks to the east, and inferred that this metamorphic series must belong to the same system. This interpretation meant a radical change in the geological map of Scotland, for the area occupied by these altered strata amounts to about 11,000 square miles.

Professor Nicol, on the other hand, maintained that no conformable upward succession from the fossiliferous limestones and associated strata to the overlying schists is to be found. He held that the line of junction is a line of fault "everywhere indicated by proofs of fracture, contortion of the strata, and powerful igneous action."

It fell to the lot of Archibald Geikie to traverse rapidly with Murchison the line of junction in the county of Ross, where, owing to stupendous inversions and overthrusts, the prevalent dip of the fossiliferous strata and the Eastern Schists is towards the east-south-east. He was misled by the apparent superposition and especially by certain deceptive sections in which the Eastern Schists rest with similar dip and strike upon the undisturbed Silurian rocks. Eventually he accepted Murchison's interpretation.

The results of these traverses were embodied in a joint paper communicated to the Geological Society, London, in 1861, "On the Altered Rocks of the Western Islands of Scotland, and the North-western and Central Highlands." Murchison's interpretation of the structure was therein described and illustrated by sections in such convincing form that it met with general acceptance for many years.

In 1878 the controversy was reopened and Murchison's position was shown to be untenable by several investigators. Dr Hicks, Professor Bonney, and Dr Callaway made important contributions to the problem.

Professor Lapworth grasped the true solution of the geological structure of that region. In 1883 he began a series of articles in the *Geological Magazine* on "The Secret of the Highlands," based on his detailed mapping of the Durness-Eireboll region in 1882, but owing to severe illness this series was never completed. He therein demonstrated the inversion of the Silurian strata on the east side of Loch Eireboll and the unconformable junction of the basal quartzite with the old Archæan floor on the east side of the fold at Ant-Sron. From his paper on "The Close of the Highland Controversy" (*Geol. Mag.*, 1885), and from the "Obituary Notice of Charles Lapworth," by Professor Watts and Sir Jethro Teall (*Proc. Roy. Soc.*, 1921), it is clear that he recognised that the Archæan Gneiss had been driven over the fossiliferous quartzites on Ben Arnaboll by a gently-inclined overthrust fault. Along this plane of movement, and at other localities, the original rocks had been crushed and rolled out into types which he termed mylonites. On the shore at Heilim, on the east side of Loch Eireboll, he observed that the serpulite grit had been repeated many times by clean-cut faults, a striking illustration of imbricate structure. All these phenomena were shown in the field in 1883 to Sir Jethro Teall by Professor Lapworth.

The Durness-Eireboll region was mapped by us in 1883–4, when we reached conclusions practically identical with those of Professor Lapworth regarding the stratigraphy and metamorphism of the rocks, in complete ignorance of his results (see "Close of the Highland Controversy," p. 98, *Geol. Mag.*, 1885). It was then proved in the course of the Geological Survey work that under extreme lateral pressure the rocks behaved like brittle rigid bodies; they snapped and were driven westwards in successive slices, so that crystalline gneiss and schist are made to rest upon fossiliferous strata of Silurian age. It was further shown that the Eastern Schists were driven westwards by the Moine thrust—the most easterly and most powerful of the series—for a minimum distance of ten miles over all underlying thrust masses till they rest upon the unmoved Silurian (Cambrian) Limestone in the Durness basin.

The evidence proving these conclusions was carefully inspected in the field by Archibald Geikie, who had never had an opportunity of examining the Eireboll sections. He was completely convinced that Murchison's interpretation of the structure must be abandoned, and he took the earliest opportunity of making a public declaration to this effect. A report giving the results of the work by Peach and Horne, with a preface by Geikie, containing a frank confession that he had been misled and that he had accepted the conclusions of his colleagues, appeared in *Nature*, 13th Nov. 1884.

In dealing with the geological structure of the Silurian rocks of the Southern Uplands, he accepted the order of succession adopted by Nicol and Murchison, and was largely influenced in holding this opinion by the doctrine of Colonies. This theory was introduced by Barrande, the distinguished palæontologist, to explain the intercalation of fossils, belonging to higher zones, in lower portions of the Silurian succession in Bohemia. These precursory fossiliferous bands were termed Colonies. The distribution of the bands of graptolite shale in the Southern Uplands as mapped by the Geological Survey was regarded as an instance of the precursory appearance of the higher graptolite forms in the Moffat region and their disappearance in the Lead Hills district.

But the researches of Professor Lapworth furnished the key to the solution of the structure of the Chain. He demonstrated a definite faunal sequence in the graptolites which is persistent though the strata may be inverted by folding. He further proved a great variation in the character of the contemporaneous sedimentary deposits; the black shales of the Moffat region, ranging from Arenig to Llandovery time, about three hundred feet in thickness, being represented in the Girvan area by several thousand feet of sedimentary deposits.

Reference must be made to Archibald Geikie's tenure of the Chair of Geology for eleven years in Edinburgh University. In spite of the difficulties which confronted him at the outset, geology not then being required for any degree, his success was remarkable. The students formed a mixed assemblage—some who were interested in the subject and wished practical training, some who intended to follow other branches of natural science, and some who turned to geology for the pleasure it gave them. He set himself to lay the foundation of a class museum and introduced petrographical methods of studying rocks with the microscope when that branch of enquiry had not advanced far in this country.

By his lectures, which were given in a clear and attractive form, and especially by his class excursions, he was an inspiring teacher. Edinburgh is an ideal centre for practical training in field geology, but in addition, at the end of each session, he carried out a long excursion, lasting a week or ten days, to examine special problems in the field. Receptive students who took part in these excursions felt the influence of his personality. He thus describes the impression produced on one of his former students by these practical demonstrations:—

"These students' rambles and the love of geology which they fostered have dwelt ever since those days in Sir William Herdman's memory, and though he has become eminent in another field of natural science, he has assured me that it was the remembrance of his experience in the Geology Class at Edinburgh and its excursions, which led him in recent years to found and endow a Professorship of Geology in the University of Liverpool. It is not always that a teacher lives to see the fruition of his labours. Certainly no incident connected with my professorial career has given me keener pleasure than this generous liberality of a former student."

Archibald Geikie's eminent services as an exponent of geological science in his class-books and text-books demand special mention. His experience in the field during his early official career enabled him to test the truth of the principles laid down by Hutton and illustrated by Playfair regarding the evolution of the earth's surface features. The potency of the ordinary agents of denudation in hollowing out valleys was not admitted by some of the leading geologists of that time. Even

Sir Charles Lyell, who was the great exponent of the uniformitarian school, contended towards the close of his career that the principal valleys in almost every great hydrographical basin "have been due to other causes besides the mere excavating power of rivers."

Geikie was an ardent follower of the views of Hutton and Playfair. He was impressed with the remark of Desmarest when pointing out the value of Hutton's contributions to the natural history of the earth and the physical geography of Scotland: "It is to Scotland that Hutton's opponents must go to amend his results and substitute for them a more rational explanation." Geikie was convinced that Scottish topography furnished no grounds for such opposition. He produced in 1865 his volume on *The Scenery of Scotland viewed in connection with its Physical Geology*, which gave an excellent popular description of the Huttonian principles of earth-sculpture, as illustrated by the country he knew best. He showed with great clearness and artistic style how the ordinary agents of denudation—rain, rivers, the sea, the wind, and moving ice—acting upon different types of rock, had carved out the distinctive varieties of Scottish scenery. As the volume was based on personal acquaintance with the physical features of the country, and a knowledge of its geological structure then ascertained, it immediately arrested attention. Its success was marked; it ran through three editions, the later editions incorporating the more important advances in Scottish geology.

In view of the success of this volume, he then projected a series of educational works on Physical Geography and Geology to meet the public demand. In 1873 he contributed to Macmillan's elementary science series a Primer on each of these subjects—a task which cost him excessive labour. His aim in writing the *Physical Geography Primer* was to stimulate habits of observation of the common phenomena of everyday experience. It ranks as one of his best educational achievements. These Primers were followed by Class-books on the same subjects, for which there has been a great demand. The great educational value of these publications is beyond doubt. Their success is due to his teaching being permeated with the Huttonian conceptions of natural forces now in operation and to the exquisite literary form in which the lessons are presented.

These Primers and Class-books were meant to be the precursors of his great *Text-book of Geology*, the first edition of which appeared in 1882. This volume was intended for the use of students and professional workers in the science, and so well did it meet the requirements of the time that the whole issue was disposed of in a comparatively short period. One of the valuable features of the book was his analysis of the research done in other countries in each of the great divisions of the geological record. It reflected extensive reading and the careful preparation of brief summaries of the results achieved. He made a strenuous effort to keep each successive edition abreast of the research of the time, and, in dealing with his own record, he frankly abandoned positions which he had previously held when new evidence proved that they were erroneous. The fourth edition, published in 1903 after his retirement from the Survey, is notable for copious references to the geological literature of other countries, which cost him enormous labour. It is also notable for the qualification which he attached to Huttonian teaching, as he thought uniformitarianism had been pushed too far:—

"It has often been insisted upon that the Present is the key to the Past; and in a wide sense this assertion is eminently true. . . . While, however, the present condition of things is thus employed, we must obviously be on our guard against the danger of unconsciously assuming that the phase of Nature's operations which we now witness has been the same in all past time. . . . For aught we can tell, the present is an era of quietude and slow change compared with some of the eras that have preceded it."

A reprint of the fourth edition of the Text-book was issued in 1924.

His active brain and fertile pen enabled him to produce several biographies in the midst of other onerous labours. He completed the *Life of Edward Forbes,* begun by Dr Wilson; he wrote the *Memoirs* of James David Forbes, of Murchison, of Ramsay, who succeeded Murchison as Director-General of the Geological Survey. The *Life of Murchison* is of permanent interest to geologists from his vivid sketches of the Founders of Geology in Scotland and England, and from his description of the achievements of Sedgwick and Murchison in establishing the Cambrian, Silurian, and Devonian systems.

In concluding this outline of Sir Archibald Geikie's scientific career, we may express our belief that his future reputation will rest mainly upon his splendid services as an exponent of geological science rather than upon his own original researches. Be that as it may, it is certain that in the roll of eminent Scottish geologists who have passed, including Hutton, Hall and Playfair, Murchison and Lyell, Hugh Miller and Charles Maclaren, Ramsay and Nicol, James Geikie and Croll, Traquair and Kidston, he will take a prominent place as one who laboured strenuously to promote the science which he loved so well.

John James Graham Brown, M.D.

John James Graham Brown, who died at his residence in Edinburgh on 28th February 1925 at the age of 72, was elected to the Fellowship of the Royal Society of Edinburgh on 2nd July 1883. His earliest paper is published in the *Proceedings* of the Society for 1875, and is entitled "The Action of Bile Salts on the Animal Economy." He was born in Edinburgh, a son of the Reverend Thomas Brown, D.D., author of *The Annals of the Disruption.* Educated at Edinburgh Academy, he proceeded to the University of Edinburgh, where he graduated M.B., C.M., in 1875. A distinguished student, he was Senior President of the Royal Medical Society, in which he retained his interest throughout life. After graduation he worked for a time with the late Professor C. S. Roy on problems connected with the circulation; their results were published in 1879 in the *Journal of Physiology* under the title of the "Blood Pressure and its Variations in the Arterioles, Capillaries, and Smaller Veins"; this paper has become a classic. He afterwards spent some time studying medicine in the Universities of Berlin, Vienna, and Prague. In 1880 he started practice as a physician in Edinburgh. In 1882 he was elected a Fellow of the Royal College of Physicians in Edinburgh, of which body he became President in 1912. In 1885, along with C. S. Sherrington and C. S. Roy, he served as a member of a Commission appointed by the Royal Society, London, to investigate an epidemic of cholera then prevailing in Spain.

As a physician he devoted himself chiefly to diseases of the nervous system, and in 1909, when a Lectureship in Neurology was established in the University, he became the first lecturer. He had for many years given instruction in this and other subjects of clinical medicine in the wards of the Edinburgh Royal Infirmary.

Besides those already mentioned, the following are his chief publications:—Books: *A Manual of Clinical Methods*, 4th ed., 1897; *A Treatment of Nervous Diseases*, 1905. "On salicylate of quinine as an antipyretic," *Edin. Med. Journ.*, November 1876; "The therapeutics of diphtheria," *Journ. Anat. and Physiol.*, vol. xii, 1878; "On a new form of respirator for continuous inhalation," *Edin. Med. Journ.*, May 1883; "On bronchial contraction" (with C. S. Roy), *Journ. Physiol.*, vol. vi, 1885, p. xxi.; "On the changes of the circulation produced by rise in temperature," *Edin. Hosp. Reports*, vol. ii, 1894, p. 62; "The Perigraph: an instrument for delineating the shape of the thorax and for clinical mensuration generally," *Scottish Med. and Surg. Journ.*, July 1897; "Notes on the treatment of diseases of the nervous system," *Ibid.*, 1899–1903; "Note on a new form of æsthesiometer," *Journ. Physiol.*, vol. xxvii, 1901, p. 85; "Syphilis and life insurance," *Scottish Med. and Surg. Journ.*, January 1923; "Ataxia," *Edin. Med. Journ.*, 1912; "Hypertonus of the sympathetic in relation to intestinal toxæmia," *Ibid.*, February 1920; "A note on asthma: its pathology and treatment," *Ibid.*, February 1922; "Note on sthenic dyspepsia-hyperchlorhydria," *Ibid.*, October 1922.

A man of much ability and replete with original ideas, but of a modest and unassuming demeanour, Dr Graham Brown possessed a charming manner which was quite exceptional, and which, combined with his other sterling qualities, endeared him to his friends. His loss leaves a gap in the scientific and medical circles of Edinburgh which will not readily be filled.

He is survived by a daughter and three sons. His eldest son is Professor of Physiology in University College, Cardiff.

E. S. S.

John Young Buchanan, M.A., F.R.S.

John Young Buchanan was born in Glasgow on 20th February 1844. He was the second son of Mr John Buchanan of Dowanhill and the elder brother of the Right Hon. T. R. Buchanan, M.P., who represented Edinburgh in Parliament for many years. He was educated at the Glasgow High School and the University of Glasgow, where he graduated in Arts in 1863. His attention had been turned to Chemistry, and he proceeded to study this science on the Continent, spending several years in Germany at the Universities of Marburg, Bonn, and Leipzig, and going on in 1867 to Paris, where he worked in the laboratory of Würtz, a master for whose scientific genius and personal kindness he always cherished a warm admiration.

Shortly after his return to Scotland Buchanan was appointed Assistant to Professor Crum Brown, then recently established in the Chair of Chemistry in the University of Edinburgh, and in 1870, on Crum Brown's nomination, he was elected a Fellow of the Royal Society of Edinburgh.

When the voyage of H.M.S. *Challenger* was planned for the purpose of investigating the physical and biological conditions of the great ocean basins, it was decided that one representative of the physical sciences should be charged with all matters concerning physics, chemistry, and geology, except such hydrographical and meteorological observations as had been assigned to the naval officers. Buchanan was chosen for this comprehensive post by the Circumnavigation Committee of the Royal Society, who had prepared the programme of scientific work. His selection was due largely to his skill and resourcefulness in devising and making apparatus, his reputation as a practical chemist, his knowledge of mineralogy, and his interest in all natural phenomena. The wisdom of the choice was fully justified. Five years were entirely occupied by the *Challenger* work; the preparations for the cruise, and the fitting up of a chemical laboratory in the small space available on board, taking up several months before the ship sailed in December 1872, and the completion of work in hand occupying a longer time after the ship returned in June 1876.

Throughout those years Buchanan's mind was bent on the practical problems arising in the study of sea-water; but he took full advantage of the many opportunities for excursions into little-known lands when the ship was in port, and he acquired a love of travel that never left him.

He returned to Edinburgh and set up a private laboratory, in which he carried on research on his own account. The possession of ample means made it unnecessary for him to seek any professional appointment, and he continued to spend much of his time at sea. He made investigations on the west coast of Scotland and on the lochs of the Great Glen in his steam yacht, the *Mallard;* and he took part in several voyages of the cable-laying ships of the Silvertown Company on the west coast of Africa and across the Atlantic, mastering the new methods of sounding by wire, which superseded the clumsy processes of the *Challenger.*

In Edinburgh he entered into the remarkable revival in the study of physical geography and kindred sciences which was a feature of the eighth decade of the nineteenth century, and he took a share in establishing the Ben Nevis Observatory, the Scottish Geographical Society, and the Scottish Marine Station. He contributed a large quantity of material for the equipment of the Physical and Chemical laboratory of the original Marine Station at Granton in 1884, and the writer of this notice had many opportunities there of profiting by his instruction in the arts of observing and recording. In 1887 the Royal Society of Edinburgh awarded the Keith Prize to Buchanan for his work in chemical oceanography, and in London the Royal Society elected him a Fellow.

Buchanan was on terms of friendship with Professor Robertson Smith, who encouraged him in the study of physical geography, and helped to persuade him to accept the lectureship in Geography in the University of Cambridge in 1889. Buchanan was given the Cambridge degree of M.A., and took rooms in Christ's College, where Robertson Smith was already established. He resigned the lectureship after holding it for four years, but Christ's College continued to be his home for twenty years, and in his will he left substantial proof of the regard he entertained for it. On leaving Cambridge, Buchanan took up house in London; but he was much abroad, frequently visiting South America, where he had property in the Argentine, and staying at continental resorts, where he had many friends, some of them in very high positions. Kindred tastes cemented a close friendship with Prince Albert I. of Monaco, and he spent much time as the Prince's guest on shore and afloat, making many cruises in the Mediterranean, the Atlantic, and to Spitsbergen in the Prince's yachts, which were splendidly equipped for oceanographical research.

The outbreak of war in 1914 was a blow from which Buchanan never recovered. He had been on terms of personal friendship with the Kaiser and other distinguished Germans, while his old scientific associations with France had been strengthened by his membership of the governing body of the Oceanographical Institute, founded by the Prince of Monaco in Paris. In real distress of mind, Buchanan gave up his London house and betook himself to Havana, in the West Indies, until hostilities ceased.

His last work was the preparation of three volumes, published by the Cambridge University Press, containing reprints of those of his papers which appeared to him the most worthy of preservation. To these he prefixed very copious analytical Tables of Contents, with comments and criticisms often of great interest. The volumes were: *Scientific Papers*, vol. i.—*Oceanographical*, published in 1913; *Comptes Rendus of Observation and Reasoning*, in 1917; and *Accounts Rendered of Work Done and Things Seen*, in 1919.

After his return to London in broken health, the isolating shadow of old age fell on him, and he withdrew more and more from the society of his remaining scientific friends. He died at the age of eighty-one, on 16th October 1925.

Buchanan's scientific work was directed to the elucidation of practical problems presented to him during the voyage of the *Challenger*. He did not care for theoretical deductions or comprehensive generalisations. He seemed to work mainly for the satisfaction of his own mind, for once a definite result was obtained the problem lost its driving power; and only a portion of the notes which he amassed, with a care and precision that can only be described as meticulous, were ever worked up for publication. He used to say that he loved work but hated writing; still, the number of his published papers exceeds one hundred.

On the *Challenger* most of his time was filled by a routine prescribed by the Circumnavigation Committee, but the tedium was lightened by flashes of discovery. One was the fact of large concretionary deposits of manganese peroxide produced by chemical action of sea-water on minerals. Another was the discovery that the gelatinous substance found on all preserved deep-sea deposits, which Huxley had taken for a primeval organism and named *Bathybius Haeckeli*, was really a precipitate of calcium sulphate thrown down from sea-water by the addition of alcohol.

The short stay of the *Challenger* in Antarctic waters raised the question of the formation and melting of sea ice, on which Buchanan worked with great keenness for several years, producing a series of valuable papers on "Ice and Brines," in which he worked out the chemical and thermal changes accompanying the freezing of saline solutions and the temperature at which pure ice melted in them. This led to a further series of calorimetric researches on "Steam and Brines."

Another subject raised on the *Challenger* and settled in a shore laboratory, was the compressibility of glass and other substances under the pressure of the deep sea, ranging up to 5 or 6 tons per square inch. The apparatus invented for this purpose was handed over to Professor P. G. Tait, and used by him in determining the pressure corrections of the *Challenger* thermometers.

Buchanan's most important contribution to the *Challenger* Reports was a memoir on the "Salinity of Ocean Water," in which he published the first complete map of the world-wide distribution of surface salinity, the main features of which have been fully confirmed by later investigators.

His work on the Telegraph ships resulted in a study of continental slopes and oceanic shoals, which had scientific as well as practical value. One of his few general papers was a suggestive survey of Similarities in the Physical Geography of the Great Oceans, contributed to the Royal Geographical Society in 1886. In Limnology he was the first to prove that vertical circulation set up by the wind produced a winter temperature of the deep water in a lake far below the maximum density point.

The work of Buchanan was that of a pioneer breaking new ground, and it was his fate, which he recognised and perhaps resented, to make observations of fundamental importance, which were lost sight of under the superstructure raised by others whose names are better known to the scientific public. All he did was original work in the fullest sense. He owed nothing to other workers. He insisted always on going to the fountain-head, preferably by direct observation, for all data, and on handling such data in the way most likely to ascertain their true relationships even if that required difference from recognised authorities.

His character exhibited singular sincerity, and a kindliness which he did nothing to advertise. Although to acquaintances his manner may often have seemed cold and distant, his nature showed a very warm and friendly side to the few congenial comrades of his choice.

H. R. M.

Thomas Lowe Bunting, M.D.

BORN at Heanor in Derbyshire in 1868, Dr Bunting took his degree at Edinburgh in 1889 after a distinguished career as a student. Whilst at the University he became associated with Professor Patrick Geddes's work amongst the students, then just beginning, which grew into the Town and Gown movement, and had for its object to make the students take an interest in the welfare of the poor. This interest lasted with him all through his life. After leaving Edinburgh he went first to an asylum near Buxton and thence to Newcastle where he joined the old Fabian Society, and then and always took a deep interest in all social problems.

In 1904 he proceeded to his M.D., gaining a Gold Medal for his thesis which was on the Histology of Lymphatic Glands. Little had been done in this field since the classic investigations of His, which had been accepted for so very many years. But His had made his observations on the glands of the ox, the structure of which Dr Bunting found to be peculiar to the species and very different from the arrangement really found in Man.

In 1911 he took his B.Hy. and D.P.H., both with honours, at the University of Durham; but even before this he had begun to study Radiology, and he acted to the end of his life as Radiologist to the Children's Hospital at Newcastle-on-Tyne. On the outbreak of War he was appointed Radiologist to the Northumberland War Hospital, a heavy task for a man still in general practice; but he carried out the duties with the untiring devotion which made his department in the Hospital a complete success. His published papers on Radiological subjects are masterpieces of deductive reasoning.

Of his medico-political work a colleague writes: Dr Bunting was first interested in medico-political matters through his being on the old Northumberland Contract Committee before it was taken over by the B.M.A. From this, he finally became a leading figure in the North-East of England as Secretary of the Newcastle Panel Committee and one of the elected Medical representatives on the Newcastle Insurance Committee and Secretary of the North-Eastern groups of Panel Committees.

The agitation in 1912 over the National Health Insurance Act came to Dr Bunting as a call for work on behalf of the profession. This led to his devoting an increasing amount of thought and time to medical politics until he became in Newcastle the guide and adviser of the Panel Committee in general and of many individual doctors in particular.

Dr Bunting was elected a Fellow of this Society in 1905. He died of angina pectoris on the 18th of February 1925.

J. S. M'C. and H. L. R.

John Fraser, M.B., C.M., F.R.C.P.E.

Dr JOHN FRASER was born in Thornhill, Dumfriesshire, and died on the 18th January 1925, in his eightieth year, following an accident when alighting from a tramcar. His whole career was intimately associated with the treatment of the insane in Scotland. He graduated in Edinburgh University with First Class Honours in 1870. After acting as resident in the Royal Infirmary, Edinburgh, he turned his attention to mental diseases and became assistant Medical Officer to the Fife and Kinross District Asylum, Cupar. A few years later he succeeded the late Sir John Batty Tuke, whose daughter he married, as superintendent. After seven years here, he left to take up the duties of Deputy Commissioner in Lunacy, a post he held from 1878 to 1895, when he succeeded the late Sir Andrew Mitchell, K.C.B., as Commissioner. He held this appointment until 1910, when he retired on reaching the age limit. During the whole of his career he was an active force in advancing new ideas. He became particularly interested in the boarding-out system, which is so peculiar a feature of Scottish Lunacy administration, and by which mental cases of a quiet and harmless nature, capable of enjoying the liberties of a natural home life under slight supervision, are boarded out under home conditions as ideal as can be secured. He was a strong advocate of the removal of all restraint in the treatment of mental cases, and of the nursing of male cases by trained female nurses, now so characteristic of all Scottish mental hospitals. He was also largely responsible for perfecting the arrangements against fire in asylums, and in seeing that the safety of the inmates was properly safeguarded.

Since his retirement he occupied himself in assisting various charitable institutions in the city, especially the Society for Prevention of Cruelty to Children. He was a director of the Royal Scottish National Institution at Larbert for the Care of the Feeble-Minded. He was presented with his portrait, painted by Mr Fiddes Watt, R.S.A., on his retirement "in recognition of his long and useful service devoted to the interests of the insane, and of his professional and private worth." Temperamentally he was eminently suited for the work to which he had devoted his life, his genial, kindly nature and his enthusiasm proving invaluable to one occupying the positions he did, and enabling him to make for himself a prominent place among the great names associated with Scottish Lunacy administration.

Dr Fraser was elected a Fellow of this Society in 1896.

A. N. B.

Giovanni Battista Grassi.

27th March 1854—*4th May* 1925.

GIOVANNI BATTISTA GRASSI was born at Rovellasca, Province of Como, and received his training in zoology and in medicine in the University of Pavia. The whole of his subsequent work was zoological, but his medical training was of considerable advantage in his numerous researches on the Protozoa and Helminthes parasitic in man and animals.

He was appointed Professor of Zoology in the University of Catania in 1883, and remained there until 1895 when he was promoted to the Chair of Comparative Anatomy in the Regia Università, Rome, where the last thirty years of his active life were spent.

Of his many contributions to knowledge only a few of outstanding importance can be referred to in this brief notice.

His first investigation, begun on his own initiative when he was a fourth-year student, was on anæmia in the cats of Rovellasca; and in a paper published in 1878, he showed that the anæmia was due to the presence in the cat's intestine of a species of Ancylostoma (*A. balsami* = *caninum*), a nematode worm. During the course of this work he instituted the sound method of determining the presence of the worm in the intestine by finding the characteristic eggs in the fæces, and he applied the same method to detecting the infection of man with *Ancylostoma duodenale.* This worm is one of the most serious and frequent parasites of man in warmer countries; heavy infections produce severe anæmia and debility, and have resulted in numerous deaths. The discovery of a certain and rapid method of diagnosis was therefore a matter of prime importance. Grassi contributed other important helminthological observations, *e.g.* on the mode of infection of man by the round worm *Ascaris lumbricoides*, and on the tapeworm Hymenolepis, which he showed could complete its life-cycle in one host.

In protozoology he made notable advances in our knowledge of *Entamœba coli*, of *Giardia intestinalis*, and of the remarkable flagellates in the gut of Termites; and in his well-known *Studii di uno Zoologo sulla Malaria* (1900), with its beautiful illustrations, gave a wonderfully full picture of the various stages of the organisms of malaria in man and in the Anopheles mosquito.

As a student of the Arthropoda he was equally accomplished, as is evidenced by his splendid work (with Sandias) on the constitution and development of the society of Termites (1893, 1894), his investigations extending over many years culminating in a fine volume, in 1912, on the life-history and methods of control of Phylloxera—a serious enemy to vines, his account (1907) of *Phlebotomus papatasii*—the first satisfactory description of the structure and life-history of a sand fly (proved in 1909 to be the transmitter of a three-day fever which has come to be known as "sand-fly fever"), and his more recent papers on the biology of mosquitoes.

In the domain of marine biology also, Grassi made distinguished contributions. In 1883 his well-known monograph on the Chætognatha was published by the Naples Zoological Station, and not long after he went to Catania he began, with Calandruccio, a series of researches on the life-history and metamorphosis of the eels, the results of the last of which—on the metamorphosis of the Murænoids—were published in 1913. These researches largely dispelled the mystery in which the life-story of the eel had been enwrapped since the time of Aristotle.

Grassi was full of enthusiasm, and enlisted his colleagues and his pupils as collaborators in some of the investigations. In a few cases, unfortunately, this arrangement led to disagreement in regard to the apportionment of the credit for new discoveries. More serious was the controversy on the priority of the discovery of the cycle of human malaria, but this cannot prevent warm recognition of the excellence of Grassi's work on the parasites concerned.

His own researches and those which he inspired others to accomplish are of extent and importance fully to justify the statement that he was among the most fruitful zoological investigators of our time. Grassi's unbounded energy, vigorous personality, and wide interests made a deep impression on the writer on the three occasions when he visited the Institute of Comparative Anatomy in Rome.

Grassi was awarded the Darwin Medal of the Royal Society of London in 1896, was elected a Fellow of the Reale Accademia dei Lincei in 1897 (he had received the Premio Reale in 1888), and was made a Senator of the Kingdom of Italy in 1908. This Society elected him to the Honorary Fellowship in 1916.

J. H. A.

Andrew Gray, M.A., LL.D., F.R.S.

Andrew Gray was born on the 2nd of July 1847, at Lochgelly, Fifeshire, and spent his youth in his native village. The educational opportunities which the village provided were somewhat meagre, but he acquired a good knowledge of English and Mathematics and made some progress in Latin and Greek. At that time the favourite textbook in arithmetic was James Gray's *Introduction to Arithmetic*. After working through "The Gray," as the book was usually called—the name being a synonym for "arithmetic," just as "Euclid" was for "geometry,"—he was promoted to Davidson's *Practical Mathematics*; but in mastering this more advanced textbook he was left to a considerable extent to his own resources. These books are now almost unknown, but, in spite of their defects, they abounded in exercises of a practical kind that aroused the interest of a studious boy; and the custom of leaving the pupil to work by himself, and of giving help only after complete failure on his part, developed a spirit of independence that was afterwards of decided value for higher research.

Some years elapsed after Gray left school before he entered the University. Part of this period was spent in Edinburgh, and there he made the acquaintance of James Pryde, who is now perhaps best known as the editor of *Chambers's Mathematical Tables*. Pryde was one of the teachers in the old Watt Institution; Gray was an eager student, and soon attracted his teacher's attention. An invitation to Pryde's house was followed by frequent visits and by the loan of books on higher mathematics, among them Lagrange's *Mécanique Analytique*. The encouragement thus given, which was always gratefully remembered, was perhaps the most important factor in determining him to undertake a University course, and in 1872 he matriculated in the University of Glasgow as a student in the Faculty of Arts.

During his undergraduate course, like many other students at that time, he was obliged to devote a part of his day to teaching; but he overcame all obstacles, took a good position in all his classes, and distinguished himself specially in Mathematics and Natural Philosophy, finally graduating in 1876 as M.A., with Honours in these two subjects. The classes, outside those of Mathematics and Natural Philosophy, which he found most congenial were those of Latin and Greek, and he always retained an interest in these languages. There was no surer way of irritating him than to suggest that an expert in physical science could not be expected to have any knowledge or appreciation of the ancient classics.

In 1875 he was asked by Sir William Thomson to become his private assistant, and he remained with him in this capacity till 1880, when he was appointed official class assistant. From this date onwards his work lay in teaching and research in the department of Natural Philosophy.

When the University College of North Wales was established in 1884 Gray was appointed Professor of Physics. Of the other professors appointed at the same time, two, Henry Jones and James J. Dobbie, were also Glasgow graduates; another professor, not a Glasgow graduate, was G. B. Mathews, and with these three his friendship was peculiarly intimate. Though they did not remain long together at Bangor, the intimacy was constantly renewed during the vacations. Gray survived the others, and he felt their loss very keenly.

The Bangor professorship provided the opportunity for testing Gray's powers. He soon made for himself a name and position. His sound judgment, his business capacity, and his untiring industry proved to be invaluable in the adjustment of the many difficulties that beset a new college. In the wider field of University extension and secondary education he was also an active worker, and he took a leading part in the foundation of the County School for Girls at Bangor.

In 1899 Lord Kelvin retired from the Chair of Natural Philosophy in Glasgow, and Gray was appointed his successor. It was a signal honour to be selected to fill the Chair which Kelvin had occupied for fifty-three years, but Gray amply justified the choice of the University Court. In the latter part of this Notice reference is made to the development of the department under his charge. It is perhaps sufficient to say here that the qualities which had distinguished his Bangor professorship were brought into even greater prominence in Glasgow. He was regular in attendance at the meetings of the Faculties and the Senate and helpful in the discussion of all matters that came before them. For eight years he was one of the Senate's representatives on the University Court, and during that period he took an active part in the many developments that have been so marked a feature of recent years. Colleagues frequently sought his advice and gratefully acknowledged the kindness with which he helped them in their difficulties.

His activities, however, were not limited by the demands of the University. He delivered many lectures to learned societies and to audiences of electrical engineers and of shipbuilders. His University Oration in memory of Lord Kelvin was afterwards expanded, and was published in 1908 with the title *The Life and Work of Lord Kelvin.* He was a Member of Council of this Society from 1903 to 1906, and a Vice-President from 1906 to 1909. At the Bournemouth meeting of the British Association in 1919 he was President of Section A.

Gray had a fine physique, and for many years enjoyed excellent health; but in 1909 he had a serious illness from which he never fully recovered, though still able for the duties of the Chair. The death of a son in 1915, the strain of the war years, during which he was frequently called upon for work on various committees, and the heavy demands on him caused by the huge influx of students after the armistice, told very heavily on his health, and in 1923 he resigned the Chair. From the date of his retiral he was a confirmed invalid, and he passed away on the 10th of October 1925. He is survived by his widow, four daughters, and three sons, one of whom, James G. Gray, is Professor of Applied Physics in Glasgow University.

His writings cover a wide field. His first book, *Absolute Measurements in Electricity and Magnetism,* which grew out of a series of articles in *Nature,* was published in 1884. Its favourable reception led to the composition of the more comprehensive treatise, *The Theory and Practice of Absolute Measurements in Electricity and Magnetism,* which contains more than 1400 pages, and appeared in three volumes, the first in 1888 and the others in 1893. This book was written at a time when electrical lighting was coming into use and the need for accurate measurement was being felt. Owing to his association with Lord Kelvin, Professor Gray was able to speak with authority on the subject, and he spared himself no labour in reading original papers to make the work a well-arranged storehouse of information. It was consequently found useful by the members of the Electrical Standards Committee of the British Association and scientists all over the world, and undoubtedly did much to make experimental electricity an exact science. Some of the suggestions contained in the book as to standards and formulæ of inductance were taken up in great detail by the late Mr E. B. Rosa and his colleagues of the physical staff of the Bureau of Standards at Washington. A revised and rewritten edition of the book appeared in 1921.

The *Treatise on Bessel Functions and their Applications to Physics* written in collaboration with Professor G. B. Mathews, was published in 1895. The book was universally accepted as attaining the object the authors had in view, and is too well known to need any comment. A second and revised edition was brought out in 1922 by Professor Gray and Dr T. M. MacRobert.

When occupied with *The Theory and Practice of Absolute Measurements in Electricity and Magnetism*, the idea occurred to Professor Gray of writing a systematic book on electricity. *A Treatise on Magnetism and Electricity* was the result. This appeared in 1898, three years after Sir Joseph Thomson's *Elements of Electricity and Magnetism.* It covered the same range as the latter book, but the treatment was more mathematical and much more exhaustive.

In 1901 appeared the first volume of *A Treatise on Physics*, which was never completed. The first volume dealt with Dynamics and Properties of Matter, and was translated into German by Professor Auerbach of Jena. In his preface the translator says, that the book is characteristically English in its style, and that there is no doubt that the treatment is concise, clear, fair, and practical—qualities in which the British physicists are considered to excel. It is unfortunate that more time was not given to this project. We possess no book in English similar to Chwolson's *Physics*; and Professor Gray, with his great knowledge, balance, good literary style, and wide appeal, was pre-eminently the man to write one.

In 1911 appeared *A Treatise on Dynamics*, written in conjunction with his son Dr J. G. Gray; and in 1919 appeared the monumental *Gyrostatics and Rotational Motion*, the fullest treatment of this subject in any language. The subject was one that interested Professor Gray all his life; an interest that grew with the development of his son's models and his inventions during the war, and Professor Gray's knowledge of the equations involved became latterly almost intuitive. He visualised the angular momenta growing up in the different directions, even in the most complicated cases, and knew what the results were going to be, long before they were brought out by the analysis.

The work that he will be principally remembered for was his teaching and the reorganisation of the Natural Philosophy Department in the University of Glasgow. When he came to Glasgow from Bangor he found a department, the equipment of which had been allowed to fall behind the times, and laboratory classes of a couple of dozen students per year. Those who were with him during these first years remember how the appearance of the place altered visibly from week to week, and how the laboratory classes overflowed into attics and into the rooms under the tower, and the energy he put into the plans for the new Natural Philosophy Institute. As a result of his efforts the latter was opened in 1906. It is the largest building devoted to the teaching of physics in the country; and the classes in it grew steadily in numbers with the years, until at the time of his resignation, including arts, pure science, and the various branches of applied science, there were 600 or 700 students taking laboratory courses each year.

As a teacher he had rather a contempt for the narrow academic outlook. The courses had to be in touch with practical life. He exacted solid work from his students. Reproduction of bookwork never sufficed, but always some capacity had to be shown for attacking unfamiliar problems. As head of a department he was careful, far-seeing, and always full of encouragement to the advanced students and the younger members of his staff. He at times expressed his condemnation of men and things he disapproved of with a vigour that made those unacquainted with him think he might be difficult to work with. But such a conception was quite wrong. He never at any time indulged in recrimination, was always big enough to accept responsibility for the mistakes of others, and it was difficult sometimes to prevail on him to visit delinquents with the penalty of their sins, even when such action was necessary in the interests of discipline.

G. A. G.
R. A. H.

John Robertson Henderson, C.I.E., M.B., F.L.S.

BOTH in this country and in India, Professor J. R. Henderson made valued contributions to the knowledge of marine life, and for more than a quarter of a century he was an outstanding teacher of zoology in India.

Born in Melrose in 1863, John Robertson Henderson received his early education at Dulwich College and Dollar Academy, and later graduated in Medicine at the University of Edinburgh. Like many another Edinburgh graduate, he made use of his medical training as a groundwork for the study of natural history; and, influenced by the teaching of Sir Wyville Thomson and the interest aroused by the return of the *Challenger* from its memorable voyage, he turned to marine zoology. During these early years of research he worked in the Firths of Forth and Clyde, in the former from a floating marine laboratory, christened "The Ark," which lay in a disused quarry west of Granton, and in the latter from Sir John Murray's small steam-yacht *Medusa*. To the faunal lists of both areas he made several contributions, mainly relating to zoophytes, mollusca, echinoderms, and crustacea. These preliminary researches, contributed to the *Proceedings of the Royal Physical Society*, led to his being entrusted with the description of the Crustacea Anomura collected by the *Challenger*, which, embodied in a report of over two hundred pages illustrated by twenty-one plates, appeared in 1888.

Before this work was published, Henderson had already left for India, where, in 1886, he had been appointed Professor of Zoology in Madras Christian College. For twenty-six years he carried on his teaching with conspicuous success, retiring only in 1911, after he had been appointed Superintendent of the Government Museum and Principal Librarian of the Connemara Public Library, as well as Keeper of the Madras Aquarium, on the retiral of Mr Edgar Thurston in 1909. He was instrumental in obtaining an increase in the Museum staff, reorganised and extended the natural history collections, and re-equipped the Aquarium. Most notable was the success of his efforts to make both Museum and Aquarium centres of education for the people, and the school demonstrations which he inaugurated are still continued as part of the essential work of the institution.

Henderson continued in India his marine researches, and in 1897 published an account of the Hermit crabs (Paguridæ) collected by

H.M. Indian Marine Survey Steamer *Investigator*. The breadth of his interests are hinted at by another of his Indian publications, an illustrated catalogue of the coins of Tipu Sultan.

A Fellow of the Linnean Society, Henderson was elected a Fellow of Madras University in 1886, and in 1918 his services to the Government of India were recognised by the conferment of the Companionship of the Indian Empire. On his retiral, upon reaching the age limit, he returned to Edinburgh, when he resumed his activities, long interrupted, on the Royal Physical Society, of the Council of which, as well as of the Council of the Zoological Society of Scotland, he became a member. He was elected a Fellow of the Royal Society of Edinburgh in 1923.

His wide and accessible knowledge and the geniality of his disposition gained Henderson many warm friends in India and in this country. He died in Edinburgh, after a short illness, on 26th October 1925, and is survived by his widow, a daughter of the late Mr W. J. Adie, of Voe, Shetland, whom he married in 1921, and by a son and daughter of a former marriage. J. R.

Daniel Rankin Steuart, F.I.C.

Daniel Rankin Steuart, who died at his residence, Blackhall, on 1st August 1925, was well-known in scientific circles in the East of Scotland.

Born at Bogside, Lanarkshire, seventy-seven years ago, he was named after Dr Rankin of Carluke, the family physician, and as a child he came under the stimulating influence of this well-known geologist. His school education as a boy was much interrupted by ill health, and under the doctor's guidance he was introduced to the wonders of botany and geology out of doors.

During ten years of open air life devoted to farming and gardening, he built up sufficient health to enable him to take up the study of chemistry at Edinburgh under Crum Brown, and later at Glasgow under Dittmar. He also spent six months studying at Munich. For a few years he worked under the late Sir George Beilby as oilworks chemist at Oakbank, and in 1877 was appointed chief chemist of the newly started Broxburn Oil Company, in which capacity he remained for over forty years, taking a prominent part in the development of the works at Broxburn.

He was a Fellow of the Institute of Chemistry, and a member of the Chemical Societies of both Great Britain and America.

He did much original work in Shale Oils, and contributed papers to many learned societies, articles to standard chemical dictionaries, and an important memoir for the Geological Survey. One authority has bracketed Steuart, Redwood, Markovnikoff as the world's greatest petroleum technologists.

Some thirty years ago he made strenuous but unavailing efforts to have the laws altered to prohibit the use of dangerous lamp oils, but he lived to see the oils made safe through the extraction of the dangerous lighter fractions which are now required for motor spirit.

Five years ago he retired to Blackhall in failing health, having continued in harness until the War was over. He had always hoped to devote his leisure to literary work, but never regained sufficient strength to do so. He was elected a Fellow of this Society in 1916.

A. S.

Charles Tweedie, M.A., B.Sc.

The death of Charles Tweedie has removed one of the best-known names from the intimate circle of Scottish Mathematics, and has broken a mathematical link with the days of the late Professor Chrystal.

Charles Tweedie was born in Swinton, Berwickshire, in 1868, and was educated at George Watson's College, Edinburgh. In October 1885 he entered Edinburgh University as Sibbald bursar. He studied Mathematics and Natural Philosophy under Professors Chrystal and Tait. At this time Professor Chrystal was busy with the proof sheets of the second volume of his Algebra, and used to give them to his students. Tweedie's brilliant solutions of the exercises first brought him to the front. He took first place in Adv. Hons. Maths. and Adv. Hons. Nat. Phil. (equal), and in 1890 graduated M.A. with First Class Honours in these subjects. He obtained the Bruce of Grangehill and Falkland Scholarship, which was awarded to the most distinguished mathematical graduate of the year. In the same summer he graduated B.Sc.

In 1891 Mr Tweedie went to Göttingen University, where he studied higher mathematics under Professors Klein and Schwartz. Next year he continued these higher studies at the University of Berlin, under Professors Fuchs, Schlesinger, and Hettner. He then returned to Scotland, and in October 1892 was appointed Mathematical Assistant in the University of Edinburgh. Shortly afterwards, when Mr Allardyce was translated to a Chair of Mathematics in California, Mr Tweedie succeeded him, and became the Lecturer in Pure Mathematics. This appointment he held for over twenty years, and he became Professor Chrystal's right-hand man in the Department of Mathematics.

During his tenure of office Mr Tweedie had a broad experience of the Departmental work of a Scottish University. He never spared himself, and felt nothing a labour which was for the good of the Department. He won the lifelong esteem of Professor Chrystal. He was in sole charge of the mathematical classes in the old days of the "Summer Session." When the post of Official Adviser in Arts was instituted in the University Mr Tweedie was one of the two first holders of the title.

Mr Tweedie's preference was for Analysis, but his academic work lay chiefly in the branches of Higher Geometry. His lectures covered a wide field, and were highly appreciated. He was an ideal teacher, and devoted extraordinary care to the presentation of his subjects. He was greatly interested, too, in the school teaching of mathematics, and was associated with Mr Pressland in the production of a well-known Trigonometry. He was for many years a University Inspector of Schools under the Scottish Education Department. As such he did his best to promote a close intimacy between the Schools and the University, so that the school staff should be in touch with the requirements of the University, and that the transition for the pupil should not be too abrupt. On the mathematical side of the preliminary and bursary examinations his influence was marked.

Mr Tweedie was a man of extensive reading. He was a good classic, and an accomplished linguist. He was well read in the works of the Geometers of Britain, France, and Germany. In the happy days before 1914 he travelled extensively on the Continent. During his middle years at the University he devoted himself to Italian, and studied deeply the geometrical writings of the Italian school, for which he had a great admiration. With so catholic a taste in languages and books he became a keen bibliophile and specialised in the works of the Scottish mathematicians.

From 1907 to 1910 he was a Member of Council of the Royal Society of Edinburgh. He was also a contributor to its pages. Needless to say he was a Past-President of the Edinburgh Mathematical Society, of which he was one of the first and most active members. Fourteen of his papers, excluding notes, appear in its *Proceedings*. Some of his papers are printed in the *Mathematical Gazette*, and two or more in French in *L'Enseignement Mathématique*. For many years he was a member of the Circolo Matematico di Palermo, to which he contributed.

Mr Tweedie had always cherished the desire to write on the early Scottish mathematicians. After he had been compelled by illness to retire from his active duties in the University, he received a Carnegie Scholarship which encouraged him in spite of his rapidly failing health to undertake this work. In 1915 he published in the *Proceedings of the Royal Society of Edinburgh* an admirable survey of the *Geometria Organica* of Colin Maclaurin, Newton's great Scottish disciple. In this paper Mr Tweedie translated, abridged, and modernised this important work for the present-day reader. At the same time he produced in the *Mathematical Gazette* " A Study of the Life and Writings of Colin Maclaurin," in which he treats of the whole scope of Maclaurin's work.

Mr Tweedie then turned his attention to James Stirling—another great name, though known chiefly to the mathematical elect—politically a

Jacobite, mathematically an amazing computer, and later, commercially, for a living, a successful mining engineer. In 1920 Mr Tweedie published a "Life of James Stirling the Venetian," in the *Mathematical Gazette*, Two years later, after much research among Stirling's correspondence, he published a volume in the Clarendon Press, which dealt fully with the life, published works, and correspondence of Stirling. Here, again, as in the case of Maclaurin, he translated and displayed the works of Stirling in a suitable form for modern readers. Mr Tweedie has succeeded in getting some of Stirling's letters printed for the first time.

By this time his strength was almost spent. He was too ill to begin upon the Gregories, as he had hoped, so had to content himself with a briefer theme in Gray the arithmetician, the Scottish Cocker. This last paper appeared, as was fitting, in the *Proceedings of the Edinburgh Mathematical Society*. Mr Tweedie died on 14th September 1925 at the age of fifty-seven, and was unmarried.

E. M. H.

Barrett, Sir William Fletcher, Kt., F.R.S., M.I.E.E., was born on 10th February 1844, and educated at Old Trafford Grammar School, Manchester, and by private study. He was assistant to Professor Tyndall, 1863; Science Master, International College, 1867; Lecturer on Physics, Royal School of Naval Architecture, 1869; Professor of Physics, Royal College of Science, Dublin, 1873–1910; and ex-President of the Society of Psychical Research, of which he was one of the founders. He was the author of numerous important publications dealing with Physical and Psychical Research.

He was elected a Fellow of the Society in 1874, and died suddenly at his residence, 31 Devonshire Place, London, on 26th May 1925, aged 81. (For fuller details regarding his life and work, see *Proc. R.S.L.*, and *Nature*, 6th June 1925.)

Barton, Edwin Henry, D.Sc., F.R.S., was born on 23rd October 1858. He was educated at the People's College; University College, Nottingham; Imperial College of Science, London; and at the University of Bonn. For a number of years he acted as draughtsman and in other positions in engineering works in Nottingham. In 1893 he joined the staff of the Physics Department in the University College, Nottingham, and became Senior Lecturer and Demonstrator in 1895. Since 1905 he had occupied the Chair of Experimental Physics in the College. His publications include—*A Textbook of Sound*, 1908 (revised reprints 1914, 1919, and 1922); *Analytical Mechanics*, 1911; *An Introduction to the Mechanics of Fluids*, 1915 (jointly with the late Captain T. P. Black, M.Sc., Ph.D.); *Practical Physics for Colleges and Schools*, 1912 (revises 1918 and 1919); sole or joint author of over forty papers in the *Philosophical Magazine*, *Physical Society's Proceedings*, and in the publications of the Royal Societies of London and Edinburgh, 1894–1919.

He was elected a Fellow of this Society in 1895, and died on 23rd September 1925, at Nottingham. (For fuller details regarding his life and work, see *Proc. R.S.L.*, and *Nature*, 7th November 1925.)

Beddard, Frank Evers, M.A. (Oxon), D.Sc., F.R.S., was born at Dudley on 19th June 1858, and educated at Harrow and New College, Oxford. From 1882 to 1884 he was engaged in Edinburgh as a member of the staff of naturalists working at the material collected by the *Challenger* Expedition, under the direction of the late Sir John Murray. From 1884 to 1915 he held the appointment of Prosector to the Zoological Society, his duty being to make anatomical studies of animals which died in the Gardens. His reputation as a zoologist rests mainly on his investigations of the oligochæta, which gained for him the gold medal of the Linnean Society. He was elected a Fellow of the Royal Society in 1892. Dr Beddard's publications include—*Animal Coloration*, 1892; *Textbook of Zoogeography*, 1895; *Monograph of the Oligochœta*, 1895; *Structure and Classification of Birds*, 1898; many papers in the *Transactions* of the Royal Societies of London and Edinburgh, the Zoological Society, and in the *Quarterly Journal of Microscopical Science*, etc.

He was elected a Fellow of this Society in 1882, and died on 14th July 1925, at his home at Hampstead, aged 67. (For fuller details regarding his life and work, see *Proc. R.S.L.*, and *Nature*, 8th August 1925.)

BOWMAN, FREDERICK HUNGERFORD, D.Sc., F.C.S., F.I.C., F.L.S., A.M.Inst.C.E., was born of Scottish parents in Huddersfield, and spent the early part of his career in Halifax. He graduated D.Sc. at Edinburgh. He applied his scientific training to business, and became an employer in the woollen trade, a banker, and head of a chemical works in Northwich. He was also connected with the Chloride Electrical Storage Co., Clifton Junction.

Dr Bowman was elected a Fellow of the Society in 1884, and was the author of a paper "On some Variations in the Structure of Wool, and other Allied Fibres," published in the *Proceedings* (vol. xiii, 1884–6, p. 657). He died at his home in Manchester on 28th May 1925, aged 86.

CAMERON, JAMES ANGUS, graduated M.B., C.M. at the University of Edinburgh in 1867, M.D. (1876), and B.Sc., Public Health (1890). He also studied at Paris and Vienna. Retiring from the ordinary work of a medical practitioner he took up residence at Firhall, Nairn, and received the appointments of Medical Officer of Health and School Medical Officer of the County and Burgh of Nairn and Medical Officer of Health of the Counties of Banff and Elgin, which he held for a considerable period. He was an examiner for the B.Sc. in Public Health in the University of Edinburgh, a Deputy Lieutenant of Nairnshire, and a Justice of the Peace.

Dr Cameron was elected a Fellow of the Society in 1894, and died at Nairn in November 1924, in his 79th year.

DARWIN, Sir FRANCIS, Kt., D.Sc., M.B., LL.D., F.R.S., the third son of Charles and Emma Wedgwood Darwin, was born at Down, Kent, on

16th August 1848, and was educated at Trinity College, Cambridge, and St George's Hospital, London. He did not practise medicine, but became assistant to his father at Down, afterwards settling at Cambridge. Sir Francis was an Honorary Fellow of Christ's College, Cambridge, and was President of the British Association in 1908. His publications include—*Life and Letters of Charles Darwin*, 1887; *Charles Darwin*, 1892; and many papers on Physiological Botany from 1876 onwards; *Practical Physiology of Plants* (with late E. H. Acton), 1894; *Elements of Botany*, 1895; *More Letters of Charles Darwin* (with A. C. Seward), 1903; *Foundation of the Origin of Species*, 1909; *Rustic Sounds*, 1917; and *Springtime*, 1920.

He was elected an Honorary Fellow of this Society in 1916, and died at his residence, in Cambridge, on 19th September 1925. (For fuller details regarding his life and work, see *Proc. R.S.L.*, and *Nature*, 17th October 1925.)

PECK, Sir WILLIAM, had occupied the post of City Astronomer of Edinburgh, in charge of the Calton Hill Observatory, since 1889, when the erection of the new Royal Observatory on Blackford Hill set the older building with most of its instruments at liberty. He was of an active and inventive mind, and interested in all mechanical pursuits, besides astronomy. He constructed many of his own instruments. He was, in addition, a popular lecturer of considerable power and attraction, and was the author of a popular *Handbook and Atlas of Astronomy*, and other works. The City Observatory was devoted chiefly to showing the heavens to visitors—a service much appreciated by the citizens. For this purpose a six-inch photovisual telescope, presented to the Observatory, was of good service. In pursuance of the science, Sir William Peck visited Spain for the eclipse of 1905, and Egypt in 1908. He received the honour of Knighthood in 1917.

He was elected a Fellow of this Society in 1889, and died on 7th March 1925, after a long illness, aged 63 years.

ROBERTSON, JOSEPH M'GREGOR, M.A., M.B., C.M., F.R.F.P.S., received his medical education at the Universities of Glasgow and Berlin. He was Honorary Consulting Physician to the Royal Naval Hospital, Bute; visiting Physician to the Princess Louise Scottish Hospital for Limbless Sailors and Soldiers; and Visiting Physician, Stobhill Hospital, Glasgow. He had also acted as Surgeon at the Glasgow Ophthalmic Institute, as Lecturer in Physiology in the University of Glasgow, and Examiner in

Physiology to the Triple Examination Board. Dr Robertson wrote numerous contributions to Medical Journals, and was the author of *Physiological Physics*, 1884, and *An Elementary Textbook of Human Physiology*, 1887.

He was elected a Fellow of this Society in 1900, and died suddenly, while presiding at a health lecture in Glasgow, on 17th March 1925.

Thorpe, Sir Thomas Edward, Kt., C.B., Ph.D., D.Sc., Sc.D., LL.D., F.R.S., was born on 8th December 1845, near Manchester, educated at Owen's College, Manchester, and at the Universities of Heidelberg and Bonn. In 1870 he was appointed Professor of Chemistry in the Andersonian Institution, Glasgow. In 1874, on the establishment of the Yorkshire College, Leeds, he was appointed to the Chair of Chemistry, and in 1885 went to London to occupy a similar position in the Royal College of Science, South Kensington. In 1894 he was selected as Principal of the Government Laboratories, where he remained for sixteen years, afterwards returning to South Kensington as Professor of General Chemistry and Director of the Chemical Laboratories of the Imperial College of Science, receiving the title of Emeritus Professor on his retirement. In 1895 he was President of the Society of Chemical Industry; from 1899–1901, President of the Chemical Society; 1890, President, Chemical Section, British Association; Vice-President, 1900; and President of the British Association, 1921. During 1894–5 he served as Vice-President of the Royal Society.

He was an honorary or corresponding member of many scientific societies at home and abroad, received honorary degrees from numerous universities, and was knighted in 1909.

Sir Thomas Edward Thorpe was elected an Honorary Fellow of this Society in 1900, and died on 23rd February 1925, at his residence, "Whinfield," Salcombe, South Devon. (For fuller details regarding his life and work, see *Proc. R.S.L.*, ser. A, vol. cix, p. 18, and *Nature*, 7th March 1925.)

Young, George, Ph.D., F.I.C., F.C.S., died at his residence, "Bradda," 46 Church Crescent, Church End, Finchley, London, on 3rd April 1925, aged 56. He was elected a Fellow of this Society in 1892.

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PROCEEDINGS OF THE STATUTORY GENERAL MEETING

Beginning the 142nd Session, 1924–1925.

October 27, 1924.

At the Statutory Meeting of the Royal Society of Edinburgh, held in the Society's Lecture Room, 24 George Street, on Monday, October 27, 1924, at 4.30 P.M.,

Professor Frederick O. Bower, F.R.S., President, in the Chair,

the Minutes of the Meeting on October 22, 1923, were read, approved, and signed.

The **Chairman** nominated as Scrutineers the Rev. Dr A. **Mitchell Hunter** and Dr **Malcolm Wilson**.

The Ballot for the Election of Office-Bearers and Members of Council was then taken.

The **General Secretary** submitted the following Report:—

The activities of the Society have proceeded on the same general lines as in previous years. At three Meetings addresses were delivered, viz.: At the Statutory General Meeting, by the **President**, "Remarks on the Present Outlook on Descent"; and on the request of the Council, by Dr **Johs. Schmidt** of the Carlsberg Laboratorium, "Danish Researches on the Life-History of the Eel"; and by Professor T. H. **Bryce**, "The Early Development of the Human Embryo." In addition to these, 31 papers were read, as compared to 29 papers and 2 exhibitions in the previous Session. These were divided among subjects as follows:—Mathematics, 7; physics, 11; chemistry, 2; geology, 4; oceanography, 1; botany, 2; palæobotany, 2; anatomy, 1; and bacteriology, 1.

Nine papers have been published, or are in course of publication, in the *Transactions*, and 23 in the *Proceedings*—making 32 in all, besides Obituary Notices of Fellows deceased.

The Society has lost by death 22 Fellows, 1 was struck off the Roll, and 26 new Fellows were elected.

Owing to pressure of professional engagements, Professor E. M. **Wedderburn** resigned his position as the Society's representative on the Meteorological Committee of the Air Ministry, and Professor R. A. **Sampson** was appointed in his stead.

Professor W. **Peddie** has intimated his resignation as our representative on the National Committee on Physics. A successor has not yet been appointed.

It is satisfactory to note that the Society has been invited by numerous foreign and home Societies and Institutions to send a representative to their ceremonial occasions. Professor J. M. **Macfarlane** represented us at the centenary of Joseph Leidy at Philadelphia on December 6. At the cinquantenaire of the Société Française de Physique, November 30 to December 17, no representative attended, but a congratulatory letter was sent. At the jubilee of the Physical Society of London, in March, Professor C. G. **Darwin** attended, and presented an address on our behalf. At the cinquantenaire of the Société Mathématique de France, in May, Professor E. T. **Whittaker** attended, and presented an address. At the Kelvin centenary, the President, Professor **Bower**, served on the Committee, and the **President** and **General Secretary** attended the celebration in July. At the centenary of the Franklin Institute of Philadelphia, September 17 to 19, Professor A. W. C. **Menzies** attended, and presented an address.

The thanks of the Society are due to various bodies and individuals who have assisted its funds for publications. From the Carnegie Trust for the Universities of Scotland, grants, amounting in all to £48, 0s. 10d., have been received. Miss Knight has contributed £55 towards the printing of her paper, "Studies in the Ectocarpaceæ," Part i (*Trans.*, vol. lii, pp. 343-360). The Sir John Murray Scientific Trust has paid the whole cost of printing *Transactions*, vol. liv, Part i, "Deep Sea Deposits of the Atlantic Ocean," amounting to £375, 17s. 9d. (net). This work, describing material collected by Sir John Murray, and provided with an introductory discussion by Mr James Chumley, forms a fitting conclusion to the "Challenger" reports. Our thanks are also due to Mr Laurence Pullar, who offered a contribution towards the publication, which in the end was not required, and to our past President, Dr J. Horne, for his active interest in the matter.

The terms of Law XIX, restricting the length of tenure of various offices, have been changed, so as to permit the Treasurer alone, upon an emergency being declared, to be elected on more than ten successive occasions.

The Keith Prize for the period 1921–23 has been awarded to Professor J. W. GREGORY, F.R.S., and the Neill Prize for the same period to Professor J. M'LEAN THOMPSON.

The Society has been concerned by the restriction under the recent Treasury Order of the gratuitous circulation or exchange of publications embodying researches undertaken in Government institutions. A letter was addressed to the Secretary for Scotland upon the matter, but without any change of the position.

Negotiations have been in progress, and are now completed, by which the office of Assistant Secretary and Librarian is brought under the Federated Universities Superannuation System.

A scheme long debated has matured in a *World List of Scientific Periodicals,* which shall indicate the libraries in every district where each periodical is taken. It is edited from the British Museum. The local secretariat for Edinburgh has been taken by the Royal Society. With the valued co-operation of other Edinburgh libraries, the first section of 64 pages has been completed and returned, the second section is in hand, and subsequent ones will be dealt with as they arrive.

In response to a request for assistance to Tokyo libraries which suffered by the earthquake, certain books have been offered, including a long series of the *Edinburgh Medical Journal.*

The accounts for the current year present some satisfactory features. The income from contributions is increased. Contributions in arrear are reduced. A circular inviting voluntary contributions from Fellows who were exempt from the operation of Law VI, as amended in 1921, again received a liberal response, £123, 7s. being received, of which £52 consisted of commutations. In consequence, the Society has been able to replace intact the Special Subscription Fund at its original figure of £1128, 17s. 9d., besides carrying a credit balance of £259, 18s. 1d. to next year. It should, however, be remembered that this favourable financial account is in part due to a recent, and probably passing, reduction in the papers offered for publication, which is not in itself a desirable feature.

The Society has received from Sir Archibald Geikie the gift of a portrait of Sir J. J. Thomson; from the Geological Society of London, one of John MacCulloch; and from Sir Edward Sharpey Schafer, two—Sir William Huggins and Lord Rayleigh. These have been hung in our rooms.

Improvements have been made in the lighting and seating of the lecture-hall, but it is still not a wholly satisfactory apartment.

The TREASURER, in presenting his Report for the year, compared the Income and Expenditure with those of the previous years.

Dr W. B. BLAIKIE moved the adoption of the Reports, and the reappointment of Messrs LINDSAY, JAMIESON & HALDANE, C.A., as auditors of the accounts for the ensuing Session.

This was unanimously agreed to.

The PRESIDENT then presented the Keith and Neill Prizes for the periods 1921–1923 to Professor J. W. GREGORY, F.R.S., and Professor J. M'LEAN THOMPSON respectively, and in doing so read the following statements:—

KEITH PRIZE.

Professor GREGORY'S claims to the award of the Keith Prize are based partly on his papers published in the *Transactions* of the Society, and partly on his numerous contributions to Geology, extending over a period of thirty-six years.

Several papers have appeared in the *Transactions* of the Society. Of special interest there may be mentioned communications relating to the Geology of Benguella and Angola, published in 1916 and 1923. They contain much new evidence bearing on the sequence and distribution of the rocks in that part of West Africa. One important discovery involves a change of view as to the Upper Cretaceous advance of the sea on the land. Suess showed with great force that the marine transgression in Cenomanian time was almost world wide. But the fossils collected by Professor GREGORY suggest that the sea receded during that stage in the Angola region.

Professor GREGORY is widely known as a voluminous writer in different branches of Geology. He has travelled widely, and written copiously about the geological features of the regions that he has visited. He is the author of over 100 papers bearing on the Geology of the Alps, the Mediterranean basin, Africa, the West Indies, and Australia. He is the author also of the Catalogue of Fossil Bryozoa in the British Museum.

His original suggestions have stimulated discussion among geologists to a remarkable degree. The views set forth in the following works may be cited as examples:—"The Origin of the Great Rift Valley in East Africa," "The Origin of Fjords," "The Evolution of the River Systems in Chinese Tibet."

NEILL PRIZE.

The Neill Prize is awarded to Professor JOHN M'LEAN THOMPSON, D.Sc., primarily for a Memoir on "The Flowering of *Napoleona imperialis*, Beauv."; but also for other Memoirs published in the *Transactions* of the Society. These fall into two categories: those relating to the Anatomy of Ferns, and those concerned with Floral Development.

The former contain many valuable observations on rare material: in particular the ontogenetic elaboration of the stele was traced in relatively primitive ferns, and the actual progression observed from the protostele to the solenostele, together with the relation of that change to the successive leaf-traces. Though these Memoirs did not raise any new issues of prime importance, the details which they contained are essential for a satisfactory theory of the stele.

The Memoirs concerned with Floral Development have taken up the ontogenetic origin of Zygomorphy, as it is seen in the Andræcium. Two examples of distinct affinity, viz. *Greyia* and *Couroupita*, have been followed out developmentally, while the latter investigation led on naturally to that on *Napoleona*, which is the Memoir included in the period of award. These Memoirs are characterised by the great beauty of the illustrations. The facts disclosed have their own present value, but that value will be enhanced as the area of observation is further enlarged. This is already being done by the author himself.

The President then made some valedictory remarks dealing with the activities of the Society during his five years' term of office.

The following Obituary Notices were then read:—

1. Dr Robert Kidston, F.R.S. By the President. *Proc.*, vol. xliv, pp. 248–252.
2. Professor Johnson Symington, M.D., F.R.S. By Lt.-Col. J. C. Lamont, C.I.E. *Proc.*, vol. xliv, pp. 262–264.
3. Professor William Jack, M.A., D.Sc., LL.D. By Professor G. A. Gibson. *Proc.*, vol. xliv, pp. 255–257.
4. Dr David Fowler Lowe. By J. B. Clark, M.A. *Proc.*, vol. xliv, pp. 258–259.

The Scrutineers having reported that the Ballot Papers were in order, the President declared that the following Office-Bearers and Members of Council had been duly elected:—

Principal Sir James Alfred Ewing, K.C.B., M.A., D.Sc., LL.D., M.Inst.C.E., F.R.S., President.
Principal J. C. Irvine, C.B.E., Ph.D., D.Sc., LL.D., F.R.S.,
The Rt. Hon. Lord Salvesen, P.C., K.C.,
Professor J. H. Ashworth, D.Sc., F.R.S.,
Professor T. H. Beare, B.A., B.Sc., M.Inst.C.E.,
Walter Biggar Blaikie, LL.D.,
Sir Robert Blyth Greig, M.C., LL.D., F.Z.S.
} Vice-Presidents.
Professor R. A. Sampson, M.A., D.Sc., F.R.S., General Secretary.
Alexander Lauder, D.Sc.,
Professor W. Wright Smith, M.A.,
} Secretaries to Ordinary Meetings.
James Currie, M.A., LL.D., Treasurer.
A. Crichton Mitchell, D.Sc., Hon. D.Sc. (Geneva), Curator of Library and Museum.

ORDINARY MEMBERS OF COUNCIL.

Professor Thomas H. Bryce, M.A., M.D., F.R.S.
Professor J. Y. Simpson, M.A., D.Sc.
Professor D'Arcy W. Thompson, C.B., D.Litt., F.R.S.
Professor Sir James Walker, D.Sc., Ph.D., LL.D., F.R.S.
Professor E. T. Whittaker, Sc.D., F.R.S.
Professor Henry Briggs, D.Sc., A.R.S.M.
W. L. Calderwood, Esq.
Professor T. J. Jehu, M.A., M.D., F.G.S.
Professor C. G. Barkla, M.A., D.Sc., F.R.S.
Professor J. Graham Kerr, M.A., F.R.S., F.L.S., F.Z.S.
Lieut.-Col. A. G. M'Kendrick, M.B.
James Watt, W.S., F.F.A.

The President, in the name of the Society, thanked the Scrutineers for their services.

Principal Sir Alfred Ewing, K.C.B., F.R.S., the newly elected President, then took the Chair, and having made some inaugural remarks, adjourned the Meeting.

PROCEEDINGS OF THE ORDINARY MEETINGS,
Session 1924–1925.

FIRST ORDINARY MEETING.

Monday, November 3, 1924.

Principal Sir Alfred Ewing, K.C.B., F.R.S., President, in the Chair.

The following Gentlemen were formally admitted as Fellows of the Society:—Sir J. G. Frazer, O.M., Hon. Fellow, Professor W. Wilson, and Dr R. J. D. Graham.

By request of the Council the President delivered a Centenary Address on "The Life and Work of Lord Kelvin." *Proc.*, vol. xlv, pp. 1–2.

SECOND ORDINARY MEETING.

Monday, November 24, 1924.

Principal Sir Alfred Ewing, K.C.B., F.R.S., President, in the Chair.

Dr Eric Ponder was formally admitted a Fellow.

The following Communications were submitted:—

1. Note on Professor Whittaker's Atomic Model. By Mr J. A. Eldridge. Communicated by Professor E. T. Whittaker, F.R.S. *Proc.*, vol. xlv, pp. 245–248.
2. Experiments and Observations on Crustacea. Part VI: The Mechanism of Massive Movement of the Operculum of *Balanus nubilis*. By Professor J. Tait, D.Sc., M.D., and Dr W. F. Emmons. *Proc.*, vol. xlv, pp. 42–47.
3. Geology of the Outer Hebrides. Part II: South Uist and Eriskay. By Professor T. J. Jehu, M.A., M.D., F.G.S., and R. M. Craig, M.A., B.Sc. *Trans.*, vol. liii, pp. 615–641.
4. The Irreducible System of Concomitants of Two Double Binary (2, 1) Forms. By W. Saddler, M.A. Communicated by Professor H. W. Turnbull, M.A. (Read by title.) *Proc.*, vol xlv, pp. 3–13.
5. A Series Formula for the Roots of Algebraic and Transcendental Equations. By A. C. Aitken, M.A. Communicated by Professor E. T. Whittaker, F.R.S. (Read by title.) *Proc.*, vol. xlv, pp. 14–22.

SPECIAL MEETING.

Admission of H.R.H. the PRINCE OF WALES as an Honorary Fellow.

Wednesday, December 3, 1924.

Principal Sir Alfred Ewing, K.C.B., F.R.S., President, in the Chair.

The President called upon the General Secretary to make a statement regarding the constitution and aims of the Society. In reply Professor R. A. Sampson, F.R.S., made the following statement:—

> The Royal Society of Edinburgh, like its elder and greater sister of London, owes its origin to one of those private associations of an interested group, which are characteristic of the initiative of our race, and to which our common country owes so much. In 1739 there was established in Edinburgh a *Society for improving Arts and Sciences, particularly Natural Knowledge*, the members of which bound themselves to meet together and to undertake any enquiry or experiment recommended by the Society.
>
> Though the members were in the first instance limited to 45, their scope was ambitious, and ranged from Geometry and Astronomy to Medicine, Chemistry, and Botany. Among the more illustrious members of this Society were Colin Maclaurin, Joseph Black, David Hume and Henry Home, Lord Kames.
>
> Out of this germ the Royal Society of Edinburgh sprang. In 1783, King George III graciously granted it a Royal Charter of incorporation, and since then we have had the honour without inter-

mission of numbering the Head of the Royal House as an Honorary Fellow, including His present Majesty King George V. The Society was planned as an academy organised in two branches, one literary the other scientific. The names of some of its founder members—Adam Smith, Ferguson the historian, David Hume, the younger, as well as Walter Scott its third President—are evidence of this intention. The literary side fell early into some abeyance, but the Society has never taken a narrow outlook, and still draws its Fellows from men of law, letters, and music, and philosophy, as well as from the many branches of learning now designated as Science.

In another respect also the Society claims that it has taken a broad view. Though all maintenance falls upon the Fellows, it welcomes the introduction of strangers to its meetings, especially those strangers who, being students at the University, gain perhaps more than any other class from witnessing science in the making often by their own instructors and apart from crystallised handbooks. Further, it has made a practice to open its valuable library of scientific periodicals, including the exchanges of academies of the whole world, to all accredited students, whether Fellows or non-Fellows.

The Society has thus, for upwards of one hundred and forty years, furnished a house and hearth for fostering Science in Edinburgh, as well as a centre for Scotland, which is a matter of no small importance. It now numbers on its roll some 650 Fellows. But its chief and permanent monument of service is the series of its publications, which now number some 100 volumes, and are composed of memoirs communicated at its meetings. Contributors of high and recognised talent have never failed it. Especially in the subjects of Geology, Botany, Meteorology, Oceanography, and Thermodynamics, memoirs of cardinal importance are found in our volumes. In full proportion to this regular output of conscientious, highly competent work, which men of science rightly regard as their daily bread, we have had the pride from time to time of gaining the willing adhesion of men of genius, whose names are known to all the world. I would mention only Hutton the geologist and Charles Bell the physiologist in the earlier years; Lister the surgeon, James Clerk Maxwell and Lord Kelvin, mathematicians and physicists in the middle period.

The Society desires to acknowledge that it has received from outside handsome treatment. The noble apartments in which it is housed were provided at public cost by H.M. Government, which also contributes £600 per year towards the cost of its publications. The grants of the Carnegie Trustees to authors have also materially aided in producing the costly and beautiful reproductions, without which many announcements of discovery would be mere vague words.

Gratefully recognising those aids, it is none the less our pride, as it is that of other learned societies throughout the country, that without any private gain, by spontaneous association and by our own personal resources, our Fellows have not failed through a long succession of years to cherish and to promote natural knowledge, have lent a generous hand to obscure talent, have been the mouthpiece by which genius has chosen to speak, and by our labours have greatly and permanently enriched the world.

At a meeting of the Society held on June 7th, 1920, it is recorded in the minutes that His Royal Highness the Prince of Wales was elected an Honorary Fellow. This is in accordance with the terms of Law VIII, which provides that "Personages of the Blood Royal may be elected Honorary Fellows at any time on the nomination of the Council." The purpose for which the present Special Meeting of the Society is summoned is to admit His Royal Highness to the Honorary Fellowship which he has graciously accepted.

Addressing His Royal Highness the PRINCE OF WALES, Sir ALFRED EWING, then said:—

In asking Your Royal Highness to accept the Honorary Fellowship of this Society, we follow an ancient tradition which closely connects the Society with the Royal House. It is an election that does not impose or imply any scientific test. We exact no immediate contribution from you to the weighty contents of our published *Proceedings* and *Transactions*. But I do not forget that in an earlier stage of Your Royal Highness's career, it fell to me to superintend an examination in scientific and other knowledge which you passed with a degree of credit that entirely satisfied the examiners. It is not on that ordeal, however, that your right to admission here is founded. It might indeed be rash to assume that you have retained all you then knew. Nevertheless I venture to assert that no one can challenge the fitness for our Fellowship of one who has already done so much to advance the interests which our Society has at heart. There are few, if any, travellers whose journeys take so wide a sweep as yours; few, if any, ethnologists and sociologists whose understanding of peoples and their modes of life and thought is so comprehensive. And I would wish, Sir, as a charmed student of your speeches, that you would communicate to us, if not the results of your researches, at least something of your power of terse and apposite expression—your happy knack of choosing and placing the nail, and then hitting it on the head.

For though the Society is, as the Secretary has remarked, not neglectful of the claims of letters —and has numbered among its Presidents a literary star of the first magnitude—its main concern is with science. It was here that Lord Kelvin, who for many years held the same office, first made public some of his greatest discoveries. To you, Sir, these must especially appeal in one of their practical aspects, as having made it possible to link up in a network of nearly instantaneous communication the mother country with the most distant of the Dominions—Dominions which are now proud to hail you as their collective spiritual agent general.

Your Royal Highness does not need to be reminded that it is to science and the applications of science that the Empire owes in great part its security and prosperity; that alike in peace and in war, in nearly all the processes of life and conduct, science tends daily to take a more and more dominant place. In science, too, we have a world-wide interest, a common bond among all nations, a meeting-ground for agreement aloof from some, at least, of the differences that afflict and disturb in other spheres of thought. To these usually serene heights we now bid your Royal Highness welcome; and accordingly, in the name and by the authority of the Royal Society of Edinburgh, I admit you an Honorary Fellow thereof, and ask you to subscribe the roll.

His Royal Highness then signed the Roll as an Honorary Fellow, and replied as follows:—

That a Society which for over one hundred and forty years has devoted itself as a centre for Scotland to the improvement of the arts and sciences should have elected me as Honorary Fellow affords me considerable pride, and I highly appreciate the words which you have addressed to me on the occasion of my admission. Even in so universal a matter as science and learning, a local centre such as the Royal Society of Edinburgh has a very special value in encouraging local pride, for it has gained a world-wide recognition without sacrificing its national colour, and, by holding its own in the face of more powerful competitors, has successfully fostered the self-reliance of which the

Scottish people are so justly proud. The Society has to be congratulated, too, on the liberal view it has taken of its privileges, especially in the matter of opening its library to all serious scholars, and of encouraging students of the University to attend its meetings. As quite the newest member of the University of Edinburgh, it has not been possible for me to attend one of these gatherings at which papers are read by contributors of high and recognised talent. I am somewhat relieved to learn that no erudite contribution to your *Transactions* will be expected of me this afternoon, and I am grateful to you, Sir Alfred, for allowing me on this occasion to rest on the laurels of that examination over which you presided so many years ago when I was a naval cadet. But if the many journeys throughout the Empire, which I have been privileged of recent years to undertake, qualify me in any way for admission to these culture-crowded heights, I shall be more than proud.

I doubt, however, whether any scientific data which I have brought back from my various wanderings would satisfy a critical audience in this Lecture Hall, for I have no notebook full of astronomical determinations of latitude, local time, and compass variations, nor am I competent to discuss with Dr Horne any of my wanderings. But it is impossible to overestimate the value which these travels have been to me, and I am convinced that the first-hand knowledge of the peoples of the Empire which I have thereby gained will stand me in good stead in after life. This Society has ever since its foundation laboured to promote natural knowledge and pure science, the cultivation of which—for its own sake, without reckoning on profit or loss—is of vital importance to the country and to the world. May I repeat that I regard it as a high privilege to have been asked to sign the roll of such a body, on which are inscribed so many illustrious names, and I thank you very sincerely for the cordiality with which you have welcomed me on the occasion of my admission to your Honorary Fellowship.

THIRD ORDINARY MEETING.

Monday, December 8, 1924.

Principal Sir Alfred Ewing, K.C.B., F.R.S., President, in the Chair.

The following Communications were submitted :—

1. Obituary Notices :—

(*a*) The Rt. Hon. Lord Abercromby, LL.D. By A. O. Curle, Esq. *Proc.*, vol. xliv, pp. 268–270.

(*b*) Sir J. J. Dobbie, M.A., D.Sc., LL.D. By Dr Alexander Lauder. *Proc.*, vol. xliv, pp. 271–272.

2. The Electrolysis of Salts of Alkyloxyacids. By David A. Fairweather, B.Sc., Ph.D. Communicated by Professor Sir James Walker, F.R.S. *Proc.*, vol. xlv, pp. 23–33.

3. The Ionisation of Iodine Vapour by Ultra-Violet Light. By W. West, B.Sc., and E. B. Ludlam, M.A., D.Sc. *Proc.*, vol. xlv, pp. 34–41.

4. The Stability of Suspensions. I. The Rate of Precipitation of Kaolin Suspensions by Salts at varying Hydrogen Ion Concentrations. By William Ogilvy Kermack and William T. H. Williamson. Communicated by Dr Alexander Lauder. *Proc.*, vol. xlv, pp. 59–70.

5. The Influence of Gelatin on the Stability of a Colloidal Solution of Cholesterol, and on the Charge on the Particles. By Peter MacCallum and William Ogilvy Kermack. Communicated by Dr Alexander Lauder. (Read by title.) *Proc.*, vol. xlv, pp. 71–89.

6. The Action of Salts with Multivalent Cations on Colloidal Solutions of Gold and Gum Benzoin. By William Ogilvy Kermack and Cecil I. B. Voge. Communicated by Dr Alexander Lauder. (Read by title.) *Proc.*, vol. xlv, pp. 90–101.

FOURTH ORDINARY MEETING.

Monday, January 12, 1925.

Principal Sir Alfred Ewing, K.C.B., F.R.S., President, in the Chair.

The following Communications were submitted :—

1. The Permian Fish *Dorypterus*. By E. Leonard Gill, M.Sc. Communicated by Dr James Ritchie. *Trans.*, vol. liii, pp. 643–661.

2. The Law of Blackening of the Photographic Plate at Low Densities. By E. A. Baker, B.Sc. Communicated by Professor R. A. Sampson, F.R.S. *Proc.*, vol. xlv, pp. 166–186.

3. The Modes of Vibration of a Stretched Membrane with a Particular Law of Density. By Dr E. L. Ince. *Proc.*, vol. xlv, pp. 102–116.

FIFTH ORDINARY MEETING.

Monday, February 9, 1925.

Principal Sir Alfred Ewing, K.C.B., F.R.S., President, in the Chair.

The following Communications were submitted:—

1. Obituary Notices:—

(*a*) Professor Charles Hunter Stewart. *Proc.*, vol. xliv, pp. 280–281.
(*b*) Major-General W. B. Bannerman. *Proc.*, vol. xliv, pp. 273–278.
(*c*) Hector Macpherson. *Proc.*, vol. xliv, p. 279.

2. The Relation of Sea Growth to the Spawning Frequency in *Salmo salar.* By W. L. Calderwood, Esq., Fishery Board for Scotland. *Proc.*, vol. xlv, pp. 142–148.

3. A Monograph on the General Morphology of the Myxinoid Fishes based on a Study of Myxine. Part VI. The Blood Vascular and Lymphatic Systems. By Professor F. J. Cole, D.Sc. Communicated by Professor J. H. Ashworth, F.R.S. *Trans.*, vol. liv, pp. 309–342.

4. The Theory of Compound Determinants from 1900 to 1920. By Sir Thomas Muir, C.M.G., LL.D., F.R.S. (Read by title.) *Proc.*, vol. xlv, pp. 187–212.

5. Addendum to Dr E. B. Ludlam's and Mr West's paper, "The Ionisation of Iodine Vapour by Ultra-violet Light." *Proc.*, vol. xlv, pp. 34–41.

SIXTH ORDINARY MEETING.

Monday, February 23, 1925.

Principal Sir Alfred Ewing, K.C.B., F.R.S., President, in the Chair.

By request of the Council Dr B. N. Peach, F.R.S., and Dr John Horne, F.R.S., gave an address on "The Scientific Career of Sir Archibald Geikie, O.M., K.C.B., F.R.S." *Proc.*, vol. xlv, pp. 346–361.

SEVENTH ORDINARY MEETING.

Monday, March 9, 1925.

Principal Sir Alfred Ewing, K.C.B., F.R.S., President, in the Chair.

The Ballot for the election of Ordinary Fellows was taken, Mr H. M. Cadell and Dr G. G. Chisholm being named as Scrutineers. The following Candidates, who had been recommended by the Council, were declared by the President duly elected:—Alexander Craig Aitken, The Rt. Hon. The Earl of Balfour, Thomas Wm. Naylor Barlow, Albert William Borthwick, Sahay Ram Bose, George Stuart Carter, James Chumley, The Hon. Lord Constable, Robert Meldrum Craig, Alexander John Dey, Alexander Graham Donald, Henry Dryerre, John Samuel Dunkerly, George Samuel Eastwood, William Leonard Ferrar, William King Gillies, Arch. Hayman Robertson Goldie, David Middleton Greig, Alex. Macmillan Heron, The Hon. Lord Sands, Charles Norman Kemp, William Ogilvy Kermack, Leonard Augustus Lucas King, William Henry Lang, James Malloch, The Rt. Hon. The Viscount Novar, William Gammie Ogg, David Penman, Harold Robinson, William Saddler, Robert Arnot Staig, Alexander Stevens, David Smith Stewart, Richard L. Sutton, Harry Moreton Stanley Turner, Frederick Walker, David Percival Dalbreck Wilkie, Harold A. Wilson.

The following Communications were submitted:—

1. Discontinuities in the Atmosphere. By Major A. H. R. Goldie, B.A. *Proc.*, vol. xlv, pp. 213–229.

2. Stone Decay and the Preservation of Buildings. By Principal A. P. Laurie, D.Sc.

3. An Investigation of the Absorption of Superposed X-Radiations. By W. H. Watson, M.A. Communicated by Professor C. G. Barkla, F.R.S. *Proc.*, vol. xlv, pp. 48–58.

4. Note on Whittaker's Quantum Mechanism. By Professor H. S. Allen, M.A., D.Sc. *Proc.*, vol. xlv, pp. 245–248 (Eldridge).

5. The Minimum System of Two Quadratic Forms. By Professor H. W. Turnbull, M.A. and Mr J. Williamson, M.A. (Read by title.) *Proc.*, vol. xlv, pp. 149–165.

6. The Equation of Conduction of Heat. By Marion O. Gray, M.A. Communicated by Professor E. T. Whittaker, F.R.S. (Read by title.) *Proc.*, vol. xlv, pp. 230–244.

EIGHTH ORDINARY MEETING.

Monday, March 23, 1925.

Professor J. H. Ashworth, D.Sc., F.R.S., Vice-President, in the Chair.

The following Gentlemen were formally admitted as Ordinary Fellows of the Society:—The Hon. **Lord Constable**, The Hon. **Lord Sands**, **Alex. John Dey**, **Alex. Graham Donald**, **William King Gillies**, **William Ogilvy Kermack**, **William Gammie Ogg**, **David Smith Stewart**, and Dr **Joseph Beard**.

The **General Secretary** read the following Obituary Notices:—

Sir W. A. Herdman, F.R.S. *Proc.*, vol. xliv, pp. 282-285.
Sir Jethro Teall, F.R.S. *Proc.*, vol. xliv, pp. 286-287.

By request of the Council, Dr **Andrew Balfour**, C.B., C.M.G., of the London School of Hygiene and Tropical Medicine, gave an Address on "Reflections on Malaria," illustrated by a Rockefeller Foundation Educational Motion Picture and Lantern Slides.

NINTH ORDINARY MEETING.

Monday, May 11, 1925.

Principal Sir Alfred Ewing, K.C.B., F.R.S., President, in the Chair.

The following Gentlemen were formally admitted as Ordinary Fellows of the Society:—Dr G. S. **Carter**, Mr R. A. **Staig**, and Dr H. **Dryerre**.

The **President** intimated that the Council had made the following awards:—

The **Gunning Victoria Jubilee Prize** for the period 1920-1924, to Sir **Joseph John Thomson**, O.M., F.R.S., in recognition of his great discoveries in Physical Science.

The **Makdougall-Brisbane Prize** for the periods 1920-1924, to Professor H. **Stanley Allen**, D.Sc., for his papers on the Quantum and Atomic Theory, published in the Society's *Proceedings* within the periods.

The following Communications were submitted:—

1. Unilateral Vasoligation on the Senile Male of the Domestic Fowl. By Dr F. A. E. **Crew**. *Proc.*, vol. xlv., pp. 249-251.
2. A Survey of Clyde Plankton. By Miss **Sheina M. Marshall**. Communicated by Professor J. **Graham Kerr**, F.R.S. *Proc.*, vol. xlv, pp. 117-141.
3. The Continuity of the Vertebrate Nervous System. Studies on *Lepidosiren paradoxa*. By Miss **Frances M. Ballantyne**. Communicated by Professor J. **Graham Kerr**, F.R.S. *Trans.*, vol. liii, pp. 663-670.
4. Perthshire Tectonics: Loch Tummel, Blair Atholl, and Glen Shee. By E. B. **Bailey**, B.A., M.C. *Trans.*, vol. liii, pp. 671-698.

TENTH ORDINARY MEETING.

Monday, May 25, 1925.

Sir Robert Greig, LL.D., Vice-President, in the Chair.

The following Gentlemen were formally admitted Ordinary Fellows of the Society:—Professor L. A. L. **King** and Professor H. A. **Wilson**.

The following Communications were submitted:—

1. Size in Relation to Internal Morphology. No. 2: The Vascular System of *Selaginella*. By **Claude W. Wardlaw**, B.Sc. Communicated by Professor F. O. **Bower**, F.R.S. *Trans.*, vol. liv, pp. 281-307.
2. The Parasitism of *Cuscuta reflexa*, Roxb. By J. **Thomson**, M.A., B.Sc. Communicated by Professor F. O. **Bower**, F.R.S. *Trans.*, vol. liv, pp. 343-356.
3. Some Points in the Anatomy of *Dicksonia*. By S. **Williams**, M.Sc. Communicated by Professor F. O. **Bower**, F.R.S. *Proc.*, vol. xlv, pp. 286-296.
4. Studies in Ammonites of the Family Echioceratidæ. By Dr A. E. **Trueman** and Miss **Daisy M. Williams**. Communicated by Dr **John Horne**, F.R.S. *Trans.*, vol. liii, pp. 699-739.

ELEVENTH ORDINARY MEETING.

Monday, June 8, 1925.

Principal Sir Alfred Ewing, K.C.B., F.R.S., President, in the Chair.

Professor T. Hudson Beare, V.-P., took the Chair while the President lectured.

The following Gentlemen were formally admitted Ordinary Fellows of the Society:—Mr A. H. R. Goldie, Dr H. Robinson, and Mr W. L. Ferrar.

The following Communications were submitted:—

1. A Ball-and-Tube Flowmeter. (With Experiments.) By Principal Sir Alfred Ewing, K.C.B., F.R.S., President. Vol. xlv, pp. 308–321.
2. L'Entretien des Pendules au Moyen de Cellules Photo-électriques. By M. le Général G. Ferrié. Communicated by Professor R. A. Sampson, F.R.S. *Proc.*, vol. xlv, pp. 261–268.
3. A Colour Vision Spectrometer. By Professor W. Peddie, D.Sc. Vol. xlv, pp. 302–307.
4. Magnetic Quality in Crystals: Discrimination of, and Stability in, Molecular Lattices. By J. Forrest, M.A., B.Sc. Communicated by Professor W. Peddie, D.Sc.
5. On the Cardinal Function of Interpolation-Theory. By W. L. Ferrar, M.A. (Read by title.) *Proc.*, vol. xlv, pp. 269–282.
6. The Electrosynthesis of *n*— Duotriacontane Dicarboxylic Acid. By Dr D. A. Fairweather. Communicated by Professor Sir J. Walker, F.R.S. (Read by title.) *Proc.*, vol. xlv, pp. 283–285.

TWELFTH ORDINARY MEETING.

Monday, June 22, 1925.

Principal Sir Alfred Ewing, K.C.B., F.R.S., President, in the Chair.

The following Gentlemen were formally admitted Ordinary Fellows of the Society:—Professor W. H. Lang and Dr A. C. Aitken.

The following Communications were submitted:—

1. Contributions to the Flora of the Old Red Sandstone of Scotland. (*a*) On Plant-Remains from the Fish-Beds of Cromarty. (*b*) On a Sporangium-bearing Branch-System from the Stromness Beds. By Professor W. H. Lang, F.R.S. *Trans.*, vol. liv, pp. 253–279.
2. Rejuvenation of the Aged Fowl through Thyroid Medication. By Dr F. A. E. Crew. *Proc.*, vol. xlv, pp. 252–260.
3 .On the Vertical Force Changes during the "Sudden Commencement" of a Magnetic Storm. By Dr A. Crichton Mitchell. *Proc.*, vol. xlv, pp. 297–301.

THIRTEENTH ORDINARY MEETING.

Monday, July 6, 1925.

Principal Sir Alfred Ewing, K.C.B., F.R.S., President, in the Chair.

The Makdougall-Brisbane Prize for the Period 1922 to 1924 was presented by the President to Professor H. Stanley Allen, M.A., D.Sc., for his investigations in theoretical physics, particularly for his communications to this Society on the magnetic character of the quantum, and on static molecular models of hydrogen and helium.

The Viscount Novar was formally admitted a Fellow of the Society.

The following Communications were submitted:—

1. The Alimentary and Respiratory Systems of *Calamoichthys calabaricus* Smith. By George Leslie Purser, M.A., F.Z.S.
2. Salmon (*Salmo salar*) of the River Moisie (E. Canada). By Mr W. J. M. Menzies. Communicated by W. L. Calderwood, I.S.O. *Proc.*, vol. xlv, pp. 334–345. (Postponed until a later meeting.)
3. Precipitation of Sols by Polyvalent Ions. By Dr W. W. Taylor. *Proc.*, vol. xlv, pp. 323–333.
4. A Contribution to the Life History of Bowenia. By Professor A. Anstruther Lawson, D.Sc. *Trans.*, vol. liv, pp. 357–394.
5. Ammonites from the Upper Kimmeridge Clay. By Dr E. Neaverson. Communicated by Professor W. T. Gordon, D.Sc.
6. Meteorological Factors affecting Fertility in Sheep. By Mr J. E. Nichols. Communicated by Dr F. A. E. Crew.

PROCEEDINGS OF THE STATUTORY GENERAL MEETING

Ending the 142nd Session, 1924–1925.

At the Statutory Meeting of the Royal Society of Edinburgh, held in the Society's Lecture Room, 24 George Street, on Monday, October 26, 1925, at 4.30 P.M.,

Principal Sir Alfred Ewing, K.C.B., F.R.S., President, in the Chair,

the Minutes of the Statutory Meeting, on October 27, 1924, were read, approved, and signed.

The PRESIDENT then delivered an Address on "SOME MODERN ASPECTS OF PHYSICAL RESEARCH." *Proc.*, vol. xlvi, pp. 1–8.

The GENERAL SECRETARY submitted the following Report:—

At three Meetings addresses were delivered, viz.—at the first Ordinary Meeting, by the PRESIDENT, on "The Life and Work of Lord Kelvin"; on February 23, by Dr B. N. PEACH, F.R.S., and Dr JOHN HORNE, F.R.S., on "The Scientific Career of Sir Archibald Geikie, O.M., K.C.B., F.R.S."; and on March 23, by Dr ANDREW BALFOUR, "Reflections on Malaria."

At a Special Meeting of the Society, held on December 3, 1924, before a distinguished gathering, His Royal Highness the Prince of Wales was formally admitted to Honorary Fellowship, having been elected an Honorary Fellow on June 7, 1920.

In addition to the above-mentioned addresses, 46 papers were read, as compared to 31 papers in the previous session. These were divided among subjects as follows:—Mathematics, 8; physics, 7; meteorology, 2; chemistry, 8; geology 4, botany, 4; palæobotany, 2; zoology, 8; and agriculture (animal breeding), 3. 13 papers have been published, or are in course of publication, in the *Transactions*, and 29 in the *Proceedings*—making 42 in all, besides Obituary Notices of Fellows deceased.

The Society has lost by death 18 Ordinary Fellows and 3 Honorary Fellows, and 38 new Fellows were elected.

The following Gentlemen were appointed Members of National Committees:—

Astronomy . . .	vice Professor G. FORBES, F.R.S.—Sir J. LARMOR, F.R.S.
Geodesy and Geophysics .	vice Dr A. C. MITCHELL—Sir J. S. FLETT, F.R.S.
Mathematics . . .	vice Professor E. T. WHITTAKER, F.R.S.—Professor B. B. BAKER.
Physics	vice Professor W. PEDDIE—Professor H. S. ALLEN.
Radio-Telegraphy . .	vice Dr J. R. ERSKINE MURRAY—Professor C. T. R. WILSON, F.R.S.

Professor T. HUDSON BEARE and Mr W. A. TAIT, D.Sc., M.Inst.C.E., represented the Society at the Congress of the Royal Sanitary Institute, held in Edinburgh from July 20 to 25, 1925. At the International Congress of Geography, held in Cairo from April 1 to 9, 1925, Dr W. F. HUME represented the Society. At the opening of the Hebrew University of Jerusalem, April 1, 1925, Professor PATRICK GEDDES represented the Society, and presented an address on its behalf. At the celebration of the centenary of the Accademia Gioenia di Scienze Naturali, Catania, no representative attended, but a congratulatory letter was sent. At the 200th anniversary of the Academy of Sciences, Leningrad, held from September 6 to 14, 1925, Professor D'ARCY W. THOMPSON, C.B., F.R.S., attended, and presented an address from the Society. An invitation was also received to be represented at the centenary celebrations in Paris, on October 11, 1925, of Chevreul, the great French chemist. A delegate could not be found at short notice to attend, but a congratulatory letter was sent.

The thanks of the Society are due to the Carnegie Trust for the Universities of Scotland, for grants towards illustrations, amounting to £115, 18s. 11d.; to Mr Laurence Pullar for gifts, amounting to £40, towards the cost of illustrating two papers in the *Transactions*; and to Dr D. A. Allan, for a contribution of £5, 1s. 3d. towards the illustration of his paper in the *Transactions*.

The Gunning Victoria Jubilee Prize for the period 1920 to 1924 has been awarded to Sir J. J. Thomson, O.M., F.R.S.; and the Makdougall-Brisbane Prize for the period 1922 to 1924 to Professor H. Stanley Allen, D.Sc.

During the session the number of *Proceedings* printed was increased from 1360 copies to 1410 copies.

The Society has received from the late President, Professor F. O. BOWER, F.R.S., a gift of his portrait; and from His Grace the Duke of Buccleuch, a picture of a group of men of science, published about 1860. These have been hung in our rooms.

A photographic portrait of Dr John Horne, F.R.S., past President, has been made by Mr A. Swan Watson, at the request of the Council, and on completion will be hung in the Society's rooms.

A resolution was forwarded to the Town Clerk, and to the Society's representative on the Advisory Committee of the Meteorological Office, Edinburgh, welcoming the suggestion that the Calton Hill Observatory should be permanently attached to the Meteorological Service, and trusting that the Town Council on the one hand, and the Air Ministry on the other, may find a satisfactory basis for arrangement, and further, that the Astronomical services to the citizens of Edinburgh, so long associated with the Calton Hill, might be continued in some effective form.

The sum of £16, 13s. 1d. was received from the Cambridge University Press on account of sales of "The Collected Papers of Dr John Aitken, F.R.S.," and £4, 6s. 1d. from Messrs Longmans, Green, & Co., Ltd., on account of sales of the Napier Tercentenary Memorial Volume.

The sum of £150 was spent during the session on binding books in the library.

It was decided, after careful consideration, to dispose of the literary magazines up to the end of 1924, which have been kept in O Room (top flat),—*Blackwood's Magazine*, *Contemporary Review*, etc., etc., to Mr John Grant, bookseller, George IV. Bridge, Edinburgh, for £45. These magazines will still be taken by the Society, and the numbers kept for one year before disposal.

An accumulated mass of dissertations and theses for degrees, presented by foreign universities, has been dispersed by presentation to departments and persons interested.

It is with satisfaction that I am able to report the virtual completion of the Edinburgh Section of the *World List of Scientific Periodicals*, which is in course of preparation by a Committee of Management in London. The volume of titles is already published, and the second volume containing reference numbers, additions, and symbols of libraries in which the various sets are to be found, will be published early next year. The information collected supplies material for a complete local list of periodicals, the treatment of which will be considered later.

The Treasurer, in presenting his Report for the Session, compared the Income and Expenditure with those of the previous year.

The Chairman then asked Non-Fellows to withdraw.

The Chairman nominated as Scrutineers Dr E. M. Horsburgh and Mr A. H. R. Goldie.

The Ballot for the Election of Office-Bearers and Members of Council was then taken.

Sir George Berry moved the adoption of the Reports, and the reappointment of Messrs Lindsay, Jamieson & Haldane, C.A., as auditors of the accounts for the ensuing Session.

This was unanimously agreed to.

The Scrutineers reported that the Ballot Papers were in order, and the President declared that the following Office-Bearers and Members of Council had been duly elected:—

Principal Sir James Alfred Ewing, K.C.B., F.R.S., President.
Professor J. H. Ashworth, D.Sc., F.R.S.,
Professor T. H. Beare, B.A., B.Sc., M.Inst.C.E.,
Walter Biggar Blaikie, LL.D.,
Sir Robert Blyth Greig, M.C., LL.D., } Vice-Presidents.
Professor T. H. Bryce, M.A., M.D., F.R.S.,
Professor E. T. Whittaker, Sc.D., F.R.S.,
Professor R. A. Sampson, M.A., D.Sc., F.R.S., General Secretary.
Alexander Lauder, D.Sc., } Secretaries to Ordinary
Professor W. Wright Smith, M.A., } Meetings.
James Currie, M.A., LL.D., Treasurer.
A. Crichton Mitchell, D.Sc., Hon. D.Sc. (Geneva), Curator of Library and Museum.

ORDINARY MEMBERS OF COUNCIL.

Professor Henry Briggs, D.Sc., A.R.S.M.
W. L. Calderwood, I.S.O.
Professor T. J. Jehu, M.A., M.D., F.G.S.
Professor C. G. Barkla, M.A., D.Sc., F.R.S.
Professor J. Graham Kerr, M.A., F.R.S., F.Z.S.
Lieut.-Col. A. G. M'Kendrick, M.B.
James Watt, W.S., F.F.A., LL.D.
Professor George Barger M.A., D.Sc., F.R.S.
John Bartholomew, M.C., M.A., F.R.G.S.
Professor A. R. Cushny, M.A., M.D., LL.D., F.R.S.
Professor C. G. Darwin, M.A., F.R.S.
Professor David Waterston, M.A., M.D., F.R.C.S.E.

The Chairman, in the name of the Society, thanked the Scrutineers for their services, and closed the meeting.

THE KEITH, MAKDOUGALL-BRISBANE, NEILL, GUNNING VICTORIA JUBILEE, JAMES SCOTT, AND BRUCE PRIZES.

The above Prizes will be awarded by the Council in the following manner:—

I. KEITH PRIZE.

The Keith Prize, consisting of a Gold Medal and from £40 to £50 in Money, will be awarded in the Session 1927–1928 for the "best communication on a scientific subject, communicated,* in the first instance, to the Royal Society of Edinburgh during the Sessions 1925–1926 and 1926–1927." Preference will be given to a paper containing a discovery. (See also resolutions on p. 404.)

II. MAKDOUGALL-BRISBANE PRIZE

This Prize is to be awarded biennially by the Council of the Royal Society of Edinburgh to such person, for such purposes, for such objects, and in such manner as shall appear to them the most conducive to the promotion of the interests of science; with the *proviso* that the Council shall not be compelled to award the Prize unless there shall be some individual engaged in scientific pursuit, or some paper written on a scientific subject, or some discovery in science made during the biennial period, of sufficient merit or importance in the opinion of the Council to be entitled to the Prize.

1. The Prize, consisting of a Gold Medal and a sum of Money, will be awarded before the close of the Session 1926–1927, for an Essay or Paper having reference to any branch of scientific inquiry, whether Material or Mental.

2. Competing Essays to be addressed to the Secretary of the Society, and transmitted not later than 8th July 1926.

3. The Competition is open to all men of science.

4. The Essays may be either anonymous or otherwise. In the former case, they must be distinguished by mottoes, with corresponding sealed billets, superscribed with the same motto, and containing the name of the Author.

5. The Council impose no restriction as to the length of the Essays, which may be, at the discretion of the Council, read at the Ordinary Meetings of the Society. They wish also to leave the property and free disposal of the manuscripts to the Authors; a copy, however, being deposited in the Archives of the Society, unless the paper shall be published in the Transactions.

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* For the purposes of this award the word "communicated" shall be understood to mean the date on which the manuscript of a paper is received in its final form for printing, as recorded by the General Secretary or other responsible official.

6. In awarding the Prize, the Council will also take into consideration any scientific papers presented * to the Society during the Sessions 1924–25, 1925–26, whether they may have been given in with a view to the prize or not. (See also resolutions on p. 404.)

III. NEILL PRIZE.

The Council of the Royal Society of Edinburgh having received the bequest of the late Dr Patrick Neill of the sum of £500, for the purpose of "the interest thereof being applied in furnishing a Medal or other reward every second or third year to any distinguished Scottish Naturalist, according as such Medal or reward shall be voted by the Council of the said Society," hereby intimate:

1. The Neill Prize, consisting of a Gold Medal and a sum of Money, will be awarded during the Session 1927–1928.

2. The Prize will be given for a Paper of distinguished merit, on a subject of Natural History, by a Scottish Naturalist, which shall have been presented * to the Society during the two years preceding the fourth Monday in October 1927,—or failing presentation of a paper sufficiently meritorious, it will be awarded for a work or publication by some distinguished Scottish Naturalist, on some branch of Natural History, bearing date within five years of the time of award. (See also resolutions on p. 404.)

IV. GUNNING VICTORIA JUBILEE PRIZE.

This Prize, founded in the year 1887 by Dr R. H. Gunning, is to be awarded quadrennially by the Council of the Royal Society of Edinburgh, in recognition of original work in Physics, Chemistry, or Pure or Applied Mathematics.

Evidence of such work may be afforded either by a Paper presented to the Society, or by a Paper on one of the above subjects, or some discovery in them elsewhere communicated or made, which the Council may consider to be deserving of the Prize.

The Prize consists of a sum of money, and is open to men of science resident in or connected with Scotland. The first award was made in the year 1887. The next award will be made in 1928–1929.

In accordance with the wish of the Donor, the Council of the Society may on fit occasions award the Prize for work of a definite kind to be undertaken during the three succeeding years by a scientific man of recognised ability.

V. JAMES SCOTT PRIZE.

This Prize, founded in the year 1918 by the Trustees of the James Scott Bequest, is to be awarded triennially, or at such intervals as the Council of the Royal Society of Edinburgh may decide, "for a lecture or essay on the fundamental concepts of Natural Philosophy."

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* For the purposes of this award the word "presented" shall be understood to mean the date on which the manuscript of a paper is received in its final form for printing, as recorded by the General Secretary or other responsible official.

VI. BRUCE PRIZE.

The Royal Society is trustee of a fund, instituted in 1923, to commemorate the work of Dr W. S. Bruce as an explorer and scientific investigator in polar regions.

The Committee of Award is appointed jointly by the Royal Society, the Royal Physical Society, and the Royal Scottish Geographical Society.

The Prize consists of a Bronze Medal and sum of money. It is open to workers of all nationalities, with a preference, *ceteris paribus*, for those of Scottish birth or origin, and is to be awarded biennially for some valuable contribution to Natural Sciences, such as Zoology, Botany, Geology, Meteorology, Oceanography, and Geography; the contribution to be in the nature of new knowledge, the outcome of a personal visit to polar regions on the part of the recipient.

The first award will be made in 1926. Papers for the consideration of the Committee should be in the hands of the General Secretary of the Royal Society, 22 George Street, Edinburgh, not later than March 31 of that year.

RESOLUTIONS OF COUNCIL IN REGARD TO THE MODE OF AWARDING PRIZES.

(*See Minutes of Meeting of January* 18, 1915.)

I. With regard to the Keith and Makdougall-Brisbane Prizes, which are open to all Sciences, the mode of award will be as follows:—

1. Papers or essays to be considered shall be arranged in two groups, A and B,—Group A to include Astronomy, Chemistry, Mathematics, Metallurgy, Meteorology and Physics; Group B to include Anatomy, Anthropology, Botany, Geology, Pathology, Physiology, and Zoology.
2. These two Prizes shall be awarded to each group in alternate biennial periods, provided papers worthy of recommendation have been communicated to the Society.
3. Prior to the adjudication the Council shall appoint, in the first instance, a Committee composed of representatives of the group of Sciences which did not receive the award in the immediately preceding period. The Committee shall consider the Papers which come within their group of Sciences, and report in due course to the Council.
4. In the event of the aforesaid Committee reporting that within their group of subjects there is, in their opinion, no paper worthy of being recommended for the award, the Council, on accepting this report, shall appoint a Committee representative of the alternate group to consider papers coming within their group and to report accordingly.
5. Papers to be considered by the Committees shall fall within the period dating from the last award in groups A and B respectively.

II. With regard to the Neill Prize, the term "Naturalist" shall be understood to include any student in the Sciences composing group B, namely, Anatomy, Anthropology, Botany, Geology, Pathology, Physiology, Zoology.

AWARDS OF THE KEITH, MAKDOUGALL-BRISBANE, NEILL, GUNNING, JAMES SCOTT, AND BRUCE PRIZES.

I. KEITH PRIZE.

1st Biennial Period, 1827-29.—Dr **Brewster**, for his papers "on his Discovery of Two New Immiscible Fluids in the Cavities of certain Minerals," published in the Transactions of the Society.

2nd Biennial Period, 1829-31.—Dr **Brewster**, for his paper "on a New Analysis of Solar Light," published in the Transactions of the Society.

3rd Biennial Period, 1831-33.—**Thomas Graham**, Esq., for his paper "on the Law of the Diffusion of Gases," published in the Transactions of the Society.

4th Biennial Period, 1833-35.—Professor **J. D. Forbes**, for his paper "on the Refraction and Polarization of Heat," published in the Transactions of the Society.

5th Biennial Period, 1835-37.—**John Scott Russell**, Esq., for his researches "on Hydrodynamics," published in the Transactions of the Society.

6th Biennial Period, 1837-39.—Mr **John Shaw**, for his experiments "on the Development and Growth of the Salmon," published in the Transactions of the Society.

7th Biennial Period, 1839-41.—Not awarded.

8th Biennial Period, 1841-43.—Professor **James David Forbes**, for his papers "on Glaciers," published in the Proceedings of the Society.

9th Biennial Period, 1843-45.—Not awarded.

10th Biennial Period, 1845-47.—General Sir **Thomas Brisbane**, Bart., for the Makerstoun Observations on Magnetic Phenomena, made at his expense, and published in the Transactions of the Society.

11th Biennial Period, 1847-49.—Not awarded.

12th Biennial Period, 1849-51.—Professor **Kelland**, for his papers "on General Differentiation, including his more recent Communication on a process of the Differential Calculus, and its application to the solution of certain Differential Equations," published in the Transactions of the Society.

13th Biennial Period, 1851-53.—**W. J. Macquorn Rankine**, Esq., for his series of papers "on the Mechanical Action of Heat," published in the Transactions of the Society.

14th Biennial Period, 1853-55.—Dr **Thomas Anderson**, for his papers "on the Crystalline Constituents of Opium, and on the Products of the Destructive Distillation of Animal Substances," published in the Transactions of the Society.

15th Biennial Period, 1855-57.—Professor **Boole**, for his Memoir "on the Application of the Theory of Probabilities to Questions of the Combination of Testimonies and Judgments," published in the Transactions of the Society.

16th Biennial Period, 1857-59.—Not awarded.

17th Biennial Period, 1859-61.—**John Allan Broun**, Esq., F.R.S., Director of the Trevandrum Observatory, for his papers "on the Horizontal Force of the Earth's Magnetism, on the Correction of the Bifilar Magnetometer, and on Terrestrial Magnetism generally," published in the Transactions of the Society.

18th Biennial Period, 1861-63.—Professor **William Thomson**, of the University of Glasgow, for his Communication "on some Kinematical and Dynamical Theorems."

19th Biennial Period, 1863-65.—Principal **Forbes**, St Andrews, for his "Experimental Inquiry into the Laws of Conduction of Heat in Iron Bars," published in the Transactions of the Society.

20th Biennial Period, 1865-67.—Professor **C. Piazzi Smyth**, for his paper "on Recent Measures at the Great Pyramid," published in the Transactions of the Society.

21st Biennial Period, 1867-69.—Professor **P. G. Tait**, for his paper "on the Rotation of a Rigid Body about a Fixed Point," published in the Transactions of the Society.

22nd Biennial Period, 1869-71.—Professor **Clerk Maxwell**, for his paper "on Figures, Frames, and Diagrams of Forces," published in the Transactions of the Society.

23rd Biennial Period, 1871–73.—Professor P. G. Tait, for his paper entitled "First Approximation to a Thermo-electric Diagram," published in the Transactions of the Society.

24th Biennial Period, 1873–1875.—Professor Crum Brown, for his Researches "on the Sense of Rotation, and on the Anatomical Relations of the Semicircular Canals of the Internal Ear."

25th Biennial Period, 1875–77.—Professor M. Forster Heddle, for his papers "on the Rhombohedral Carbonates," and "on the Felspars of Scotland," published in the Transactions of the Society.

26th Biennial Period, 1877–79.—Professor H. C. Fleeming Jenkin, for his paper "on the Application of Graphic Methods to the Determination of the Efficiency of Machinery," published in the Transactions of the Society; Part II having appeared in the volume for 1877–78.

27th Biennial Period, 1879–81.—Professor George Chrystal, for his paper "on the Differential Telephone," published in the Transactions of the Society.

28th Biennial Period, 1881–83.—Thomas Muir, Esq., LL.D., for his "Researches into the Theory of Determinants and Continued Fractions," published in the Proceedings of the Society.

29th Biennial Period, 1883–85.—John Aitken, Esq., for his paper "on the Formation of Small Clear Spaces in Dusty Air," and for previous papers on Atmospheric Phenomena, published in the Transactions of the Society.

30th Biennial Period, 1885–87.—John Young Buchanan, Esq., for a series of communications, extending over several years, on subjects connected with Ocean Circulation, Compressibility of Glass, etc.; two of which, viz., "On Ice and Brines," and "On the Distribution of Temperature in the Antarctic Ocean," have been published in the Proceedings of the Society.

31st Biennial Period, 1887–89.—Professor E. A. Letts, for his papers on the Organic Compounds of Phosphorus, published in the Transactions of the Society.

32nd Biennial Period, 1889–91.—R. T. Omond, Esq., for his contributions to Meteorological Science, many of which are contained in vol. xxxiv of the Society's Transactions.

33rd Biennial Period, 1891–93.—Professor Thomas R. Fraser, F.R.S., for his papers on *Strophanthus hispidus*, Strophanthin, and Strophanthidin, read to the Society in February and June 1889 and in December 1891, and printed in vols. xxxv, xxxvi, and xxxvii of the Society's Transactions.

34th Biennial Period, 1893–95.—Dr Cargill G. Knott, for his papers on the Strains produced by Magnetism in Iron and in Nickel, which have appeared in the Transactions and Proceedings of the Society.

35th Biennial Period, 1895–97.—Dr Thomas Muir, for his continued communications on Determinants and Allied Questions.

36th Biennial Period, 1897–99.—Dr James Burgess, for his paper "on the Definite Integral $\frac{2}{\sqrt{\pi}}\int_0^t \epsilon^{-t^2}dt$, with extended Tables of Values," printed in vol. xxxix of the Transactions of the Society.

37th Biennial Period, 1899–1901.—Dr Hugh Marshall, for his discovery of the Persulphates, and for his Communications on the Properties and Reactions of these Salts, published in the Proceedings of the Society.

38th Biennial Period, 1901–03.—Sir William Turner, K.C.B., LL.D., F.R.S., etc., for his memoirs entitled "A Contribution to the Craniology of the People of Scotland," published in the Transactions of the Society, and for his "Contributions to the Craniology of the People of the Empire of India," Parts I, II, likewise published in the Transactions of the Society.

39th Biennial Period, 1903–05.—Thomas H. Bryce, M.A., M.D., for his two papers on "The Histology of the Blood of the Larva of *Lepidosiren paradoxa*," published in the Transactions of the Society within the period.

40th Biennial Period, 1905–07.—Alexander Bruce, M.A., M.D., F.R.C.P.E., for his paper entitled "Distribution of the Cells in the Intermedio-Lateral Tract of the Spinal Cord," published in the Transactions of the Society within the period.

41st Biennial Period, 1907–09.—Wheelton Hind, M.D., B.S., F.R.C.S., F.G.S., for a paper published in the Transactions of the Society, "On the Lamellibranch and Gasteropod Fauna found in the Millstone Grit of Scotland."

42nd Biennial Period, 1909–11.—Professor Alexander Smith, B.Sc., Ph.D., of New York, for his researches upon "Sulphur" and upon "Vapour Pressure," appearing in the Proceedings of the Society.

43rd Biennial Period, 1911–1913.—James Russell, Esq., for his series of investigations relating to magnetic phenomena in metals and the molecular theory of magnetism, the results of which have been published in the Proceedings and Transactions of the Society, the last paper having been issued within the period.

44th Biennial Period, 1913–15.—James Hartley Ashworth, D.Sc., for his papers on "Larvæ of Lingula and Pelagodiscus," and on "Sclerocheilus," published in the Transactions of the Society, and for other papers on the Morphology and Histology of Polychæta.

45th Biennial Period, 1915–17.—Robert C. Mossman, for his work on the Meteorology of the Antarctic Regions, which originated with the important series of observations made by him during the voyage of the "Scotia" (1902–1904), and includes his paper "On a Sea-Saw of Barometric Pressure, Temperature, and Wind Velocity between the Weddell Sea and the Ross Sea," published in the Proceedings of the Society.

46th Biennial Period, 1917–19.—John Stephenson, Lt.-Col. I.M.S., for his series of papers on the Oligochæta and other Annelida, several of which have been published in the Transactions of the Society.

47th Biennial Period, 1919–21.—Ralph Allen Sampson, F.R.S., for his Astronomical Researches, including the papers "Studies in Clocks and Time-keeping: No. 1, Theory of the Maintenance of Motion; No. 2, Tables of the Circular Equation," published in the Proceedings of the Society within the period of the award.

48th Biennial Period, 1921–23.—John Walter Gregory, F.R.S., for his papers published in the Transactions of the Society, and in recognition of his numerous contributions to Geology, extending over a period of thirty-six years.

II. MAKDOUGALL-BRISBANE PRIZE.

1st Biennial Period, 1859.—Sir Roderick Impey Murchison, on account of his Contributions to the Geology of Scotland.

2nd Biennial Period, 1860–62.—William Seller, M.D., F.R.C.P.E., for his "Memoir of the Life and Writings of Dr Robert Whytt," published in the Transactions of the Society.

3rd Biennial Period, 1862–64.—John Denis Macdonald, Esq., R.N., F.R.S., Surgeon of H.M.S. "Icarus," for his paper "on the Representative Relationships of the Fixed and Free Tunicata, regarded as Two Sub-classes of equivalent value; with some General Remarks on their Morphology," published in the Transactions of the Society.

4th Biennial Period, 1864–66.—Not awarded.

5th Biennial Period, 1866–68.—Dr Alexander Crum Brown and Dr Thomas Richard Fraser, for their conjoint paper "on the Connection between Chemical Constitution and Physiological Action," published in the Transactions of the Society.

6th Biennial Period, 1868–70.—Not awarded.

7th Biennial Period, 1870–72.—George James Allman, M.D., F.R.S., Emeritus Professor of Natural History, for his paper "on the Homological Relations of the Cœlenterata," published in the Transactions, which forms a leading chapter of his Monograph of Gymnoblastic or Tubularian Hydroids—since published.

8th Biennial Period, 1872–74.—Professor Lister, for his paper "on the Germ Theory of Putrefaction and the Fermentive Changes," communicated to the Society, 7th April 1873.

9th Biennial Period, 1874–76.—Alexander Buchan, A.M., for his paper "on the Diurnal Oscillation of the Barometer," published in the Transactions of the Society.

10th Biennial Period, 1876–78.—Professor Archibald Geikie, for his paper "on the Old Red Sandstone of Western Europe," published in the Transactions of the Society.

11th Biennial Period, 1878–80.—Professor Piazzi Smyth, Astronomer-Royal for Scotland, for his paper "on the Solar Spectrum in 1877–78, with some Practical Idea of its probable Temperature of Origination," published in the Transactions of the Society.

12th Biennial Period, 1880–82.—Professor James Geikie, for his "Contributions to the Geology of the North-West of Europe," including his paper "on the Geology of the Faroes," published in the Transactions of the Society.

13th Biennial Period, 1882–84.—Edward Sang, Esq., LL.D., for his paper "on the Need of Decimal Subdivisions in Astronomy and Navigation, and on Tables requisite therefor," and generally for his Recalculations of Logarithms both of Numbers and Trigonometrical Ratios,—the former communication being published in the Proceedings of the Society.

14th Biennial Period, 1884-86.—John Murray, Esq., LL.D., for his papers "On the Drainage Areas of Continents, and Ocean Deposits," "The Rainfall of the Globe, and Discharge of Rivers," "The Height of the Land and Depth of the Ocean," and "The Distribution of Temperature in the Scottish Lochs as affected by the Wind."

15th Biennial Period, 1886-88.—Archibald Geikie, Esq., LL.D., for numerous Communications, especially that entitled "History of Volcanic Action during the Tertiary Period in the British Isles," published in the Transactions of the Society.

16th Biennial Period, 1888-90.—Dr Ludwig Becker, for his paper on "The Solar Spectrum at Medium and Low Altitudes," printed in vol xxxvi, Part I, of the Society's Transactions.

17th Biennial Period, 1890-92.—Hugh Robert Mill, Esq., D.Sc., for his papers on "The Physical Conditions of the Clyde Sea Area," Part I being already published in vol. xxxvi of the Society's Transactions.

18th Biennial Period, 1892-94.—Professor James Walker, D.Sc., Ph.D., for his work on Physical Chemistry, part of which has been published in the Proceedings of the Society, vol. xx, pp. 255-263. In making this award, the Council took into consideration the work done by Professor Walker along with Professor Crum Brown on the Electrolytic Synthesis of Dibasic Acids, published in the Transactions of the Society.

19th Biennial Period, 1894-96.—Professor John G. M'Kendrick, for numerous Physiological papers, especially in connection with Sound, many of which have appeared in the Society's publications.

20th Biennial Period, 1896-98.—Dr William Peddie, for his papers on the Torsional Rigidity of Wires.

21st Biennial Period, 1898-1900.—Dr Ramsay H. Traquair, for his paper entitled "Report on Fossil Fishes collected by the Geological Survey in the Upper Silurian Rocks of Scotland," printed in vol. xxxix of the Transactions of the Society.

22nd Biennial Period, 1900-02.—Dr Arthur T. Masterman, for his paper entitled "The Early Development of *Cribrella oculata* (Forbes), with remarks on Echinoderm Development," printed in vol. xl of the Transactions of the Society.

23rd Biennial Period, 1902-04.—Mr John Dougall, M.A., for his paper on "An Analytical Theory of the Equilibrium of an Isotropic Elastic Plate," published in vol. xli of the Transactions of the Society.

24th Biennial Period, 1904-06.—Jacob E. Halm, Ph.D., for his two papers entitled "Spectroscopic Observations of the Rotation of the Sun," and "Some Further Results obtained with the Spectroheliometer," and for other astronomical and mathematical papers published in the Transactions and Proceedings of the Society within the period.

25th Biennial Period, 1906-08.—D. T. Gwynne-Vaughan, M.A., F.L.S., for his papers, 1st, "On the Fossil Osmundaceæ," and 2nd, "On the Origin of the Adaxially-curved Leaf-trace in the Filicales," communicated by him conjointly with Dr R. Kidston.

26th Biennial Period, 1908-10.—Ernest MacLagan Wedderburn, M.A., LL.B., for his series of papers bearing upon "The Temperature Distribution in Fresh-water Lochs," and especially upon "The Temperature Seiche."

27th Biennial Period, 1910-12.—John Brownlee, M.A., M.D., D.Sc., for his contributions to the Theory of Mendelian Distributions and cognate subjects, published in the Proceedings of the Society within and prior to the prescribed period.

28th Biennial Period, 1912-14.—Professor C. R. Marshall, M.D., M.A., for his studies "On the Pharmacological Action of Tetra-alkyl-ammonium Compounds."

29th Biennial Period, 1914-16.—Robert Alexander Houstoun, Ph.D., D.Sc., for his series of papers on "The Absorption of Light by Inorganic Salts," published in the Proceedings of the Society.

30th Biennial Period, 1916-18.—Professor A. Anstruther Lawson for his Memoirs on "The Prothalli of *Tmesipteris Tannensis* and of *Psilotum*," published in the Transactions of the Society, together with previous papers on Cytology and on The Gametophytes of various Gymnosperms.

31st Biennial Period, 1918-20.—Professor J. H. Maclagan Wedderburn of Princeton University for his Memoirs in Universal Algebra, etc., published in the Transactions and Proceedings of the Society, and elsewhere.

32nd Biennial Period, 1920-22.—Professor W. T. Gordon, M.A., D.Sc., for his paper on "Cambrian Organic Remains from a Dredging in the Weddell Sea," published in the Transactions of the Society within the period, and for his investigations on the Fossil Flora of the Pettycur Limestone, previously published in the Transactions.

33rd Biennial Period, 1922-24.—Professor H. Stanley Allen, D.Sc., for his papers on the "Quantum and Atomic Theory," published in the Society's Proceedings within the period.

III. THE NEILL PRIZE.

1st Triennial Period, 1856-59.—Dr W. Lauder Lindsay, for his paper "on the Spermogones and Pycnides of Filamentous, Fruticulose, and Foliaceous Lichens," published in the Transactions of the Society.

2nd Triennial Period, 1859-61.—Robert Kaye Greville, LL.D., for his contributions to Scottish Natural History, more especially in the department of Cryptogamic Botany, including his recent papers on Diatomaceæ.

3rd Triennial Period, 1862-65.—Andrew Crombie Ramsay, F.R.S., Professor of Geology in the Government School of Mines, and Local Director of the Geological Survey of Great Britain, for his various works and memoirs published during the last five years, in which he has applied the large experience acquired by him in the Direction of the arduous work of the Geological Survey of Great Britain to the elucidation of important questions bearing on Geological Science.

4th Triennial Period, 1865-68.—Dr William Carmichael M'Intosh, for his paper "on the Structure of the British Nemerteans, and on some New British Annelids," published in the Transactions of the Society.

5th Triennial Period, 1868-71.—Professor William Turner, for his papers "on the Great Finner Whale; and on the Gravid Uterus, and the Arrangement of the Fœtal Membranes in the Cetacea," published in the Transactions of the Society.

6th Triennial Period, 1871-74.—Charles William Peach, Esq., for his Contributions to Scottish Zoology and Geology, and for his recent contributions to Fossil Botany.

7th Triennial Period, 1874-77.—Dr Ramsay H. Traquair, for his paper "on the Structure and Affinities of *Tristichopterus alatus* (Egerton)," published in the Transactions of the Society, and also for his contributions to the Knowledge of the Structure of Recent and Fossil Fishes.

8th Triennial Period, 1877-80.—John Murray, Esq., for his paper "on the Structure and Origin of Coral Reefs and Islands," published (in abstract) in the Proceedings of the Society.

9th Triennial Period, 1880-83.—Professor Herdman, for his papers "on the Tunicata," published in the Proceedings and Transactions of the Society.

10th Triennial Period, 1883-86.—B. N. Peach, Esq., for his Contributions to the Geology and Palæontology of Scotland, published in the Transactions of the Society.

11th Triennial Period, 1886-89.—Robert Kidston, Esq., for his Researches in Fossil Botany, published in the Transactions of the Society.

12th Triennial Period, 1889-92.—John Horne, Esq., F.G.S., for his Investigations into the Geological Structure and Petrology of the North-West Highlands.

13th Triennial Period, 1892-95.—Robert Irvine, Esq., for his papers on the Action of Organisms in the Secretion of Carbonate of Lime and Silica, and on the solution of these substances in Organic Juices. These are printed in the Society's Transactions and Proceedings.

14th Triennial Period, 1895-98.—Professor Cossar Ewart, for his recent Investigations connected with Telegony.

15th Triennial Period, 1898-1901.—Dr John S. Flett, for his papers entitled "The Old Red Sandstone of the Orkneys" and "The Trap Dykes of the Orkneys," printed in vol. xxxix of the Transactions of the Society.

16th Triennial Period, 1901-04.—Professor J. Graham Kerr, M.A., for his Researches on *Lepidosiren paradoxa*, published in the Philosophical Transactions of the Royal Society, London.

17th Triennial Period, 1904-07.—Frank J. Cole, B.Sc., for his paper entitled "A Monograph on the General Morphology of the Myxinoid Fishes, based on a Study of Myxine," published in the Transactions of the Society, regard being also paid to Mr Cole's other valuable contributions to the Anatomy and Morphology of Fishes.

1st Biennial Period, 1907-09.—Francis J. Lewis, M.Sc., F.L.S., for his papers in the Society's Transactions "On the Plant Remains of the Scottish Peat Mosses."

2nd Biennial Period, 1909-11.—James Murray, Esq., for his paper on "Scottish Rotifers collected by the Lake Survey (Supplement)," and other papers on the "Rotifera" and "Tardigrada," which appeared in the Transactions of the Society—(this Prize was awarded after consideration of the papers received within the five years prior to the time of award: see Neill Prize Regulations).

3rd Biennial Period, 1911–13.—Dr W. S. Bruce, in recognition of the scientific results of his Arctic and Antarctic explorations.

4th Biennial Period, 1913–15.—Robert Campbell, D.Sc., for his paper on "The Upper Cambrian Rocks at Craigeven Bay, Stonehaven," and "Downtonian and Old Red Sandstone Rocks of Kincardineshire," published in the Transactions of the Society.

5th Biennial Period, 1915–17.—W. H. Lang, F.R.S., M.B., D.Sc., for his paper in conjunction with Dr R. Kidston, F.R.S., on *Rhynia Gwynne-Vaughani*, Kidston and Lang, published in the Transactions of the Society, and for his previous investigations on Pteridophytes and Cycads.

6th Biennial Period, 1917–19.—John Tait, D.Sc., M.D., for his work on Crustacea, published in the Proceedings of the Society, and for his papers on the blood of Crustacea.

7th Biennial Period, 1919–21.—Sir Edward A. Sharpey Schafer, F.R.S., for his recent contributions to our knowledge of Physiology, and in recognition of his published work extending over a period of fifty years.

8th Biennial Period, 1921–23.—John M'Lean Thompson, M.A., D.Sc., University of Liverpool, for his series of Memoirs on Staminal Zygomorphy, and on the Anatomy of the Filicales.

IV. GUNNING VICTORIA JUBILEE PRIZE.

1st Triennial Period, 1884–87.—Sir William Thomson, Pres. R.S.E., F.R.S., for a remarkable series of papers "on Hydrokinetics," especially on Waves and Vortices, which have been communicated to the Society.

2nd Triennial Period, 1887–90.—Professor P. G. Tait, Sec. R.S.E., for his work in connection with the "Challenger" Expedition, and his other Researches in Physical Science.

3rd Triennial Period, 1890–93.—Alexander Buchan, Esq., LL.D., for his varied, extensive, and extremely important Contributions to Meteorology, many of which have appeared in the Society's publications.

4th Triennial Period, 1893–96.—John Aitken, Esq., for his brilliant Investigations in Physics, especially in connection with the Formation and Condensation of Aqueous Vapour.

1st Quadrennial Period, 1896–1900.—Dr T. D. Anderson, for his discoveries of New and Variable Stars.

2nd Quadrennial Period, 1900–04.—Sir James Dewar, LL.D., D.C.L., F.R.S., etc., for his researches on the Liquefaction of Gases, extending over the last quarter of a century, and on the Chemical and Physical Properties of Substances at Low Temperatures: his earliest papers being published in the Transactions and Proceedings of the Society.

3rd Quadrennial Period, 1904–08.—Professor George Chrystal, M.A., LL.D., for a series of papers on "Seiches," including "The Hydrodynamical Theory and Experimental Investigations of the Seiche Phenomena of Certain Scottish Lakes."

4th Quadrennial Period, 1908–12.—Professor J. Norman Collie, Ph.D., F.R.S., for his distinguished contributions to Chemistry, Organic and Inorganic, during twenty-seven years, including his work upon Neon and other rare gases. Professor Collie's early papers were contributed to the Transactions of the Society.

5th Quadrennial Period, 1912–16.—Sir Thos. Muir, C.M.G., LL.D., F.R.S., for his series of Memoirs upon "The Theory and History of Determinants and Allied Forms," published in the Transactions and Proceedings of the Society between the years 1872 and 1915.

6th Quadrennial Period, 1916–20.—C. T. R. Wilson, Esq., F.R.S., in recognition of his important discoveries in relation to Condensation Nuclei, Ionisation of Gases, and Atmospheric Electricity.

7th Quadrennial Period, 1920–24.—Sir J. J. Thomson, O.M., F.R.S., in recognition of his great discoveries in Physical Science.

V. JAMES SCOTT PRIZE.

1st Award, 1918–22.—Professor A. N. Whitehead, F.R.S., for his lecture delivered on June 5, 1922, "On the Relatedness of Nature."

VI. BRUCE PRIZE.

1st Award will be made in 1926.

ABSTRACT

OF

THE ACCOUNTS OF JAMES CURRIE, ESQ., LL.D.

As Treasurer of the Royal Society of Edinburgh.

SESSION 1924-1925.

I. GENERAL FUND.

CHARGE.

1. Arrears of Contributions at 30th September 1924			£111 6 0
			£111 6 0
2. Contributions for present Session :—			
1. 308 Fellows at £3, 3s. each		£970 4 0	
1 Fellow at £2, 2s.		2 2 0	
17 Fellows at £4, 4s. each		71 8 0	
		£1043 14 0	
2. Fees of Admission and Contributions of thirty-six new Fellows at £6, 6s. each		226 16 0	
3. Commutation Fees in lieu of future Contributions of two Fellows		111 6 0	
			1381 16 0
3. Extra Contributions for 1924-1925 under Amended Law, No. 6—			
Voluntary Contributions		£24 3 0	
Commutations		52 0 0	
			76 3 0
4. Interest received—			
Interest on £7830 five per cent. War Loan, 1929-47, Untaxed		£391 10 0	
Interest on Deposit Receipts		52 13 3	
Interest on £2100 2½ per cent. Consols, less Tax, £11, 9s. 6d		41 0 6	
			485 3 9
5. Transactions and Proceedings sold			168 11 6
6. Annual Grant from Government			600 0 0
7. Income Tax repaid :—			
Year to 5th April 1925		£11 16 0	
			11 16 0
8. Receipts from Sale of Napier Tercentenary Memorial Volume			4 6 1
Amount of the Charge			**£2839 2 4**

DISCHARGE.

1. Taxes, Insurance, Coal and Lighting :—			
Insurance		£21 18 10	
Coal, etc., to 9th May 1925		43 10 0	
Gas to 6th May 1925		3 17 0	
Electric Light to 14th April 1925		8 7 10	
Water, 1924-25		4 4 0	
Equitable Life Assurance Co., Ltd., for Superannuation Scheme	£45 0 0		
Less—Received from Assistant Secretary	15 0 0		
		30 0 0	
			£111 17 8
Carry forward			£111 17 8

Brought forward			£111 17 8
2. SALARIES:—			
General Secretary, 1924-25		£100 0 0	
Librarian and Assistant Secretary		300 0 0	
Assistant Librarian		86 13 4	
Office Keeper		147 12 0	
Treasurer's Clerk		35 0 0	
			669 5 4
3. EXPENSES OF TRANSACTIONS:—			
Neill & Co., Ltd., Printers		£284 2 7	
Hislop & Day, Ltd., Engravers		92 5 4	
The Zinco-Collotype Co.		54 0 0	
G. Waterston & Sons, Ltd.		16 2 6	
		£446 10 5	
Less Receipts—			
Grants by Carnegie Trustees towards Dr Tyrrell, Dr Allan, Professor Bryce, Dr Kidston, and Professor Lang, Professor Jehu, and C. W. Wardlaw's Papers	£115 18 11		
Received from Dr Allan	5 1 3		
Mr L. Pullar towards Dr Trueman's and Mr Bailey's Plates	40 0 0		
		161 0 2	
			285 10 3
4. EXPENSES OF PROCEEDINGS:—			
Neill & Co., Ltd., Printers		£791 4 6	
Hislop & Day, Ltd., Engravers		41 7 1	
			832 11 7
5. BOOKS, PERIODICALS, NEWSPAPERS, ETC.:—			
James Thin, Bookseller		£216 19 3	
R. Grant & Son, Booksellers		1 12 0	
Robertson & Scott, News Agents		7 7 8	
Ray Society, Subscription		1 1 0	
Berwickshire Naturalists' Club, Do.		0 10 0	
Palæontographical Society, Do.		1 1 0	
W. Blackwood & Sons, Printers		3 17 6	
Williams & Norgate, Ltd.		2 0 0	
Edinburgh and Leith Post Office Directory, Ltd.		0 13 6	
World's List of Scientific Periodicals		2 10 0	
			237 11 11
6. LOAN TO MAKERSTOUN MAGNETIC METEOROLOGICAL OBSERVATION FUND £100, *less* repaid £98, 12s. 11d.		£1 7 1	
7. OTHER PAYMENTS:—			
Neill & Co., Ltd., Printers		£129 10 6	
A. Cowan & Sons, Ltd.		10 18 0	
S. Duncan & Sons, Tailors (uniform)		9 9 0	
Macvitties, Guest & Co., Ltd.		51 4 9	
Andrew H. Baird		15 15 9	
Lindsay, Jamieson & Haldane, C.A., Auditors		10 10 0	
The Window Cleaning Co., Ltd.		12 16 0	
Orrock & Son, Bookbinders		150 0 0	
Gillies & Wright, Joiners		38 0 10	
Federated Superannuation System for Universities		5 0 0	
George Cairns		4 13 6	
G. Waterston & Sons, Ltd.		23 16 0	
James Dunn		4 10 0	
A. Black & Co., Brushmakers		4 11 6	
Edward & Co., Electrical Engineers		48 10 10	
Petty Expenses, Postages, Carriage, etc.		136 18 6	
Presentation to Professor F. O. Bower		10 10 0	
			666 15 2
Carry forward			£2803 11 11

Brought forward				£2803	11	11
8. ARREARS OF CONTRIBUTIONS written off				38	17	0
9. ARREARS of CONTRIBUTIONS outstanding at 30th September 1925 :—						
Present Session	£67	4	0			
Previous Sessions	28	10	0			
				95	14	0
Amount of the Discharge				**£2938**	**2**	**11**

Amount of the Charge	**£2839**	**2**	**4**
Amount of the Discharge	**2938**	**2**	**11**
Excess of Payments over Receipts	**£99**	**0**	**7**
Balance of General Fund at 30th September 1924	**259**	**18**	**1**
Balance of General Fund at 30th September 1925	**£160**	**17**	**6**

Made up thus :—

Due by Union Bank of Scotland, Ltd., on Account Current	£148	12	2			
Balance of Loan to Makerstoun Magnetic Meteorological Observation Fund	1	7	1			
Due by Treasurer	10	18	3			
				£160	17	6

SPECIAL SUBSCRIPTION FUND

To 30th September 1925.

CHARGE.

Balance at 30th September 1924	£1128	17	9

DISCHARGE.

Balance at 30th September 1925 :—			
Due by Union Bank of Scotland, Ltd., on Deposit Receipt	£1128	17	9

Note :—

Total Subscriptions received towards Special Subscription Fund							£1128	17	9
Sums transferred to General Fund to meet Deficiency on Accounts—									
To 30th September 1919				£301	16	11			
To 30th September 1920				34	9	5			
To 30th September 1922				408	11	1½			
				£744	17	5½			
Sums transferred from General Fund :—									
To 30th September 1921	£244	19	9						
To 30th September 1923	344	4	7½						
To 30th September 1924	155	13	1						
				£744	17	5½			
Amount of Special Subscription Fund at 30th September 1925, as above							£1128	17	9

II. KEITH FUND

To 30*th September* 1925.

CHARGE.

1.	BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1924		£39 11 6
2.	INTEREST RECEIVED :—		
	On £650 five per cent. War Loan, 1929-47, Untaxed	£32 10 0	
	On Deposit Receipt of £39, 11s. 6d., Uplifted	0 19 3	
			33 9 3
			£73 0 9

DISCHARGE.

1.	BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1925	£73 0 9

III. NEILL FUND

To 30*th September* 1925.

CHARGE.

1.	BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1924		£15 8 10
2.	INTEREST RECEIVED :—		
	On £300 five per cent. War Loan, 1929-47, Untaxed	£15 0 0	
	On Deposit Receipt of £15, 8s. 10d., Uplifted	0 7 4	
			15 7 4
			£30 16 2

DISCHARGE.

1.	BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1925	£30 16 2

IV. MAKDOUGALL-BRISBANE FUND

To 30*th September* 1925.

CHARGE.

1.	BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1924		£64 11 11
2.	INTEREST RECEIVED :—		
	On £400 five per cent. War Loan, 1929-47, Untaxed	£20 0 0	
	On Deposit Receipts of £64, 11s. 11d., £23, 18s. 2d., Uplifted	1 8 5	
			21 8 5
			£86 0 4

DISCHARGE.

1.	Alex Kirkwood & Son, for Gold Medal	£18 0 0
	Prof. H. Stanley Allen, Money Portion of Prize, 1920-24	23 9 6
		£41 9 6
2.	BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1925	44 10 10
		£86 0 4

V. MAKERSTOUN MAGNETIC METEOROLOGICAL OBSERVATION FUND

To 30th September 1925.

CHARGE.

1.	INTEREST RECEIVED:—		
	On £250 five per cent. War Loan, 1929–47, Untaxed		£12 10 0
2.	BORROWED from General Fund £100, *less* repaid £98, 12s. 11d.	£1 7 1	
3.	BALANCE due to General Fund at 30th September 1925		1 7 1
			£13 17 1

DISCHARGE.

1.	Balance due to General Fund at 30th September 1924		£13 17 1
2.	Repaid on Account of Loan	£12 10 0	
			£13 17 1

VI. GUNNING VICTORIA JUBILEE PRIZE FUND

To 30th September 1925.

CHARGE.

1.	BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1924		£147 7 0
2.	INTEREST RECEIVED:—		
	On £570 five per cent. War Loan, 1929–47, Untaxed	£28 10 0	
	On Deposit Receipt of £147, 7s., Uplifted	3 12 6	
			32 2 6
			£179 9 6

DISCHARGE.

1.	Sir J. J. Thomson, Money Prize (not yet paid), 1920–24		£105 0 0
2.	BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1925	£179 9 6	
	Less—Due for Prize	105 0 0	
			74 9 6
			£179 9 6

VII. JAMES SCOTT PRIZE FUND

To 30th September 1925.

CHARGE.

1.	BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1924	£266 3 9
2.	INTEREST RECEIVED:—	
	On Deposit Receipt of £266, 3s. 9d., Uplifted	6 12 8
		£272 16 5

DISCHARGE.

1.	BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1925	£272 16 5

VIII. DR JOHN AITKEN FUND

To 30*th September* 1925.

CHARGE.

1. Balance due by Union Bank of Scotland, Ltd., at 30th September 1924:—			
On Deposit Receipt	£464	0	0
On Account Current	8	0	1
	£472	0	1
2. Interest Received:—			
On Deposit Receipt of £464, Uplifted	11	14	1
3. Received from Sale of Volumes to Fellows	17	10	0
	£501	4	2

DISCHARGE.

1. Balance due by Union Bank of Scotland, Ltd., at 30th September 1925:—			
On Deposit Receipt	£475	14	1
On Account Current	25	10	1
	£501	4	2

STATE OF THE FUNDS BELONGING TO THE ROYAL SOCIETY OF EDINBURGH

As at 30th *September* 1925.

1. GENERAL FUND—

1. £7830 five per cent. War Loan, 1929–47, at 102 per cent.	£7986	12	0
2. £2100 2½ per cent. Consolidated Stock at 56 per cent.	1176	0	0
3. Deposit Receipt Union Bank of Scotland, Ltd., being balance of Legacy received during 1917–18, from the Trustees of the late Mr Robert Mackay Smith, £500 *less* legacy duty £50	450	0	0
4. Arrears of Contributions, as per preceding Abstract of Accounts	95	14	0
5. Balance of Special Subscription Fund	1128	17	9
6. Balance in Bank and due by Treasurer	159	10	5
7. Balance of Loan to Makerstoun Magnetic Meteorological Observation Fund	1	7	1
Amount	£10,998	1	3

Exclusive of Library, Museum, Pictures, etc., and Furniture in the Society's Rooms at George Street, Edinburgh.

2. KEITH FUND—

1. £650 five per cent. War Loan, 1929–47, at 102 per cent.	£663	0	0
2. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt	73	0	9
Amount	£736	0	9

3. NEILL FUND—

1. £300 five per cent. War Loan, 1929–47, at 102 per cent.	£306	0	0
2. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt	30	16	2
Amount	£336	16	2

4. MAKDOUGALL-BRISBANE FUND—

1. £400 five per cent. War Loan, 1929–47, at 102 per cent.	£408 0 0
2. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt . .	44 10 10
Amount . . .	£452 10 10

5. MAKERSTOUN MAGNETIC METEOROLOGICAL OBSERVATION FUND—

1. £250 five per cent. War Loan, 1929–47, at 102 per cent.	£255 0 0
Less—Balance of Loan from General Fund	1 7 1
Amount . . .	£253 12 11

6. GUNNING VICTORIA JUBILEE PRIZE FUND—Instituted by Dr Gunning of Edinburgh and Rio de Janeiro—

1. £570 five per cent. War Loan, 1929–47, at 102 per cent.		£581 8 0
2. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt	£179 9 6	
Less—Due for Prize 1920–24	105 0 0	
		74 9 6
Amount . . .		£655 17 6

7. JAMES SCOTT PRIZE FUND—

Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt . .		£272 16 5
Whereof—Applicable to Capital	£250 0 0	
Applicable to Revenue	22 16 5	
	£272 16 5	

8. TAIT MEMORIAL FUND—

This Fund consists of War Loan, and is to mature for a period of about ten years from 1918, when it is expected to yield about £75 per annum.

9. DR JOHN AITKEN FUND—

Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt	£475 14 1
Balance due do. do. on Account Current . .	25 10 1
	£501 4 2

Edinburgh, 16*th October* 1925.—We have examined the preceding Accounts of the Treasurer of the Royal Society of Edinburgh for the Session 1924–1925, and have found them to be correct. The securities of the various Investments, as noted in the foregoing Statement of Funds (with the exception of No. 8), have been verified by us as at 30th September 1925.

LINDSAY, JAMIESON & HALDANE, C.A.,
Auditors.

LIST OF VOLUNTARY CONTRIBUTORS who have made a Single Payment under Law VI (end of para. 3), up to 30th September 1925.

Professor F. G. Baily.
Professor S. C. Mahalanobis.
Professor W. H. Metzler.
Rev. Professor Alex. Moffat.
Sir David Prain, F.R.S.

LIST OF VOLUNTARY CONTRIBUTORS under Law VI (end of para. 3) up to 30th September 1925.

Professor T. H. Beare.
Dr W. B. Blaikie.
Dr H. Bolton.
Professor F. O. Bower, F.R.S.
Dr F. E. Beddard, F.R.S.
Dr James Currie.
Dr T. W. Dewar.
Professor J. M. Dixon.
Mr D. B. Dott.
Dr A. Freeland Fergus.
Mr P. S. Ford.
Professor J. G. Gray.
Sir R. B. Greig.
Dr John Horne, F.R.S.
Dr W. F. Hume.
Sir J. H. R. Kemnal.
Dr George M'Gowan.
Professor Wm. Peddie.
Mr Ralph Richardson.
Professor S. Simpson.
Professor J. Arthur Thomson, 1923–24, 1924–25.
Mr J. C. Wright.

THE COUNCIL OF THE SOCIETY.

October 1925.

President.

Principal Sir ALFRED EWING, K.C.B., F.R.S.

Vice-Presidents.

Professor J. H. ASHWORTH, D.Sc., F.R.S.
Professor T. HUDSON BEARE, B.A., B.Sc., M.Inst.C.E.
WALTER BIGGAR BLAIKIE, LL.D.
Sir ROBERT BLYTH GREIG, M.C., LL.D., F.Z.S.
Professor THOMAS H. BRYCE, M.A., M.D., F.R.S.
Professor E. T. WHITTAKER, Sc.D., F.R.S.

General Secretary.

Professor R. A. SAMPSON, M.A., D.Sc., F.R.S.

Secretaries to Ordinary Meetings.

ALEXANDER LAUDER, D.Sc.
Professor W. WRIGHT SMITH, M.A.

Treasurer.

JAMES CURRIE, M.A., LL.D.

Curator of Library and Museum.

A. CRICHTON MITCHELL, D.Sc., Hon. D.Sc. (Geneva).

Councillors.

Professor HENRY BRIGGS, D.Sc., A.R.S.M.
W. L. CALDERWOOD, Esq., I.S.O.
Professor T. J. JEHU, M.A., M.D., F.G.S.
Professor C. G. BARKLA, M.A., D.Sc., F.R.S.
Professor J. GRAHAM KERR, M.A., F.R.S.
Lt.-Col. A. G. M'KENDRICK, M.B.
JAMES WATT, W.S., F.F.A., LL.D.
Professor GEORGE BARGER, M.A., D.Sc., F.R.S.
JOHN BARTHOLOMEW, M.C., M.A., F.R.G.S.
Professor A. R. CUSHNY, M.A., M.D., LL.D., F.R.S.
Professor C. G. DARWIN, M.A., F.R.S.
Professor DAVID WATERSTON, M.A., M.D., F.R.C.S.E.

Office Staff.

Assistant Secretary and Librarian, G. A. STEWART.
Assistant Librarian, R. J. B. MUNRO.
Housekeeper, SAMUEL HEDDLE.

ALPHABETICAL LIST OF THE ORDINARY FELLOWS OF THE SOCIETY,

Corrected to October 26, 1925.

N.B.—*Those marked* * *are Annual Contributors.*
,, ,, † *have commuted Voluntary Contribution* (*see* 3*rd Paragraph, Law VI*).

B. prefixed to a name indicates that the Fellow has received a Makdougall-Brisbane Medal.
K. ,, ,, ,, Keith Medal.
N. ,, ,, ,, Neill Medal.
V. J. ,, ,, ,, the Gunning Victoria Jubilee Prize.
C. ,, ,, ,, contributed one or more Communications to the Society's TRANSACTIONS or PROCEEDINGS.

Date of Election.			Service on Council, etc.
1922		*Abernethy, Charles Lawrence, M.A. (Hons.), B.Sc., Research Physicist, "Exnaboe," Craiglockhart Avenue, Slateford, Edinburgh	
1898		Adami, J. G., C.B.E., M.A., M.D. (Camb., M'Gill, and Belfast), LL.D., F.R.S., Vice-Chancellor of the University of Liverpool	
1925	C.	*Aitken, Alexander Craig, M.A., Lecturer in Mathematics in the University of Edinburgh. Westfield House, Cramond, Greater Edinburgh	
1889		†Alison, John, M.A., LL.D., Head Master, George Watson's College, Edinburgh	
1894		†Allan, Francis John, M.D., C.M., Edin., M.O.H. City of Westminster. Lincluden, 33 Cromwell Road, Teddington, Middlesex 5	
1888	C.	Allardice, R. E., M.A., Professor of Mathematics in Stanford University, Palo Alto, Santa Clara Co., California	
1920	C. B.	*Allen, Herbert Stanley, M.A. (Cambridge), D.Sc. (London), Professor of Natural Philosophy in the University of St Andrews	1921-24.
1906		Anderson, Daniel E., M.D., B.A., B.Sc., 3 North Grove, Hampstead Lane, Highgate, London, N. 6	
1920	C.	*Anderson, Ernest Masson, M.A., B.Sc., F.G.S., Geologist, H. M. Geological Survey of Scotland, 7 Braidburn Terrace, Edinburgh	
1905		*Anderson, William, M.A., Head Science Master, George Watson's College, Edinburgh, 6 Lockharton Crescent, Edinburgh 10	
1903		Anderson-Berry, David, M.D., LL.D., F.R.S.L., M.R.A.S., F.S.A. (Scot.), Versailles, Highgate, London, N.	
1905		*Andrew, George, M.A., B.A., H.M.I.S., Royal Technical College, George Street, Glasgow. Hamewith, Kilmacolm, Renfrewshire	
1881	C.	Anglin, A. H., M.A., LL.D., M.R.I.A., formerly Professor of Mathematics, Queen's College, Cork	
1915		Anthony, Charles, M.Inst.C.E., M. Am. Soc. C.E., F.R.San.I., F.R.Met.S., F.R.A.S., F.C.S., Royal Society, Edinburgh, until further notice	
1906		Appleton, Colonel Arthur Frederick, F.R.C.V.S., Nutwell, 34 Shortlands Road, Shortlands, Kent 15	
1910	C.	Archibald, E. H., B.Sc., Professor of Chemistry, University of British Columbia, Vancouver, Canada	
1921		*Arthur, William, M.A., Assistant to the Professor of Mathematics in the University of Glasgow. 149 Stanmore Road, Mount Florida, Glasgow	
1911	C. K.	*Ashworth, James Hartley, D.Sc., F.R.S., Professor of Zoology, University of Edinburgh (VICE-PRESIDENT), 69 Braid Avenue, Edinburgh	1912-14, 1915-18. Sec. 1918-23. V-P 1923-
1920		*Bagnall, Richard Siddoway, Member of the Entomological and other Scientific Societies of Britain, France, Italy, Belgium, Russia, Spain, and the U.S.A., c/o Middleton & Middleton, 13 Granger Street West, Newcastle-on-Tyne	

Date of Election.			Service on Council, etc.
1920	C.	* Bailey, Edward Battersby, M.C., B.A., F.G.S., District Geologist, H.M. Geological Survey of Scotland, 23 Pentland Terrace, Edinburgh 20	
1896	C.	† Baily, Francis Gibson, M.A., M.Inst.E.E., Professor of Electrical Engineering, Heriot-Watt College, Edinburgh, Newbury, Colinton, Midlothian	1909-12, 1920-23.
1921		* Baker, Bevan Braithwaite, M.A., D.Sc., Professor of Mathematics, Royal Holloway College, Englefield Green, London	
1925		* Balfour, Arthur James, The Rt. Hon. the Earl of Balfour, K.G., O.M., F.R.S., Whittinghame, Prestonkirk, East Lothian	
1905	C.	Balfour-Browne, William Alexander Francis, M.A., F.Z.S., F.E.S., Barrister-at-Law, formerly Lecturer in Zoology (Entomology) in the University of Cambridge. Winscombe Court, Winscombe, Somerset	
1918		* Balsillie, David, B.Sc., F.G.S., Department of Mineralogy and Geology, Royal Scottish Museum, Edinburgh 25	
1923		* Bannerman, David Armitage, B.A. (Cantab.), M.B.E., F.R.G.S., Special Assistant (Ornithology) in the Department of Zoology, British Museum (Natural History), London. 60 Addison Road, London, W. 14	
1889		† Barbour, A. H. F., M.A., M.D., LL.D., F.R.C.P.E., 4 Charlotte Square, Edinburgh	
1886		Barclay, A. J. Gunion, M.A., 3 Chandos Avenue, Oakleigh Park, London, N.	
1903		Bardswell, Noël Dean, M.V.O., M.D., M.R.C.P. Ed. and Lond., New County Hall, Westminster Bridge Road, London, S.E. 1	
1922		* Barger, George, M.A., D.Sc., F.R.S., Professor of Chemistry (Medical) in the University of Edinburgh, 48 St Alban's Road, Edinburgh 30	1925-
1914	C.	* Barkla, Charles Glover, M.A., D.Sc., F.R.S., Professor of Natural Philosophy in the University of Edinburgh, Nobel Laureate, Physics, 1917, The Hermitage of Braid, Edinburgh	1915-18, 1924-
1925		* Barlow, Thomas William Naylor, O.B.E., M.R.C.S., D.P.H., Barrister-at-Law, Past President of the Incorporated Society of Medical Officers of Health. 23 North Drive, Wallasey	
1921		* Barr, Archibald, D.Sc., LL.D., F.R.S. (Glasgow and Birmingham), Em.-Professor of Engineering in the University of Glasgow, Westerton of Mugdock, Milngavie	
1904		Barr, Sir James, C.B.E., M.D., LL.D., F.R.C.P. Lond., 72 Rodney Street, Liverpool	
1921		* Bartholomew, John, M.C., M.A., F.R.G.S., Geographical Institute, Duncan Street, Edinburgh. Nairne Lodge, Duddingston 35	1925-
1913		Beard, Joseph, F.R.C.S. (Edin.), M.R.C.S. (Eng.), L.R.C.P. (Lond.), D.P.H. (Camb.), Medical Officer of Health and School Medical Officer, City of Carlisle, 8 Carlton Gardens, Carlisle	
1888		Beare, Thomas Hudson, B.A., B.Sc., M.Inst.C.E., J.P., D.L. (VICE-PRESIDENT), Professor of Engineering in the University of Edinburgh	1907-09. V-P 1909-15, 1923-
1897	C.	Beattie, Sir John Carruthers, K.B., D.Sc., Vice-Chancellor and Principal, The University, Cape Town	
1893	C. B.	Becker, Ludwig, Ph.D., Regius Professor of Astronomy in the University of Glasgow, The Observatory, Glasgow, Millbank Terrace, Crieff	
1887		Begg, Ferdinand Faithfull, 15 Wilbury Road, Hove, Sussex 40	
1906		Bell, John Patrick Fair, F.Z.S., Springbank, Ayton, Berwickshire	
1916		* Bell, Robert John Tainsh, M.A., D.Sc., Professor of Mathematics in the University of Otago, Dunedin, New Zealand	
1915		Bell, Walter Leonard, M.D.Edin., F.S.A.Scot., 123 London Road, North Lowestoft, Suffolk	
1893	C.	Berry, Sir George A., M.B., C.M., LL.D., F.R.C.S.E., M.P., King's Knoll, North Berwick	1916-19. V-P 1919-22.
1897	C.	Berry, Richard J. A., M.D., F.R.C.S.E., Professor of Anatomy in the University of Melbourne, Victoria, Australia 45	
1880	C.	Birch, De Burgh, C.B., M.D., Emeritus Professor of Physiology in the University of Leeds	
1907		* Black, Frederick Alexander, Solicitor, 59 Academy Street, Inverness	
1897	C.	Blaikie, Walter Biggar, LL.D. (VICE-PRESIDENT), Firbank, Colinton	1914-17. V-P 1924-
1904	C.	* Bles, Edward J., M.A., D.Sc., Elterholm, Madingley Road, Cambridge	
1918		* Blight, Francis James, Chairman and Managing Director of Charles Griffin & Co., Ltd., Publishers, Tregenna, Wembley, Middlesex 50	

Date of Election.			Service on Council, etc.
1894		Bolton, Herbert, D.Sc., F.G.S., F.Z.S., Director of the Bristol Museum and Art Gallery, Bristol, 58 Coldharbour Road, Redland, Bristol	
1915		* Boon, Alfred Archibald, D.Sc., B.A., F.I.C., Professor of Chemistry, Heriot-Watt College, Edinburgh	
1925		* Borthwick, Albert William, O.B.E., D.Sc., Civil Servant under the Forestry Commission. c/o W. H. Borthwick, Esq., Crookston, Heriot, Midlothian	
1925		* Bose, Sahay Ram, M.A., F.L.S., D.Sc., Professor of Botany in the Carmichael Medical College, Belgachia, Calcutta, India	
1872	C.	Bottomley, J. Thomson, M.A., D.Sc., LL.D., F.R.S., F.C.S., 13 University Gardens, Glasgow 55	
1886	C.	Bower, Frederick O., M.A., D.Sc., LL.D., F.R.S., F.L.S., Emeritus Regius Professor of Botany in the University of Glasgow. 2 The Crescent, Ripon, Yorks	1887-90, 1893-96, 1907-09, 1917-19. V-P 1910-16. P 1919-24.
1924		* Bowman, Alexander, D.Sc., Scientific Superintendent, Marine Laboratory, Fishery Board for Scotland, Torry, Aberdeen	
1901		Bradbury, J. B., M.D., Downing Professor of Medicine, University of Cambridge	
1916		Bradley, His Honour Judge (Francis Ernest), M.A., M.Com., LL.D., Member of the Court of Governors of Manchester University. 8 Balmoral Road, St Annes-on-the-Sea	
1903	C.	* Bradley, O. Charnock, M.D., D.Sc., Principal, Royal (Dick) Veterinary College, Edinburgh 60	1907-10, 1915-17.
1886		Bramwell, Sir Byrom, M.D., F.R.C.P.E., LL.D., D.C.L., 10 Heriot Row, Edinburgh	1890-93.
1907		* Bramwell, Edwin, Professor of Clinical Medicine in the University of Edinburgh, M.D., F.R.C.P.E., F.R.C.P. Lond., 23 Drumsheugh Gardens, Edinburgh	
1918		* Bremner, Alexander, M.A., D.Sc., Headmaster, Demonstration School, Training Centre, Aberdeen, 13 Belgrave Terrace, Aberdeen	
1916	C.	* Briggs, Henry, D.Sc., Ph.D., A.R.S.M., James A. Hood Professor of Mining in the University of Edinburgh. Allermuir, Liberton, Midlothian	1923-
1895		Bright, Sir Charles, M.Inst.C.E., M.Inst.E.E., F.R.Aë.S., F.Inst.Radio.E., F.R.A.S., F.R.G.S., Little Brewers', Hatfield Heath, Harlow, Essex, and Athenæum Club, Pall Mall, London, S.W. 65	
1893		Brock, G. Sandison, M.D., 78 Coleherne Court, London, W. 5	
1901	C.	Brodie, W. Brodie, M.D., Camden House, Bletchingley, Surrey	
1907		Brown, Alexander, M.A., B.Sc., Professor of Applied Mathematics, The University, Cape Town	
1885	C.	Brown, J. Macdonald, M.D., F.R.C.S., 64 Upper Berkeley Street, Portman Square, London, W.	
1924		* Brown, Thomas Arnold, M.A., Professor of Mathematics in University College, Exeter 70	
1923		* Brown, Walter, M.A., B.Sc., Professor of Mathematics, The University, Hong Kong, China	
1921		* Bruce, Alexander, B.Sc. (Edin.), Government Agricultural Chemist and City Analyst, Colombo, Ceylon	
1912		* Bruce, Alexander Ninian, D.Sc., M.D., 8 Ainslie Place, Edinburgh	
1898	C. K.	† Bryce, T. H., M.A., M.D. (Edin.), F.R.S. (VICE-PRESIDENT), Professor of Anatomy in the University of Glasgow. 2 The University, Glasgow	1911-14, 1922-25, V-P 1925-
1902		* Burgess, A. G., M.A., D.Sc., Rector of The Academy, Rothesay, Blythswood, Rothesay 75	
1887		† Burnet, Sir John James, A.R.A., R.S.A., LL.D., Architect, 1 Montague Place, Bedford Square, London, W.C. 1	
1888		Burns, Rev. T., D.D., J.P., F.S.A.Scot., Minister of Lady Glenorchy's Parish Church, Croston Lodge, Chalmers Crescent, Edinburgh	
1917		* Burnside, George Barnhill, M.I.Mech.E., 104 Beechwood Drive, Glasgow, W.	
1915		* Butchart, Raymond Keiler, B.Sc., Ph.D., University College, Dundee, 5 Briarwood Terrace, West Park Road, Dundee	
1896		Butters, J. W., M.A., B.Sc., Rector of Ardrossan Academy 80	
1887	C.	Cadell, Henry Moubray, of Grange, B.Sc., D.L., Linlithgow	1919-22.
1910		* Calderwood, Rev. Robert Sibbald, Minister of Cambuslang, The Manse, Cambuslang, Lanarkshire	

Date of Election.				Service on Council, etc.
1893	C.	Calderwood, W. L., I.S.O., Inspector of Salmon Fisheries of Scotland, South Bank, Canaan Lane, Edinburgh		1923-
1905	C.	Cameron, John, M.D., D.Sc., M.R.C.S. Eng., Professor of Anatomy, Dalhousie University, Halifax, Nova Scotia		
1921		*Campbell, Andrew, Advisory Chemist, Burmah Oil Co., Ltd., and Anglo-Persian Oil Co., Ltd. Lindum, 9 Foxgrove Road, Beckenham, Kent	85	
1918		*Campbell, John Menzies, L.D.S. (Glas.), D.D.S. (Toronto), L.D.S. (Ontario), 14 Buckingham Terrace, Glasgow, W.		
1915	C. N.	*Campbell, Robert, M.A., D.Sc., F.G.S., Lecturer in Petrology, University of Edinburgh, 2 Woodhall Road, Colinton		1920-28.
1899	C.	Carlier, Edmund W. W., M.D., M.Sc., F.E.S., Professor of Physiology, University, Birmingham		
1910		Carnegie, Col. David, C.B.E., M.Inst.C.E., M.Inst.Mech.E., M.I.S.Inst., Haven, Seasalter, Whitstable		
1920	C.	*Carruthers, R. G., F.G.S., District Geologist, H. M. Geological Survey, High Barn, Stocksfield-on-Tyne	90	
1905	C.	*Carse, George Alexander, M.A., D.Sc., Reader in Natural Philosophy, University of Edinburgh, 3 Middleby Street, Edinburgh		
1901		Carslaw, H. S., M.A., D.Sc., Professor of Mathematics in the University of Sydney, New South Wales		
1925		*Carter, George Stuart, M.A., Ph.D., Lecturer in Zoology in the University of Glasgow. 25 Belhaven Terrace, Great Western Road, Glasgow		
1905		Carter, Joseph Henry, F.R.C.V.S., Avalon, Western Road, Henley-on-Thames		
1898		Carus-Wilson, Cecil, F.R.G.S., F.G.S., Waldegrave Park, Strawberry Hill, Middlesex, and Sandacres Lodge, Parkstone-on-Sea, Dorset	95	
1882		Cay, W. Dyce, M.Inst.C.E., Junior Carlton Club, Pall Mall, London, S.W. 1		
1899		Chatham, James, Actuary, c/o Robert Murrie, Esq., 28 St Andrew Square, Edinburgh		
1912		Chaudhuri, Banawari Lal, B.A. (Cal.), D.Sc. (Edin.), Assistant Superintendent, Natural History Section, Indian Museum, Sherpurtown P.O., India		
1924		*Chisholm, George Goudie, M.A., B.Sc., LL.D., Em. Reader in Geography in the University of Edinburgh. 12 Hallhead Road, Edinburgh		
1925	C.	*Chumley, James, formerly Assistant in the "Challenger" Office, now Research Fellow in the Zoological Department of the University of Glasgow. 565 Crow Road, Jordanhill, Glasgow	100	
1891		Clark, John B., M.A., J.P., Head Master of Heriot's Hospital School, Lauriston, Garleffin, 146 Craiglea Drive, Edinburgh		
1911		*Clark, William Inglis, D.Sc., 22 Buckingham Terrace, Edinburgh		
1903		Clarke, William Eagle, I.S.O., LL.D., F.L.S., Honorary Supervisor of the Bird Collection and formerly Keeper of the Natural History Collections in the Royal Scottish Museum, Edinburgh. 8 Grosvenor Street, Edinburgh		
1909		Clayton, Thomas Morrison, M.D., D.Hy., B.Sc., D.P.H., Medical Officer of Health, Gateshead, 13 The Crescent, Gateshead-on-Tyne		
1922		*Clerk, Sir Dugald, K.B.E., D.Sc., LL.D., F.R.S., M.Inst.C.E., etc., Lukyns, Ewhurst, Surrey	105	
1904	C.	Coker, Ernest George, M.A., D.Sc., Hon. D.Sc. (Sydney), F.R.S., M.Inst.C.E., M.Inst.E.E., Professor of Civil and Mechanical Engineering, University of London, University College, Gower Street, London, W.C.		
1904		Coles, Alfred Charles, M.D., D.Sc., York House, Poole Road, Bournemouth, W.		
1888	V. J. C.	Collie, John Norman, Ph.D., D.Sc., LL.D., F.R.S., F.C.S., F.I.C., F.R.G.S., Professor of Organic Chemistry in the University College, Gower Street, London		
1904		*Colquhoun, Walter, M.A., M.B., 18 Walmer Crescent, Ibrox, Glasgow		
1909	C.	*Comrie, Peter, M.A., B.Sc., Head Master of Leith Academy, 19 Craighouse Terrace, Edinburgh	110	
1886		Connan, Daniel M., M.A.		
1925		*Constable, Arthur Henderson Briggs, The Hon. Lord Constable, M.A., LL.B., Senator of the College of Justice. 23 Royal Circus, Edinburgh		
1923	C.	*Cooper, James, Associate, Heriot-Watt College, M.I.M.E., Assistant Professor of Mining, Heriot-Watt College, Edinburgh. 35 Lauriston Gardens, Edinburgh		
1924		*Copson, Edward Thomas, M.A., Lecturer in Mathematics in the University of Edinburgh		
1905		*Corrie, David, F.C.S., 6 Carlton Hill, London, N.W. 8	115	
1914		*Coutts, William Barron, M.A., B.Sc., Senior Lecturer in Range Finding and Optics, Artillery College, Red Barracks, Woolwich, S.E. 18. 74 Vanbrugh Park, Blackheath, S.E. 3		

Date of Election.			Service on Council, etc.
1911		* Cowan, Alexander, Papermaker, Valleyfield, Penicuik, Midlothian	
1920		Craib, William Grant, M.A. (Aberdeen), Regius Professor of Botany in the University of Aberdeen	
1916	C.	† Craig, E. H. Cunningham, B.A. (Cambridge), Geologist and Mining Engineer, The Dutch House, Beaconsfield	
1908		Craig, James Ireland, M.A., B.A., Woolwich House, The Drive, Sydenham, London, S.E. 26 120	
1925	C.	* Craig, Robert Meldrum, M.A., B.Sc., F.G.S., Lecturer in Economic Geology in the University of Edinburgh	
1903		Crawford, Lawrence, M.A., D.Sc., Professor of Pure Mathematics, The University, Cape Town	
1922	C.	* Crew, Francis Albert Eley, M.D., Ch.B., D.Sc., Ph.D., Lecturer in Genetics in the University of Edinburgh, and Director of the Animal Breeding Research Station. 17 Salisbury Road, Edinburgh	
1870		Crichton-Browne, Sir Jas., Kt., M.D., LL.D., D.Sc., F.R.S., Vice-President and Treasurer of the Royal Institution of Great Britain. 45 Hans Place, London, S.W.	
1916		* Crombie, James Edward M.A., LL.D., Millowner, Parkhill House, Dyce, Aberdeenshire 125	
1914		* Cumming, Alexander Charles, O.B.E., D.Sc., Roselands, Crescent Road, Blundell Sands, Liverpool	
1917		* Cunningham, Brysson, D.Sc., B.E., M.Inst.C.E., Lecturer on Waterways, Harbours, and Docks, University College, London, 141 Copers Cope Road, Beckenham, Kent	
1898		Currie, James, M.A. Cantab., LL.D. (TREASURER), Larkfield, Goldenacre, Edinburgh	Treas. 1906–
1919		* Cushny, Arthur Robertson, M.A., M.D., LL.D., F.R.S., Professor of Materia Medica and Pharmacology, University, Edinburgh	1919-22. 1925–
1904		* Cuthbertson, John, Secretary, West of Scotland Agricultural College, 6 Charles Street, Kilmarnock 130	
1885		Daniell, Alfred, M.A., LL.B., D.Sc., Advocate, The Athenæum Club, Pall Mall, London	
1924		* Darwin, Charles Galton, M.A., F.R.S., Tait Professor of Natural Philosophy in the University of Edinburgh	1925–
1921		* Datta, Rasik Lal, D.Sc., Industrial Chemist to the Government of Bengal Department of Industries. 78 Manicktola Street, Calcutta, India	
1917		* Day, T. Cuthbert, Partner of the firm of Hislop & Day, 36 Hillside Crescent, Edinburgh	
1923		* Deane, Arthur, M.R.I.A., Curator, Public Art Gallery and Museum, Belfast. Mount Nephin, Newtownbreda, Belfast 135	
1894		Denny, Sir Archibald, Bart., LL.D., Spencer House, Park Side, Wimbledon, S.W. 19	
1905		* Dewar, James Campbell, C.A., 27 Douglas Crescent, Edinburgh	
1906		† Dewar, Thomas William, M.D., F.R.C.P., Kincairn, Dunblane	
1925		Dey, Alexander John, Managing Director of T. & H. Smith, Ltd., Manufacturing Chemists, Blandfield Works, Edinburgh. Rothiemay, Corstorphine, Edinburgh	
1876	C.	Dickson, J. D. Hamilton, M.A., Senior Fellow and formerly Tutor, St Peter's College, Cambridge 140	
1924		* Dinham, C. H., B.A., District Geologist, Geological Survey, 33 George Square, Edinburgh	
1885	C.	Dixon, James Main, M.A., Litt. Hum. Doctor, Professor of English, University of Southern California, University Avenue, Los Angeles, California, U.S.A.	
1923		* Dixon, Ronald Audley Martineau, F.G.S., F.S.A. (Scot.), F.R.G.S., Thearne Hall near Beverley	
1897		Dobbie, James Bell, F.Z.S., 12 South Inverleith Avenue, Edinburgh	
1881	C.	Dobbin, Leonard, Ph.D., formerly Lecturer in Chemistry in the University of Edinburgh. Faladam, Blackshiels, Midlothian 145	1904-07, 1913-16.
1918		* Dodd, Alexander Scott, B.Sc., F.I.C., F.C.S., City Analyst for Edinburgh, 20 Stafford Street, Edinburgh	
1925		* Donald, Alexander Graham, M.A., F.F.A., F.S.A. (Scot.), Secretary of the Scottish Provident Institution, Edinburgh. 18 Carlton Terrace, Edinburgh	
1905		* Donaldson, Rev. Wm. Galloway, J.P., F.R.G.S., F.E.I.S., The Manse of Forfar, Forfarshire	
1882	C.	Dott, David B., F.I.C., Memb. Pharm. Soc., Ravenslea, Musselburgh	
1921	B. C.	* Dougall, John, M.A., D.Sc., Publisher's Reader, 47 Cornwall Avenue, Jordanhill, Glasgow 150	

Date of Election.			Service on Council, etc.
1901 & 1918		* Douglas, Carstairs Cumming, M.D., D.Sc., Professor of Medical Jurisprudence and Hygiene, Anderson's College, Glasgow, 110 South Brae Drive, Jordanhill, Glasgow	
1910		* Douglas, Loudon MacQueen, Author and Lecturer, Newpark, West Calder, Midlothian	
1923	C.	* Drever, James, M.A., B.Sc., D.Phil., Reader in Psychology, University of Edinburgh. South Cottage, Boswell Road, Edinburgh	
1908	C.	Drinkwater, Harry, M.D., M.R.C.S. (Eng.), F.L.S., Lister House, Wrexham, North Wales	
1901		Drinkwater, Thomas W., L.R.C.P.E., L.R.C.S.E., Chemical Laboratory, Surgeons' Hall, Edinburgh 155	
1917		* Dron, Robert W., M.A., M.Inst.C.E., M.Inst.M.E., Professor of Mining in the University of Glasgow. 17 Blythswood Square, Glasgow	
1923		* Drummond, J. Montagu F., M.A. (Cantab.), Regius Professor of Botany in the University of Glasgow	
1925		* Dryerre, Henry, M.R.C.S., Ph.D., Lecturer in Physiology in the University of Edinburgh. Kenmore, Lasswade	
1921		* Drysdale, Charles Vickery, O.B.E., D.Sc. (Lond.), M.I.E.E., F.Inst.P., Superintendent of the Admiralty Research Laboratory, Teddington, Middlesex	
1925	C.	* Dunkerly, John Samuel, D.Sc., Ph.D., Lecturer in Zoology in the University of Glasgow 160	
1904		* Dunlop, William Brown, M.A., 4A St Andrew Square, Edinburgh	
1908		Dunstan, John, M.R.C.V.S., Inversnaid, Liskeard, Cornwall	
1892	C.	Dunstan, M. J. R., M.A., F.I.C., F.C.S., Royal Agricultural College, Cirencester	
1906	C.	Dyson, Sir Frank Watson, Kt., M.A., D.Sc., LL.D., F.R.S., Astronomer Royal, Royal Observatory, Greenwich	1907–10.
1925		* Eastwood, George Samuel, B.Sc., Principal Teacher of Mathematics, Beath Secondary School, Cowdenbeath, Fife. Craigie Lea, Cowdenbeath, Fife 165	
1893		Edington, Alexander, M.D., C.M., D.P.H., D.T.M. and E. (Lieut.-Col.) Ethandune, Greytown, Natal	
1904		* Edwards, John, LL.D., 4 Great Western Terrace, Kelvinside, Glasgow	
1904		* Elder, William, M.D., F.R.C.P.E., 4 John's Place, Leith	
1913 & 1921		* Elliot, George Francis Scott, M.A. (Cantab.), B.Sc., F.R.G.S., F.L.S., Drumwhill, Mossdale	
1924		* Elliot, Walter Elliot, M.C., M.P., M.B., Ch.B., D.Sc., 14 Markham Square, Chelsea, London 170	
1906	C.	* Ellis, David, D.Sc., Ph.D., Professor of Botany and Bacteriology, Royal Technical College, Glasgow	
1897	C.	Erskine-Murray, James Robert, D.Sc., H.M. Signal School, Portsmouth	
1924		* Evans, Arthur Humble, M.A., Sc.D., Lecturer in English History (under Special Board), 9 Harvey Road, Cambridge	
1924		Evans, William Edgar, B.Sc., Assistant in charge of Herbarium, Royal Botanic Garden, Edinburgh	
1879	C. N.	Ewart, James Cossar, M.D., F.R.C.S.E., F.R.S., F.Z.S., Regius Professor of Natural History, University of Edinburgh, Craigybield, Penicuik, Midlothian 175	1882–85, 1904–07. V-P 1907–12.
1902		* Ewen, John Taylor O.B.E., B.Sc., M.I.Mech.E., J.P., H.M. Inspector of Schools, Clairmont, 54 Albert Drive, Pollokshields, Glasgow	
1878	C.	† Ewing, Sir James Alfred, K.C.B., M.A., D.Sc., LL.D., M.Inst.C.E., F.R.S., J.P. (PRESIDENT) Principal of the University of Edinburgh, formerly Director of Naval Education, Admiralty, 16 Moray Place, Edinburgh	1888–91, 1919–20. V-P 1920–23. P 1924–
1900	C.	Eyre, John W. H., M.D., M.S. (Dunelm), D.P.H. (Camb.), Professor of Bacteriology, Guy's Hospital, London	
1910	C.	* Fairgrieve, Mungo M'Callum, M.A. (Camb. and Glasg.), Master at the Edinburgh Academy, 37 Queen's Crescent, Edinburgh	
1907	C.	Falconer, John Downie, M.A., D.Sc., F.G.S., Director of the Geological Survey of Nigeria. 58 Prince of Wales Road, Battersea Park, London, S.W. 11 180	
1888	C.	Fawsitt, Charles A., Glenfruin, Milton Road, Harpenden, Herts	
1923		* Feldman, William Moses, M.D., B.S. (Lond.), M.R.C.S., L.R.C.P., Physician, St Mary's Hospital for Women and Children, Plaistow. 851 Finchley Road, London, N.W. 11	
1883	C.	Felkin, Robert W., M.D., F.R.G.S., Whare Ra, Havelock North, Hawkes Bay, New Zealand	

Date of Election.			Service on Council, etc.
1899		Fergus, Andrew Freeland, M.D., LL.D., 14 Newton Place, Charing Cross, Glasgow	
1907		* Fergus, Edward Oswald, c/o Messrs M'Kay & Boyd, Solicitors, 50 Wellington Street, Glasgow 185	
1904		* Ferguson, James Haig, M.D., F.R.C.P.E., F.R.C.S.E., 7 Coates Crescent, Edinburgh	
1925	C.	* Ferrar, William Leonard, M.A., Fellow and Tutor of Hertford College, Oxford	
1898		† Findlay, Sir John R., Bt., K.B.E., M.A. Oxon., LL.D., 3 Rothesay Terrace, Edinburgh	
1911		Fleming, John Arnold, F.C.S., etc., Pottery Manufacturer, Locksley, Helensburgh, Dumbartonshire	
1906		* Fleming, Robert Alexander, M.A., M.D., F.R.C.P.E., Physician, Royal Infirmary, 10 Chester Street, Edinburgh 190	
1900	C. N.	Flett, Sir John S., K.B.E., M.A., D.Sc., LL.D., F.R.S., Director of the Geological Survey of Great Britain and of the Museum of Practical Geology, London, 28 Jermyn Street, S.W. 1	1916–19.
1872		Forbes, George, M.A., M.Inst.C.E., M.Inst.E.E., F.R.S., F.R.A.S., formerly Professor of Natural Philosophy in Anderson's College, Glasgow. 11 Little College Street, Westminster, S.W.	
1892		Ford, John Simpson, F.C.S., 7 Corrennie Drive, Edinburgh	
1921		* Forrest, George Topham, F.R.I.B.A., F.G.S., F.S.Arc., Architect to the London County Council, and Superintending Architect of Metropolitan Buildings, The County Hall, Westminster Bridge, London, S.E. 1	
1920	C.	* Franklin, Thomas Bedford, B.A. (Hons. Mathematics), Cambridge, Stancliffe Hall, near Matlock, Derbyshire 195	
1910		* Fraser, Alexander, Actuary, 15 S. Learmonth Gardens, Edinburgh	
1915		* Fraser, Rev. Joseph Robert, U.F. Manse, Kinneff, Bervie	
1914		* Fraser, William, Managing Director, Neill & Co., Ltd., Printers, 212 Causewayside, Edinburgh	
1891		Fulton, T. Wemyss, M.D., Scientific Superintendent, Scottish Fishery Board, 41 Queen's Road, Aberdeen	
1907		* Galbraith, Alexander, "Ravenswood," Dalmuir, Glasgow 200	
1888	C.	Galt, Alexander, D.Sc., late Keeper of the Department of Technology, Royal Scottish Museum, Edinburgh, c/o Clydesdale Bank, 1 Melville Place, Edinburgh	
1901		Ganguli, Sanjiban, M.A., Principal, Maharaja's College, and Director of Public Instruction, Jaipur State, Jaipur, India	
1923		* Gardiner, Frederick, M.D., B.Sc., F.R.C.S.E., F.R.S.M., Lecturer in Diseases of the Skin in the University of Edinburgh; Physician to the Royal Infirmary. 60 George Square, Edinburgh	
1909	C.	* Geddes, Rt. Hon. Sir Auckland C., G.C.M.G., K.C.B., P.C., M.D., D.C.L., formerly British Ambassador to the U.S.A. Athenæum Club, Pall Mall, London	
1880	C.	Geddes, Patrick, Professor of Sociology and Civics, University of Bombay, India 205	
1914		Gemmell, John Edward, M.B., C.M., Hon. Surgeon, Hospital for Women and Maternity Hospital; Hon. Gynæcologist, Victoria Central Hospital, Liscard, 28 Rodney Street, Liverpool	
1923		* Gemmill, James Fairlie, M.A., M.D., D.Sc., F.R.S., Professor of Natural History, University College, Dundee. Hawkhill House, Dundee	
1909		* Gentle, William, B.Sc., 10 West Savile Road, Edinburgh	
1920	C.	* Ghosh, Sudhamoy, M.Sc. (Cal.), D.Sc. (Edin.), F.C.S., Professor of Chemistry, School of Tropical Medicine and Hygiene, Central Avenue, Calcutta, India	
1914		* Gibb, Sir Alexander, G.B.E., C.B., M.Inst.C.E., formerly Director-General of Civil Engineering, Ministry of Transport. Queen Anne's Lodge, Westminster, London, S.W. 1 210	
1916		* Gibb, A. W., M.A., D.Sc., Professor of Geology in the University of Aberdeen, 1 Belvidere Street, Aberdeen	
1910	C.	* Gibb, David, M.A., B.Sc., Lecturer in Mathematics, Edinburgh University, 15 South Lauder Road, Edinburgh	
1917	C.	* Gibson, Alexander, M.B., Ch.B., F.R.C.S. (Eng.), 661 Broadway, Winnipeg, Canada	
1910		* Gibson, Charles Robert, Lynton, Mansewood, by Pollokshaws	
1890		Gibson, George A., M.A., LL.D., Professor of Mathematics in the University of Glasgow, 10 The University, Glasgow 215	1905–08, 1912–13. V-P 1917–20.

Date of Election.			Service on Council, etc.
1921		*Gibson, Walcot, D.Sc., F.R.S., F.G.S., formerly Assistant Director, H.M. Geological Survey (Scotland), 31 Clarendon Road, Putney, London, S.W. 15	
1911		Gidney, Lt.-Col. Henry A. J., F.R.C.S., D.O., D.P.H., J.P., M.L.A., I.M.S. (retired), Army Specialist Public Health, c/o The Allahabad Bank, Ltd., Calcutta, India	
1900		Gilchrist, Douglas A., B.Sc., Professor of Agriculture and Rural Economy, Armstrong College, Newcastle-upon-Tyne	
1925		*Gillies, William King, M.A., B.A., F.E.I.S., Rector of the Royal High School, Edinburgh. Davaar, 12 Suffolk Road, Edinburgh	
1907		Gilruth, John Anderson, M.R.C.V.S., D.V.Sc. (Melb.), Administrator, Government House, Darwin Northern Territory, Australia 220	
1909		*Gladstone, Hugh Steuart, M.A., M.B.O.U., F.Z.S., Capenoch, Thornhill, Dumfriesshire	
1911		Gladstone, Reginald John, M.D., F.R.C.S. (Eng.), Lecturer and Senior Demonstrator of Anatomy, King's College, University of London, 22 Regent's Park Terrace, London, N.W.	
1898		Glaister, John, M.D., F.R.F.P.S. Glasgow, D.P.H. Camb., Regius Professor of Forensic Medicine and Public Health in the University of Glasgow, 3 Newton Place, Glasgow	
1925	C.	*Goldie, Archibald Hayman Robertson, M.A., B.A., Superintendent, Meteorological Office, Air Ministry, Edinburgh, 6 Drumsheugh Gardens, Edinburgh	
1910		Goodall, Joseph Strickland, M.B. (Lond.), M.R.C.P., F.R.C.S.Ed., M.S.A. (Eng.), Professor of Physiology and Biology, City of London Hospital. 136 Harley, Street, London, W. 1 225	
1901		Goodwillie, James, M.A., B.Sc., Liberton, Edinburgh	
1920	C.	*Gordon, William, B.Sc., Ph.D., A.M.I.Mech.E., Lecturer in Engineering in the University of Edinburgh, 3 Wellington Street, Edinburgh	
1913	C. B.	*Gordon, William Thomas, M.A., D.Sc. (Edin.), M.A. (Cantab.), Professor of Geology, University of London, King's College, Strand, W.C.	
1897		Gordon-Munn, John Gordon, M.D., Croys, Dalbeattie	
1923		*Grabham, George Walter, M.A. (Cantab.), Government Geologist, Anglo-Egyptian Sudan. Box 178, Khartoum 230	
1924		Graham, Robert James Douglas, M.A., D.Sc., Lecturer in Plant Physiology in the University of Edinburgh. Royal Botanic Garden, Edinburgh	
1898	C.	Gray, Albert A., M.D., 4 Clairmont Gardens, Glasgow	
1909	C.	*Gray, James Gordon, D.Sc., Professor of Applied Physics in the University of Glasgow, 11 The University, Glasgow	1913-15.
1910		Gray, Robert Bruce M'Gregor, A.M.Inst.C.E., Westbourne Grove, Selby, Yorkshire	
1918		*Gray, Wm. Forbes, F.S.A. (Scot.), Editor and Author, 8 Mansionhouse Road, Edinburgh 235	
1897		Greenlees, Thomas Duncan, M.D. Edin., 23 St Matthew's Gardens, St Leonards-on-Sea, Sussex	
1922		*Greenwood, William Osborne, M.D. (Leeds), B.S. (Lond.), L.S.A., Obstetric Surgeon and Physician, Woodroyd, 17 Ripon Road, Harrogate, Yorks	
1905	C. K.	*Gregory, John Walter, D.Sc., F.R.S., Professor of Geology in the University of Glasgow, 4 Park Quadrant, Glasgow	1908-11 V-P 1920-23.
1925		*Greig, David Middleton, M.B., C.M., F.R.C.S.E., Conservator of the Museum of the Royal College of Surgeons of Edinburgh. 12 Abbotsford Crescent, Edinburgh	
1906		Greig, Edward David Wilson, C.I.E., M.D., D.Sc., Lt.-Col., H.M. Indian Medical Service, 38 Coates Gardens, Edinburgh 240	
1905		Greig, Sir Robert Blyth, M.C., LL.D., F.Z.S. (VICE-PRESIDENT), Chairman of the Board of Agriculture for Scotland, York Buildings, Queen Street, Edinburgh	1921-24. V-P 1924-
1910		*Grimshaw, Percy Hall, Assistant Keeper, Natural History Department, The Royal Scottish Museum, 49 Comiston Drive, Edinburgh	
1899		Guest, Edward Graham, M.A., B.Sc., City Treasurer, Edinburgh. 5 Newbattle Terrace, Edinburgh	
1907	C.	*Gulliver, Gilbert Henry, D.Sc., A.M.I.Mech.E., 99 Southwark Street, London, S.E.	
1888	C.	Guppy, Henry Brougham, M.B., C.M., F.R.S., Red House, Fowey, Cornwall 245	
1922		*Guthrie, William Alexander, M.B.E., F.I.C., Memb. Inst. Petroleum Technologists, Oil Refining Expert; at present Petroleum Research Chemist with the Egyptian Government. Turf Club, Cairo	

Date of Election.			Service on Council, etc.
1911		* Guy, William, F.R.C.S., L.R.C.P., L.D.S.Ed., Consulting Dental Surgeon, Edinburgh Royal Infirmary; Dean, Edinburgh Dental Hospital and School; Lecturer on Human and Comparative Dental Anatomy and Physiology, 11 Wemyss Place, Edinburgh	
1911		Hall-Edwards, John Francis, L.R.C.P. (Edin.), Hon. F.R.P.S., Senior Medical Officer in charge of X-ray Department, General Hospital, Birmingham, 141A and 141B Great Charles Street (Newhall Street), Birmingham	
1922		* Hannay, Robert Kerr, M.A., Professor of Ancient History and Palæography in the University of Edinburgh. 5 Royal Terrace, Edinburgh	
1923		* Hanneford-Smith, William, Member of the Institute of Metals, 3 The Avenue, Gravesend, Kent 250	
1918		* Hardie, P. S., M.A., B.Sc., Physics Department, School of Medicine, Cairo, Egypt	
1896	C.	Harris, David Fraser, B.Sc. (Lond.), D.Sc. (Birm.), M.D., F.S.A. Scot., formerly Professor of Physiology in the Dalhousie University, Halifax, Nova Scotia. Dane Hill, East Grinstead, Sussex	
1923	C.	* Harris, Robert Graham, M.A., D.Sc. (Edin.), (Aeronautical) Research Physicist, Lorraine, Manor Road, Farnborough, Hants	
1914		Harrison, Edward Philip, Ph.D., F.Inst.P., Chief Scientist, H.M.S. "Vernon," Portsmouth	
1921		* Harrison, John William Heslop, D.Sc. (Durham), Lecturer in Genetics, Armstrong College, Newcastle. The Avenue, Birtley, Co. Durham 255	
1914	C.	Harvey-Gibson, Robert John, C.B.E., D.L., J.P., M.A., D.Sc., Emer. Professor of Botany, University of Liverpool. Grasmere, Westmorland	
1892	C.	Heath, Thomas, B.A., formerly Assistant Astronomer, Royal Observatory, Edinburgh, 7 Comiston Road, Edinburgh	
1893		Hehir, Sir Patrick, K.C.I.E., C.B., C.M.G., F.R.C.P.E., F.R.C.S.E., D.P.H. (Camb.), D.T.M. (Liv. Univ.), Maj.-General I.M.S. (retired), Ennisfarne, Westward Ho! Devon	
1900		Henderson, John, D.Sc., A.Inst.E.E., Kinnoul, Gregory's Road, Beaconsfield, Bucks	
1908		* Henderson, William Dawson, M.A., B.Sc., Ph.D., Lecturer, Zoological Laboratories, University, Bristol 260	
1890	C.	Hepburn, David, C.M.G., M.D., Professor of Anatomy in the University College of South Wales and Monmouthshire, Cardiff	
1925		* Heron, Alexander Macmillan, D.Sc., Officiating Superintendent, Geological Survey of India, Calcutta, India	
1916		* Herring, Percy Theodore, M.D., F.R.C.P.Ed., Professor of Physiology, University of St Andrews, Linton, St Andrews	1917–20.
1922		Hindle, Edward, M.A. (Cantab.), Ph.D., A.R.C.Sc., London School of Hygiene and Tropical Medicine. 23 Endsleigh Gardens, London, N.W. 1	
1902		Hinxman, Lionel W., B.A., formerly of the Geological Survey of Scotland. 4 Morant Gardens, Ringwood, Hants 265	
1904		Hobday, Frederick T. G., C.M.G., F.R.C.V.S., Officier du Merite Agricole, Cavaliere dei S.S. Maurizio e Lazaro, Hon. Veterinary Surgeon to H.M. the King, Editor of the Veterinary Journal, 165 Church Street, Kensington, London, W.	
1885		Hodgkinson, W. R., C.B.E., M.A., Ph.D., F.I.C., F.C.S., formerly Professor of Chemistry and Physics at the Ordnance College, Woolwich. 89 Shooter's Hill Road, Blackheath, Kent	
1923		* Hogben, Lancelot Thomas, M.A., D.Sc., Department of Zoology, M'Gill University, Montreal, Canada	
1911		Holland, William Jacob, LL.D. (St Andrews), Director, Carnegie Institute, Pittsburg, Pa. 5545 Forbes Street, Pittsburg, Pa., U.S.A.	
1920	C.	* Horne, Alexander Robert, O.B.E., B.Sc., M.I.Mech.E., A.M.I.C.E., Head of the School of Engineering, The Polytechnic, Regent Street, London. 31 Methuen Park, Muswell Hill, London, N. 10 270	
1881	C. N.	Horne, John, LL.D., F.R.S., F.G.S., formerly Director of the Geological Survey of Scotland, 20 Merchiston Gardens, Edinburgh	1902-05, 1906-07, 1914-15. V-P 1907-13. P 1915-19.
1896		Horne, J. Fletcher, M.D., F.R.C.S.E., The Poplars, Barnsley	
1904	C.	* Horsburgh, Ellice Martin, M.A., D.Sc., Reader in Technical Mathematics, University of Edinburgh, 11 Granville Terrace, Edinburgh	

Date of Election.			Service on Council, etc.
1897		Houston, Sir Alex. Cruikshanks, K.B.E., C.V.O., M.B., C.M., D.Sc., 19 Fairhazel Gardens, South Hampstead, London, N.W.	
1912	C. B.	* Houstoun, Robert Alexander, M.A., Ph.D., D.Sc., Lecturer in Physical Optics, University, Glasgow, 45 Kirklee Road, Glasgow 275	
1893		Howden, Robert, M.A., M.B., C.M., D.Sc., Professor of Anatomy in the University of Durham, 14 Burdon Terrace, Newcastle-upon-Tyne	
1883	C.	Hoyle, William Evans, M.A., D.Sc., M.R.C.S., formerly Director of the Welsh National Museum. Glenroy, Blundell Avenue, Porthcawl, Glam., Wales	
1910		Hume, William Fraser, D.Sc. (Lond.), Director, Geological Survey of Egypt, Helwân, Egypt. 4 Ma'adi, near Cairo, Egypt	
1923		* Hunter, Rev. Adam Mitchell, M.A., D.Litt., Librarian of New College, Edinburgh. 54 Morningside Drive	
1916		* Hunter, Charles Stewart, L.R.C.P.E., L.R.C.S.E., D.P.H., Cotswold, 36 Streatham Hill, London, S.W. 2 280	
1911		Hunter, Gilbert Macintyre, M.Inst.C.E., M.Inst.E.S., M.Inst.M.E., Resident Engineer, Nitrate Railways, Iquique, Chile, and Maybole, Ayrshire	
1887	C.	Hunter, William, C.B., M.D., M.R.C.P.L. and E., M.R.C.S., 103 Harley Street, London	
1908		Hyslop, Theophilus Bulkeley, M.D., M.R.C.P.E., 5 Portland Place, London, W.	
1923	C.	* Ince, Edward Lindsay, M.A. (Cantab.), D.Sc. (Edin.), Lecturer in Mathematics, University of Liverpool. 2B Grosvenor Terrace, Ullet Road, Liverpool	
1920		* Inglis, James Gall, Publisher and Editor of Educational Works, 36 Blacket Place, Edinburgh 285	
1912		* Inglis, Robert John Mathieson, M.Inst.C.E., District Engineer, W. Division, Lond. & N.E. Railway, S. Section. 15 Beaumont Gate, Glasgow, W. 2	
1904	C.	Innes, R. T. A., Director, Government Observatory, Johannesburg, Transvaal	
1917		* Irvine, Sir James Colquhoun, Kt., C.B.E., Ph.D., D.Sc., LL.D., F.R.S., Principal of the University of St Andrews	1920-22. V-P 1922-25.
1923		* Jack, John Louttit, Solicitor, Director of Housing and Town Planning, Scottish Board of Health, 125 George Street, Edinburgh	
1914		Jack, John Noble 290	
1889	C.	James, Alexander, M.D., F.R.C.P.E., 9 Randolph Crescent, Edinburgh	
1901		Jardine, Robert, M.D., M.R.C.S., F.R.F.P.S. Glas., 20 Royal Crescent, Glasgow, W.	
1912	C.	* Jeffrey, George Rutherford, M.D. (Glasg.), F.R.C.P. (Edin.), etc., Bootham Park Private Mental Hospital, York	
1906	C.	* Jehu, Thomas John, M.A., M.D., F.G.S., Professor of Geology in the University of Edinburgh. 35 Great King Street, Edinburgh	1917-20, 1923-
1900		Jerdan, David Smiles, M.A., D.Sc., Ph.D., 26 Avenue du Château d'Eau, Saventhem, Belgium 295	
1925		* Johnston, Christopher Nicholson, The Hon. Lord Sands, Kt., LL.D., Senator of the College of Justice. 4 Heriot Row, Edinburgh	
1895		Johnston, Col. Henry Halcro, C.B., D.Sc., M.D., F.L.S., late A.M.S., Orphir House, Kirkwall, Orkney	
1924		* Jones, David Thomas, C.B.E., Chairman, H.M. Fishery Board for Scotland, 101 George Street, Edinburgh	
1888		Jones, John Alfred, M.Inst.C.E., Fellow of the University of Madras, Sanitary Engineer to the Government of Madras, c/o Messrs Parry & Co., 70 Gracechurch Street, London	
1922		* Juritz, Charles Frederick, M.A., D.Sc., F.I.C., Chief of the Union Department of Chemistry, Villa Marina, Three Anchor Bay, Cape Town, South Africa 300	
1915		Kemnal, Sir James Hermann Rosenthal, Managing Director and Engineer-in-Chief of Babcock & Wilcox, Ltd., Kemnal Manor, Chislehurst, Kent	
1925		* Kemp, Charles Norman, B.Sc., Secretary of the Royal Scottish Society of Arts, Ivy Lodge, Laverockbank Road, Edinburgh	
1912		Kennedy, Robert Foster, M.D. (Queen's Univ., Belfast), M.B., B.Ch. (R.U.I.), Assoc. Professor of Neurology, Cornell University, New York, 20 West 50th Street, New York, U.S.A.	
1909		Kenwood, Henry Richard, C.M.G., M.B., C.M., Chadwick Emeritus Professor of Hygiene in the University of London, "Wadhurst," Queen's Road, Finsbury Park, London, N.	
1925	C.	* Kermack, William Ogilvy, M.A., B.Sc., Chemist to the Research Laboratory of the Royal College of Physicians, 2 Forrest Road, Edinburgh 305	
1908		* Kerr, Andrew William, F.S.A.Scot., 81 Great King Street, Edinburgh	

Date of Election.			Service on Council, etc.
1903 & 1923	C. N.	* Kerr, John Graham, M.A., F.R.S., F.L.S., F.Z.S., Regius Professor of Zoology, University of Glasgow, 9 The University, Glasgow	1904–07, 1913–16, 1924–
1891		Kerr, Joshua Law, M.D., J.P., 29 Blyth Street, Brunswick, Victoria, Australia	
1913		* Kerr, Walter Hume, M.A., B.Sc., Lecturer on Engineering Drawing and Structural Design in the University of Edinburgh	
1908		Kidd, Walter Aubrey, M.D., 2 Suffolk Square, Cheltenham 310	
1907		* King, Archibald, M.A., B.Sc., formerly Rector of the Academy, Castle Douglas; H.M. Inspector of Schools, Inverspey, Fochabers, Morayshire	
1925		* King, Leonard Augustus Lucas, M.A., Professor of Zoology in the West of Scotland Agricultural College, Glasgow. 48 University Avenue, Glasgow, W. 2	
1880		† King, W. F., Dunivard, Hunter's Quay, Argyllshire	
1918		* Kingon, Rev. John Robert Lewis, M.A. (Edin. and Cape of Good Hope), D.Sc. (Ghent), F.L.S., Observatory Congregational Church, Lower Main Road, Cape Town, South Africa	
1878		Kintore, The Right Hon. the Earl of, K.T., P.C., G.C.M.G., M.A. Cantab. LL.D., Keith Hall, Inverurie, Aberdeenshire 315	
1901		Knight, Rev. G. A. Frank, M.A., D.D., 10 Hillhead Street, Glasgow	
1907		* Knight, James, M.A., D.Sc., F.C.S., F.G.S., J.P., Rector, Queen's Park High School, Langside, Glasgow	
1921		* Lamb, James Alexander George, Banker, 11 Braid Crescent, Edinburgh	
1920		* Lamont, John Charles, Lieut.-Col., I.M.S. (retired), C.I.E., M.B., C.M. (Edin.), M.R.C.S. (Eng.), 7 Merchiston Park, Edinburgh	
1878	C.	Lang, Sir P. R. Scott, Kt., M.A., LL.D., B.Sc., Emeritus Professor of Mathematics, University of St Andrews. Mansefield, St Andrews 320	
1925	C.	* Lang, William Henry, M.B., C.M., D.Sc., F.R.S., Barker Professor of Cryptogamic Botany in the University of Manchester	
1910	C.	* Lauder, Alexander, D.Sc. (SECRETARY), Lecturer in Agricultural Chemistry, Edinburgh and East of Scotland College of Agriculture, 13 George Square, Edinburgh	1917–20. Sec. 1923–
1885	C.	Laurie, A. P., M.A., D.Sc., J.P., Principal of the Heriot-Watt College, Edinburgh	1908–11, 1913–16.
1921		* Laurie, The Rev. Albert Ernest, M.C., C.F., D.D., Rector of Old St Paul's, and Canon of St Mary's Cathedral, Edinburgh. Lauder House, Jeffrey Street, Edinburgh	
1894	C.	Laurie, Malcolm, B.A., D.Sc., F.L.S., 4 Wordsworth Road, Harpenden, Herts 325	
1910	C. B.	* Lawson, A. Anstruther, B.Sc., Ph.D., D.Sc., F.L.S., Professor of Botany, University of Sydney, New South Wales, Australia	
1905		* Lawson, David, M.A., M.D., L.R.C.P. and S.E., Druimdarroch, Banchory, Kincardineshire	
1910	C.	* Lee, Gabriel W., D.Sc., Palæontologist, Geological Survey of Scotland, 33 George Square, Edinburgh	
1903		* Leighton, Gerald Rowley, O.B.E., M.D., D.Sc., Medical Officer, Scottish Board of Health, 125 George Street, Edinburgh	
1910		Levie, Alexander, F.R.C.V.S., D.V.S.M., Balmae, Manor Road, Littleover, Derby 330	
1916	C.	* Levy, Hyman, M.A., D.Sc., Professor of Mathematics, Imperial College of Science and Technology, London, S.W. 7. 62 Kenilworth Avenue, Wimbledon Park, London, S.W. 19	
1914	C. N.	Lewis, Francis John, D.Sc., F.L.S., Professor of Biology, University of Alberta, Edmonton South, Alberta, Canada	
1918		* Lidstone, George James, F.F.A., F.I.A., LL.D., Manager and Actuary of the Scottish Widows' Fund Life Assurance Society, 9 St Andrew Square, Edinburgh	1919–22.
1905		* Lightbody, Forrest Hay, 53 Queen Street, Edinburgh	
1923		* Lim, R. K. S., M.B., Ch.B., D.Sc., Ph.D., The Medical Research Institute, The University, Amoy, China. 335	
1912		* Lindsay, John George, M.A., B.Sc. (Edin.), Rector of Dunfermline High School	
1920	C.	* Lindsay, Thomas A., M.A. (Hons.), B.Sc., Head Master, Higher Grade School, Bucksburn, Aberdeenshire	
1912		* Linlithgow, The Most Honourable the Marquis of, Hopetoun House, South Queensferry	
1903		† Liston, William Glen, C.I.E., M.D., Lt.-Col. Indian Medical Service, Milburn Tower, Gogar, Corstorphine, Edinburgh	
1908		* Littlejohn, Henry Harvey, M.A., M.B., B.Sc., F.R.C.S.E., Professor of Forensic Medicine, and late Dean of the Faculty of Medicine in the University of Edinburgh, 11 Rutland Street, Edinburgh 340	

Date of Election.			Service on Council, etc.
1898		Lothian, Alexander Veitch, M.A., B.Sc. (Abroad: no communications to be sent)	
1923	C.	* Ludlam, E. B., B.A., D.Sc., Lecturer in Chemistry in the University of Edinburgh	
1900		† Lusk, Graham, M.A., Ph.D., LL.D., Professor of Physiology, Cornell University Medical College, New York, N.Y., U.S.A.	
1923		* Lyford-Pike, James, M.A., B.Sc., Lecturer in Forestry in the University of Edinburgh. Rosetta, Liberton, Edinburgh	
1924		* Lyon, David Murray, M.D., Ch.B., F.R.C.P.E., D.P.H., Professor of Therapeutics and Clinical Medicine in the University of Edinburgh. Dreghorn Loan, Colinton 345	
1894		Mabbott, Walter John, M.A., Rector of County High School, Duns, Berwickshire	
1887		M'Aldowie, Alexander M., M.D., 8 Halland Road, Cheltenham	
1917		* Macalister, Sir Donald, Bt., K.C.B., M.D., M.A., B.Sc., Principal of the University of Glasgow, The University, Glasgow	
1907		MacAlister, Donald Alexander, A.R.S.M., F.G.S., 10 St Alban's Road, Kensington, London, W. 8	
1921		* M'Arthur, Neil, M.A., B.Sc., Lecturer in Mathematics, Glasgow University, 1 Holyrood Crescent, Glasgow 350	
1883		M'Bride, P., M.D., F.R.C.P.E., 20 South Drive, Harrogate	
1903		M'Cormick, Sir W. S., M.A., LL.D., Chairman of the Advisory Council, Department of Scientific and Industrial Research, 16–18 Old Queen Street, Westminster, S.W. 1	1910–13.
1923		* M'Cracken, William, J.P., F.S.I., F.H.A.S., Land Agent and Technical Adviser, Englesea House, Crewe	
1918		* M'Culloch, Rev. James David, 43 Brougham Street, Greenock	
1905		* Macdonald, Hector Munro, O.B.E., M.A., F.R.S., Professor of Mathematics, University of Aberdeen, 52 College Bounds, Aberdeen 355	1908–11.
1897	C.	Macdonald, James A., M.A., B.Sc., H.M. Inspector of Schools, Meldon, Southside Road, Inverness	
1904		* Macdonald, John A., M.A., B.Sc., Commercial High School, Johannesburg, Transvaal	
1920		* M'Donald, Stuart, M.A., M.D., F.R.C.P.E., Professor of Pathology, University of Durham. College of Medicine, Newcastle-on-Tyne	
1904		Macdonald, William, M.S.Agr., Sc.D., Ph.D., D.Sc., Editor, *Agricultural Journal* of South Africa, Rand Club, Johannesburg, Transvaal	
1886		Macdonald, William J., M.A., LL.D., 15 Coniston Drive, Edinburgh 360	
1901	C.	MacDougall, R. Stewart, M.A., D.Sc., Professor of Biology, Royal Veterinary College, Edinburgh, 9 Dryden Place, Edinburgh	1914–17.
1910		Macewen, Hugh Allen, O.B.E., M.B., Ch.B., D.P.H. (Lond. and Camb.), Local Government Board, Ministry of Health, Whitehall, London, S.W.	
1888	C.	M'Fadyean, Sir John, Kt., M.B., B.Sc., LL.D., Principal, and Professor of Comparative Pathology in the Royal Veterinary College, Camden Town, London	
1885	C.	† Macfarlane, J. M., D.Sc., LL.D., Emeritus Professor of Botany, 4320 Osage Avenue, Philadelphia, Pennsylvania, U.S.A.	
1897		MacGillivray, Angus, C.M., M.D., D.Sc., F.S.A. (Scot.), 23 South Tay Street, Dundee 365	
1878		M'Gowan, George, F.I.C., Ph.D., 21 Montpelier Road, Ealing, London, W. 5	
1922		* Macgregor, Murray, M.A., B.Sc., Assistant Director (Scotland), H.M. Geological Survey, 33 George Square, Edinburgh	
1903		* M'Intosh, Donald C., M.A., D.Sc., Education Offices, Elgin	
1911		M'Intosh, John William, A.R.C.V.S., Dollis Hill Farm, Cricklewood, London, N.W. 2	
1869	C. N.	† M'Intosh, William Carmichael, M.D., LL.D., D.Sc., F.R.S., F.L.S., Emeritus Professor of Natural History in the University of St Andrews, Pres. Ray Society, 2 Abbotsford Crescent, St Andrews 370	1885–88.
1895	C.	Macintyre, John, M.D., LL.D., 179 Bath Street, Glasgow	
1912	C.	M'Kendrick, Anderson Gray, M.B., Lt.-Col., Indian Medical Service, Superintendent, Research Laboratory, Royal College of Physicians, Edinburgh	1924–
1914		* M'Kendrick, Archibald, F.R.C.S.E., D.P.H., L.D.S., 12 Rothesay Place, Edinburgh	
1873	C. B.	M'Kendrick, John G., M.D., F.R.C.P.E., LL.D., F.R.S., Emeritus Professor of Physiology in the University of Glasgow.	1875–78, 1885–88, 1893–94, 1900–02. V-P 1894–1900.
1900	C.	M'Kendrick, John Souttar, M.D., F.R.F.P.S. (Glas.), 2 Buckingham Terrace, Hillhead, Glasgow 375	

Date of Election.			Service on Council, etc.
1910	C.	* Mackenzie, Alister, M.A., M.D., D.P.H., Principal, College of Hygiene and Physical Training, Dunfermline	
1916	C.	* Mackenzie, John E., D.Sc., Reader in Chemistry, University of Edinburgh, Major-Adjutant, O.T.C., 2A Ramsay Garden, Edinburgh	
1894		Mackenzie, Robert, M.D., Napier, Nairn	
1904	C.	* Mackenzie, Sir W. Leslie, M.A., M.D., D.P.H., LL.D., Medical Member of the Scottish Board of Health, 14 Belgrave Place, Edinburgh	
1918		* Mackie, Wm., M.A., M.D., LL.D., D.P.H., 40 Montgomery Street, Glasgow, N.W. 380	
1910		* MacKinnon, James, M.A., Ph.D., Professor of Ecclesiastical History, Edinburgh University. 12 Lygon Road, Edinburgh	
1904		* Mackintosh, Donald James, C.B., M.V.O., M.B., C.M., LL.D., Supt. Western Infirmary, Glasgow	
1899		Maclean, Sir Ewan John, M.D., M.R.C.P. Lond., J.P., Professor of Obstetrics and Gynæcology, Welsh National School of Medicine, 12 Park Place, Cardiff	
188	C.	Maclean, Magnus, M.A., D.Sc., LL.D., M.Inst.C.E., M.I.E.E., formerly Professor of Electrical Engineering in the Royal Technical College, 51 Kersland Terrace, Hillhead, Glasgow	1916–19.
1913		* M'Lellan, Dugald, M.Inst.C.E., Divisional Engineer, L.M. and S. Railway, Buchanan Street Station, Glasgow. 1 Queen's Gate, Dowanhill, Glasgow 385	
1916	C.	* M'Lintock, W. F. P., D.Sc. (Edin.), Museum of Practical Geology, 28 Jermyn Street, London, S.W. 1	
1923		* Macmillan, Rt. Hon. H. P., K.C., LL.D., 44 Grosvenor Road, Westminster, S.W. 1.	
1907	C.	* Macnair, Professor Peter, Curator of the Natural History Collections in the Glasgow Museums, Kelvingrove Museum, Glasgow	
1917		* Macpherson, Rev. Hector Copland, M.A., Ph.D., F.R.A.S., Guthrie Memorial U.F. Church. 7 Wardie Crescent, Edinburgh	
1921		* M'Quistan, Dougald Black, M.A., B.Sc., Associate-Professor of Natural Philosophy, Royal Technical College, Glasgow. 29 Viewpark Drive, Rutherglen 390	
1921	C.	* MacRobert, Thomas Murray, M.A., D.Sc., Lecturer in Mathematics in the University of Glasgow. 6 Lothian Gardens, Kelvinside, N., Glasgow	
1921	C.	* M'Whan, John, M.A. (Glasgow), Ph.D. (Gött.), Lecturer in Mathematics in the University of Glasgow. 59 Greenlees Road, Cambuslang	
1898	C.	† Mahâlanobis, S. C., B.Sc., Professor of Physiology, Presidency College, Calcutta, India	
1913		Majumdar, Tarak Nath, D.P.H. (Cal.), L.M.S., F.C.S., Health Officer, Calcutta, IV, 11 Belvedere Road, Calcutta, India	
1917		* Malcolm, L. W. Gordon, M.Sc. (Cantab.), F.R.G.S., F.R.S.G.S., Capt. R.G.A. (R.O.), Wellcome Historical Medical Museum, 54A Wigmore Street, London, W. 1 395	
1908		Mallik, Devendranath, Sc.D., B.A., Professor of Mathematics, Astronomical Observatory, Presidency College, Calcutta, India	
1925		* Malloch, James, M.A., J.P., F.S.A.(Scot.), Executive Officer to the National Committee for the Training of Teachers. West Croft, Cramond Bridge, Edinburgh	
1912		Maloney, William Joseph, M.D. (Edin.), Professor of Neurology at Fordham University, New York City, N.Y., U.S.A.	
1913		Marchant, Rev. Sir James, K.B.E., LL.D., F.R.A.S., F.L.S., Director, National Council for Promotion of Race-Regeneration. Rhondda House, 60 Gower Street, London, W.C. 1	
1909	C. B.	* Marshall, C. R., M.D., M.A., LL.D., Professor of Materia Medica, Marischal College, Aberdeen 400	1915–18.
1882	C.	Marshall, D. H., M.A., Em.-Professor of Physics, Queen's University, Elmtree House, Union Street, W., Kingston, Ontario, Canada	
1901	C.	Marshall, F. H. A., Sc.D., F.R.S., Lecturer on Agricultural Physiology in the University of Cambridge, Christ's College, Cambridge	
1920	C.	* Marshall, John, M.A., D.Sc. (St Andrews), B.A. (Cantab.), University Reader in Mathematics, Bedford College, Regent's Park, London, N.W. 1. Logan House, Torrington Park, North Finchley, London, N. 12	
1885		Masson, George Henry, M.D., D.Sc., M.R.C.P.E., Port of Spain, Trinidad, British West Indies	
1913	C.	Masson, Sir David Orme, K.B.E., M.A., D.Sc., LL.D., F.R.S., Professor of Chemistry in the University of Melbourne 405	
1898	C. B.	Masterman, Arthur Thomas, M.A., D.Sc., F.R.S., formerly Superintending Inspector, H.M. Board of Agriculture and Fisheries. Royal Auto Club, Pall Mall, London, S.W. 3 Kedale Road, Seaford	1902–04.
1911		Mathews, Gregory Macalister, F.L.S., F.Z.S., Foulis Court, Fair Oaks, Hants	

Date of Election.			Service on Council, etc.
1921		* Mathieson, John, F.R.S.G.S., late Division Superintendent, Ordnance Survey (retired), 42 East Claremont Street, Edinburgh	
1906		* Mathieson, Robert, F.C.S., St Serf's, Innerleithen	
1902		Matthews, Ernest Romney, A.M.Inst.C.E., F.G.S., Chadwick Professor of Municipal Engineering in the University of London, University College, Gower Street, London, W.C. 410	
1924		* Matthews, James Robert, M.A., F.L.S., Lecturer in Botany in the University of Edinburgh. Royal Botanic Garden, Edinburgh	
1917		* Maylard, A. Ernest, M.B., B.Sc. (Lond.), F.R.F.P.S. (Glasgow), Kingsmuir, Peebles	
1922		* Meakins, Jonathan Campbell, M.D., F.R.C.P.Ed., Professor of Medicine and Director of the Department of Medicine, M'Gill University, Montreal, Canada	
1901	C.	Menzies, Alan W. C., M.A., B.Sc., Ph.D., F.C.S., Professor of Chemistry, Princeton University, Princeton, New Jersey, U.S.A.	
1917		* Merson, George Fowlie, Manufacturing Technical Chemist, 9 Hampton Terrace, Edinburgh 415	
1888		Methven, Cathcart W., M.Inst.C.E., F.R.I.B.A., 194 King Edward Avenue, Scottsville, Pietermaritzburg, Natal, S. Africa	
1902	C.	† Metzler, William H., A.B., Ph.D., Corresponding Fellow of the Royal Society of Canada, Dean of the New York State College for Teachers, Albany, N.Y., U.S.A.	
1885	C. B.	Mill, Hugh Robert, D.Sc., LL.D., Hill Crest, Dormans Park, E. Grinstead	
1908		* Miller, Alexander Cameron, M.D., F.S.A. Scot., Craig Linnhe, Fort-William, Inverness-shire	
1910		* Miller, John, M.A., D.Sc., Professor of Mathematics, Royal Technical College, 2 Northbank Terrace, North Kelvinside, Glasgow 420	
1909		Mills, Bernard Langley, M.D., F.R.C.S.E., M.R.C.S., D.P.H., Lt.-Col. R.A.M.C., formerly Army Specialist in Hygiene, c/o National Provincial Bank, Fargate, Sheffield	
1905		* Milne, Archibald, M.A., D.Sc., Deputy Director of Studies, Edinburgh Provincial Training College. 38 Morningside Grove, Edinburgh	
1905		* Milne, C. H., M.A., Head Master, Daniel Stewart's College, 19 Merchiston Gardens, Edinburgh	
1904	C.	* Milne, James Robert, D.Sc., Lecturer in Natural Philosophy, University of Edinburgh. Jesmond, The Loan, Colinton, Edinburgh	
1886		Milne, William, M.A., B.Sc., 70 Beechgrove Terrace, Aberdeen 425	
1899		Milroy, T. H., M.D., B.Sc., Professor of Physiology in Queen's University, Belfast	
1889	C.	Mitchell, A. Crichton, D.Sc., Hon. Doc. Sc. (Genève), formerly Director of Public Instruction in Travancore, India (CURATOR OF LIBRARY AND MUSEUM), 246 Ferry Road, Edinburgh	1915-16. Cur. 1916-
1897		† Mitchell, George Arthur, M.A., 9 Lowther Terrace, Kelvinside, Glasgow	
1900		Mitchell, James, M.A., B.Sc., Monydrain, Lochgilphead	
1911		Modi, Edalji Manekji, D.Sc., LL.D., Litt.D., F.C.S., etc., Proprietor and Director of Arthur Road Chemical Works, Meher Buildings, Tardeo, Bombay, India 430	
1906	C.	† Moffat, Rev. Alexander, M.A., LL.D., Professor of Physics, Christian College, Madras. Dalmeny, Cathedral P.O., Madras, India	
1890	C.	† Mond, R. L., M.A. Cantab., F.C.S., Combe Bank, near Sevenoaks, Kent	
1887	C.	Moos, N. A. F., D.Sc., L.C.E., J.P., Director of Bombay and Alibag Observatories (retired), Gowalia, Tank Road, Bombay, India	
1896		Morgan, Alexander, M.A., D.Sc., formerly Principal, Edinburgh Provincial Training College, 1 Midmar Gardens, Edinburgh	
1919		* Morris, Robert Owen, M.A., M.D., C.M. (Edin.), D.P.H. (Liverpool), Tuberculosis Institute, Newtown, N. Wales 435	
1892	C.	Morrison, J. T., M.A., B.Sc., Professor of Mathematical Physics, University, Stellenbosch, Cape Colony	
1914		Mort, Spencer, M.D., Ch.B., Ch.M., F.R.C.S.E., Lieut.-Col., R.A.M.C., North Middlesex Hospital, Upper Edmonton, London, N. 18	
1901		Moses, O. St John, M.D., D.Sc., F.R.C.S., Lt.-Col. I.M.S., Professor of Medical Jurisprudence, c/o Messrs King, Hamilton & Co., 4 and 5 Koila Ghat Street, Calcutta, India	
1892	C. K.	Mossman, R. C., Oficina Meteorologica, Paseo Colon, 974, Buenos Aires	
1916		* Muir, Robert, M.A., M.D., Sc.D., LL.D., F.R.S., Professor of Pathology, University of Glasgow. 16 Victoria Crescent, Dowanhill, Glasgow 440	
1874	C. K. V. J.	Muir, Sir Thomas, C.M.G., M.A., D.Sc., LL.D., F.R.S., lately Superintendent-General of Education for Cape Colony, Elmcote, Sandown Road, Rondebosch, South Africa	1885-88. V-P 1888-91.

Date of Election.			Service on Council, etc.
1888	C.	Muirhead, George, LL.D., Commissioner to His Grace the Duke of Richmond and Gordon, K.G., Speybank, Fochabers	
1907		Muirhead, James M. P., J.P., F.R.S.L., F.S.S., P.O. Box 82, Queenstown, Cape Province, South Africa	
1887		Mukhopâdhyay, Asûtosh, M.A., LL.D., F.R.A.S., M.R.I.A., Professor of Mathematics at the Indian Association for the Cultivation of Science, 77 Russa Road North, Bhowanipore, Calcutta, India	
1894 & 1921		* Munro, J. M. M., M.Inst.E.E., Assoc.M.Inst.C.E., Consulting Electrician and Engineer, 11 Randolph Place, Edinburgh 445	
1925		* Munro-Ferguson, Ronald Craufurd, The Right Hon. The Viscount Novar, P.C., G.C.M.G., LL.D., Raith, Kirkcaldy	
1896		Murray, Alfred A., M.A., LL.B.. 20 Warriston Crescent, Edinburgh	
1923		* Murray, The Right Hon. Charles David, Lord Murray, P.C., C.M.G., K.C., M.A., LL.D., 62 Great King Street, Edinburgh	
1907		† Musgrove, James, M.D., F.R.C.S. Edin. and Eng., LL.D., Emeritus-Professor of Anatomy, University of St Andrews, The Swallowgate, St Andrews	
1888		Napier, A. D. Leith, M.D., C.M., M.R.C.P., Hampton Street, Hawthorne, Unley, S. Australia 450	
1897		Nash, Hon. Alfred George, M.L.C., B.Sc., F.R.G.S., C.E., Belretiro, Mandeville, Jamaica, W.I.	
1924		* Nelson, Philip, M.A., M.D., Ch.B., F.S.A., Beechwood, Calderstones, Liverpool	
1898		Newman, Sir George, K.C.B., M.D., D.C.L., F.R.C.P., Chief Medical Officer of the Ministry of Health and the Board of Education, Whitehall, S.W. 1	
1884		Nicholson, J. Shield, M.A., D.Sc., formerly Professor of Political Economy in the University of Edinburgh, 3 Belford Park, Edinburgh	1885-87, 1892-95, 1897-1900.
1880	C.	Nicol, W. W. J., M.A., D.Sc., 15 Blacket Place, Edinburgh 455	
1925		* Ogg, William Gammie, M.A., B.Sc., Ph.D., Advisory Officer in Soil Chemistry, Edinburgh and East of Scotland College of Agriculture, 13 George Square, Edinburgh	
1923	C.	* Ogilvie, Alan G., M.A., B.Sc. (Oxon.), Lecturer in Geography in the University of Edinburgh. 12 Ramsay Garden, Edinburgh	
1888		† Ogilvie, Sir F. Grant, Kt., C.B., M.A., B.Sc., LL.D., Principal Assistant Secretary, Department of Scientific and Industrial Research, Dewdney, Shere, Guildford	1901-03.
1886		Oliver, James, M.D., F.L.S., Physician to the London Hospital for Women, 123 Harley Street, London, W.	
1895	C.	Oliver, Sir Thomas, Kt., M.D., LL.D., F.R.C.P., Professor of Physiology in the University of Durham, 7 Ellison Place, Newcastle-upon-Tyne 460	
1924		* Orr, John Boyd, D.S.O., M.C., M.A., D.Sc., M.D., Director of Rowett Research Institute for Research in Animal Nutrition. Research Lecturer in Physiology of Nutrition in the University of Aberdeen	
1915		* Orr, Lewis P., F.F.A., Manager of Scottish Life Assurance Co., 19 St Andrew Square, Edinburgh, 3 Belgrave Place, Edinburgh	
1908		Page, William Davidge, F.C.S., F.G.S., M.Inst.M.E., 10 Clifton Dale, York	
1905		Pallin, Lt.-Col. William Alfred, C.B.E., D.S.O., F.R.C.V.S., Headquarters, Southern Command, India. c/o Glyn, Mills & Co., Holts Branch, 3 Whitehall Place, London, S.W. 1	
1924		* Parker, Joseph, D.Sc., Principal, Fife Mining School, Cowdenbeath 465	
1901		Paterson, David, F.C.S., Leewood, Rosslyn Castle, Midlothian	
1918		* Paterson, Rev. William Paterson, D.D., LL.D., Professor of Divinity, University, Edinburgh. 3 Royal Terrace, Edinburgh	
1886	C.	Paton, D. Noël, M.D., B.Sc., LL.D., F.R.C.P.E.. F.R.S., Professor of Physiology in the University of Glasgow	1894-97, 1904-06, 1909-12. V.P 1918-21.
1919	C.	* Patterson, Thomas Stewart, D.Sc. (London and Glasgow), Ph.D. (Heidelberg), Professor of Organic Chemistry in the University of Glasgow. 10 Oakfield Terrace, Hillhead, Glasgow	
1892		Paulin, Sir David, Actuary, 6 Forres Street, Edinburgh 470	
1881	C. N.	Peach, Benjamin N., LL.D., F.R.S., F.G.S., formerly District Superintendent and Acting Palæontologist of the Geological Survey of Scotland, c/o Wm. Marshall, Esq., 33 Comiston Drive, Edinburgh	1905-08, 1911-12. V.P 1912-17.

Date of Election.			Service on Council, etc.
1923		* Peacock, Alexander David, M.Sc., Lecturer in Zoology, Armstrong College, University of Durham. 6 Prudhoe Terrace, Tynemouth	
1907		* Pearce, John Thomson, B.A., B.Sc., School House, Tranent	
1914		Pearson, Joseph, D.Sc., F.L.S., Director of the Colombo Museum, and Marine Biologist to the Ceylon Government, Colombo Museum, Ceylon	
1904		* Peck, James Wallace, C.B., M.A., Scottish Board of Health. 125 George Street, Edinburgh 475	
1887	C. B.	Peddie, Wm., D.Sc., Professor of Natural Philosophy in University College, Dundee. The Weisha, Ninewells, Dundee	1904-07, 1908-11. V-P 1919-22.
1925		* Penman, David, D.Sc., M.Inst.M.E., Principal Dhanbad School of Mines, India; recently H.M. Acting-Chief Inspector of Mines, India. Mines Department, Dhanbad, East Indian Railway, India	
1893		Perkin, Arthur George, F.R.S., F.I.C., Grosvenor Lodge, Grosvenor Road, Leeds	
1913	C.	† Philip, Alexander, M.A., LL.B., Writer, The Mary Acre, Brechin	
1889		† Philip, Sir R. W., Kt., M.A., M.D., LL.D., F.R.C.P.E., Professor of Tuberculosis, University of Edinburgh. 45 Charlotte Square, Edinburgh 480	
1907	C.	† Phillips, Major Charles E. S., O.B.E., Castle House, Shooters Hill, Woolwich, S.E. 18	
1914		* Pilkington, Basil Alexander, "Kambla," Davidson's Mains	
1905		* Pinkerton, Peter, M.A., D.Sc., Rector, High School, Glasgow. 7 Park Quadrant, Glasgow, C. 3	
1908	C.	* Pirie, James Hunter Harvey, B.Sc., M.D., F.R.C.P.E., Superintendent of the Routine Division of The South African Institute for Medical Research, P.O. Box 1038, Johannesburg, South Africa	
1911		* Pirie, James Simpson, M.Inst.C.E., 25 Grange Road, Edinburgh 485	
1906		Pitchford, Herbert Watkins, C.M.G., F.R.C.V.S., Lt.-Col., Oaklands Drive, Weybridge, Surrey	
1921		* Pollard, Sir George Herbert, K.B., M.D., C.M. (Edin.), Barrister-at-Law, Inner Temple. 79 Albert Road, Southport	
1924		* Ponder, Eric, M.B., Ch B., Lecturer in Physiology and Biochemistry in the University of Edinburgh	
1919		* Porritt, B. D., D.Sc. (Lond.), F.I.C., Research Association of British Rubber and Tyre Manufacturers, 105-7 Lansdowne Road, Croydon, London	
1888		† Prain, Sir David, Lt.-Col., Indian Medical Service (retired), C.M.G., C.I.E., M.A., M.B., LL.D., F.R.S., F.L.S., For. Memb. K. Svensk. Vetensk. Akad.; Hon. Memb. Soc. Lett. ed Arti d. Zelanti, Acireale; Pharm. Soc. Gt. Britain; Corr. Memb. Bayer Akad. Wiss., etc.; formerly Director, Royal Botanic Gardens, Kew, Surrey. The Well Farm, Warlingham, Surrey 490	
1902		* Preller, Charles du Riche, M.A., Ph.D., A.M.Inst.C.E., M.I.E.E., F.G.S., 61 Melville Street, Edinburgh	
1892	C.	Pressland, Arthur J., M.A. Camb., Edinburgh Academy	
1915		Price, Frederick William, M.D., M.R.C.P. Edin., Physician to the Great Northern Hospital, London. 133 Harley Street, London, W.	
1903		* Pullar, Laurence, Dunbarney, Bridge of Earn, Perthshire	
1911		Purdy, John Smith, D.S.O., M.D., C.M. (Aberd.), D.P.H. (Camb.), F.R.G.S., Town Hall, Sydney, N.S.W., Australia 495	
1920		* Purser, George Leslie, M.A. (Cantab.), F.Z.S., Lecturer in the Natural History Department of the University of Edinburgh. c/o Muir, 21 Buccleuch Place, Edinburgh	
1898		Purves, John Archibald, D.Sc., "Thorpe Hall," Eaton Rise, Ealing, London, W. 5	
1899	C.	Ramage, Alexander G., Marchfield, Davidson's Mains, Midlothian	
1914		* Ramsay, Peter, M.A., B.Sc., Head Mathematical Master, George Watson's College, 63 Comiston Drive, Edinburgh	
1911		* Rankin, Adam A., British Astronomical Association, West of Scotland Branch, 24 Woodend Drive, Jordanhill, Glasgow 500	
1904		Ratcliffe, Joseph Riley, M.B., C.M., c/o The Librarian, The University, Birmingham	
1900		Raw, Nathan, C.M.G., M.D., Ashfields, Loughton, Essex	
1924		* Read, John, M.A., B.Sc., Ph.D., Professor of Chemistry in the University of St Andrews	
1883	C.	Readman, J. B., D.Sc., F.C.S., Frankleigh House, Bradford-on-Avon, Wilts	
1902		Rees-Roberts, John Vernon, M.D., D.Sc., D.P.H., 11 Oak Hill Park, Hampstead, London, N.W. 3 505	

Date of Election.			Service on Council, etc.
1902		Reid, Sir George Archdall O'Brien, K.B.E., M.B., C.M., 20 Lennox Road South, Southsea, Hants	
1913		Reid, Harry Avery, O.B.E., F.R.C.V.S., D.V.H., Bacteriologist and Pathologist, Department of Agriculture, Wellington, New Zealand	
1924		* Reid, William Carstairs, Civil Engineer. 23 Saxe-Coburg Place, Edinburgh	
1908	C.	* Rennie, John, D.Sc., Lecturer on Parasitology and Experimental Zoology, University of Aberdeen, 60 Desswood Place, Aberdeen	
1914		Renshaw, Graham, M.D., M.R.C.S., L.R.C.P., L.S.A., Editor of the *Avicultural Magazine*, Sale Bridge House, Sale, Manchester 510	
1913		* Richardson, Major Harry, M.Inst.E.E., M.Inst.M.E., 16 Stratford Place, London, W. 1	
1908		Richardson, Linsdall, F.G.S., 10 Oxford Parade, Cheltenham, Glos.	
1875		Richardson, Ralph, W.S., 2 Parliament Square, Edinburgh	
1916	C.	* Ritchie, James, M.A., D.Sc., Keeper of the Natural History Department in the Royal Scottish Museum, 20 Upper Gray Street, Edinburgh	1921-24.
1914	C.	* Ritchie, James Bonnyman, D.Sc., Headmaster, Academy, Forres 515	
1906	C.	* Ritchie, William Thomas, M.D., F.R.C.P.E., 10 Douglas Crescent, Edinburgh	
1898	C.	Roberts, Alexander William, D.Sc., F.R.A.S., Lovedale, South Africa	
1919		* Roberts, Alfred Henry, O.B.E., M.Inst.C.E., Superintendent and Engineer, Leith Docks, 2 Cargil Terrace, Trinity, Edinburgh	
1902	C.	* Robertson, Robert A., M.A., B.Sc., Reader in Botany in the University of St Andrews	
1919		* Robertson, William Alexander, F.F.A., Century Insurance Co., Ltd., 18 Charlotte Square. Kilwarlin, Barnshot Road, Colinton 520	
1896	C.	Robertson, W. G. Aitchison, D.Sc., M.D., F.R.C.P.E., St Margarets, Boscombe Manor Estate, Bournemouth	
1910	C.	* Robinson, Arthur, M.D., M.R.C.S., Professor of Anatomy, University of Edinburgh. 35 Coates Gardens, Edinburgh	1910-12. Sec. 1912-18 V-P 1918-21.
1925		* Robinson, Harold, D.Sc., Ph.D., Reader in Physics in the University of Edinburgh. 25 Nile Grove, Edinburgh	
1916		* Ronald, David, M.Inst.C.E., Chief Engineer, Scottish Board of Health, 125 George Street, Edinburgh	
1881		Rosebery, The Right Hon. the Earl of, K.G., K.T., LL.D., D.C.L., F.R.S., Dalmeny Park, Linlithgowshire 525	
1909	C.	* Ross, Alex. David, M.A., D.Sc., F.R.A.S., Professor of Mathematics and Physics, University of Western Australia, Perth, Western Australia	
1921		* Ross, Edward Burns, M.A. (Edin. and Camb.), Professor of Mathematics in the Madras Christian College, Madras	
1906		* Russell, Alexander Durie, B.Sc., Mathematical Master, Falkirk High School. 14 Heugh Street, Falkirk	
1902	C. K.	* Russell, James, 22 Glenorchy Terrace, Edinburgh	
1925	C.	* Saddler, William, M.A., B.A., Lecturer in Mathematics and Applied Mathematics, and Assistant to the Professor in the University of St Andrews. 119 South Street, St Andrews 530	
1906		Saleeby, Caleb Williams, M.D., 13 Greville Place, Hampstead, London, N.W. 6	
1916	C.	* Salvesen, The Rt. Hon. Lord, P.C., K.C., Judge of the Court of Session (Retired), Dean Park House, Edinburgh	1920-22. V-P 1922-25.
1914		* Salvesen, Theodore Emile, 37 Inverleith Place, Edinburgh	
1912	C. K.	* Sampson, Ralph Allen, M.A., D.Sc., F.R.S. (GENERAL SECRETARY), Astronomer Royal for Scotland, Professor of Astronomy, University, Edinburgh. Royal Observatory, Edinburgh	1912-15, 1919-21. V-P 1915-18. Sec. 1922-23. Gen. Sec. 1923-
1903		* Samuel, Sir John S., K.B.E., D.L., J.P., F.S.A. (Scot.), 13 Park Circus, Glasgow, W. 535	
1922		* Sarkar, Bijali Behari, M.Sc., D.Sc. (Edin.), Assistant to the Professor of Physiology, University, Calcutta. 33/1c Lansdowne Road, Calcutta	
1903		* Sarolea, Charles, Ph.D., D.Litt., Professor of French, University of Edinburgh. 21 Royal Terrace, Edinburgh	

Date of Election.			Service on Council, etc
1924		*Scot-Skirving, Archibald Adam, C.M.G., M.B., C.M., F.R.C.S., M.R.C.S., Surgeon, Royal Infirmary, Edinburgh, Senior Lecturer in Clinical Surgery in the University of Edinburgh. 6 Rothesay Place, Edinburgh	
1885	C.	Scott, Alexander, M.A., D.Sc., F.R.S., 34 Upper Hamilton Terrace, London, N.W.	
1919		*Scott, Alexander, M.A., D.Sc., Chief Assistant to Principal, Pottery Laboratory, Stoke-on-Trent 540	
1919		*Scott, Alexander Ritchie, B.Sc. (Edin.), D.Sc. (Lond.), Principal London County Council, Beaufoy Institute, Prince's Road, Vauxhall Street, London, S.E. 11	
1917		*Scott, Henry Harold, M.D., M.R.C.P., L.R.C.P. (London), M.R.C.S. (Eng.), D.P.H., 51 New Cavendish Street, London, W. 1	
1900	C. N.	Sharpey-Schafer, Sir Edward Albert, M.D., LL.D., D.Sc., F.R.S., Corresponding Member of the French Academy of Medicine, Professor of Physiology in the University of Edinburgh	1900–03, 1906–09, 1918–19. V-P 1913–17.
1908		*Simpson, George Freeland Barbour, M.D., F.R.C.P.E., F.R.C.S.E., 43 Manor Place, Edinburgh	
1900	C.	Simpson, James Young, M.A., D.Sc., Professor of Natural Science in the New College, Edinburgh. 25 Chester Street, Edinburgh 545	1922–25.
1911	C.	Simpson, Sutherland, M.D., D.Sc. (Edin.), Professor of Physiology, Medical College, Cornell University, Ithaca, N.Y., U.S.A. 118 Eddy Street, Ithaca, N.Y., U.S.A.	
1900		Sinhjee, Sir Bhagvat, G.C.I.E., M.D., LL.D. Edin., H.H. the Thakur Sahib of Gondal, Gondal, Kathiawar, Bombay, India	
1903		*Skinner, Robert Taylor, M.A., J.P., Head Master, Donaldson's Hospital, Edinburgh	
1901		Smart, Edward, B.A., B.Sc., Tillyloss, Tullylumb Terrace, Perth	
1920		*Smellie, William Robert, M.A., D.Sc., Geologist on the Staff of the Anglo-Persian Oil Company, Ardene, Mossend, Lanarkshire 550	
1915		*Smith, James Lorrain, M.A., M.D., LL.D., F.R.S., Professor of Pathology, University of Edinburgh. 9 Carlton Terrace, Edinburgh	1918-21.
1921		*Smith, The Right Hon. James Parker, P.C., M.A. (Camb.), Ryvra, North Berwick	
1921		*Smith, Norman Kemp, M.A., D.Phil., Professor of Logic and Metaphysics, University of Edinburgh. Ellerton, Grange Loan, Edinburgh	
1923		*Smith, Percy James Lancelot, M.A. (Oxon.), F.I.C., F.C.S., Science Master, Loretto School. 47 Dalrymple Loan, Musselburgh	
1911		*Smith, Stephen, B.Sc., Engineer, 31 Grange Loan, Edinburgh 555	
1907	C.	†Smith, William Ramsay, D.Sc., M.D., C.M., Permanent Head of the Health Department, South Australia, Belair, South Australia	
1880		Smith, Sir William (Robert), M.D., D.Sc., LL.D., Principal of The Royal Institute of Public Health, Em.-Professor of Forensic Medicine and Toxicology in King's College, University of London. 36 Russell Square, London, W.C. 1	
1919		*Smith, William Wright, M.A. (Edin.) (SECRETARY), Regius Professor of Botany, University of Edinburgh, Regius Keeper of the Royal Botanic Garden, and King's Botanist in Scotland. Inverleith House, Edinburgh	Sec. 1923-
1899		Snell, Ernest Hugh, M.D., B.Sc., D.P.H. Camb., Medical Officer of Health, Coventry	
1880		Sollas, W. J., M.A., D.Sc., LL.D., F.R.S., Fellow of University College, Oxford, and Professor of Geology and Palæontology in the University of Oxford 560	
1910		*Somerville, Robert, B.Sc., 31 Cameron Street, Dunfermline	
1889		† Somerville, Wm., M.A., D.Sc., D.Oec., Sibthorpian Professor of Rural Economy and Fellow of St John's College in the University of Oxford. 121 Banbury Road, Oxford	
1911	C.	*Sommerville, Duncan M'Laren Young, M.A., D.Sc., Professor of Pure and Applied Mathematics, Victoria College, Wellington, New Zealand	
1896		Spence, Frank, M.A., B.Sc., 25 Craiglea Drive, Edinburgh	
1925		*Staig, Robert Arnot, M.A., Carnegie Teaching Fellow in the University of Glasgow. Glenlea, Lasswade, Midlothian 565	
1891		Stanfield, Richard, Professor of Mechanics and Engineering in the Heriot-Watt College, Edinburgh	
1923		*Stebbing, Edward Percy, M.A., Professor of Forestry in the University of Edinburgh. Hawthornden Castle, Lasswade	
1885 & 1915		*Steggall, John Edward Aloysius, M.A., Professor of Mathematics at University College, Dundee (St Andrews University). Woodend, Perth Road, Dundee	

Date of Election.			Service on Council, etc.
1923		* Stenhouse, Andrew G., F.G.S., 191 Newhaven Road, Leith	
1912	C. K.	† Stephenson, John, M.B., D.Sc. (Lond.), Lt.-Col. I.M.S., Zoological Department, University, Edinburgh 570	
1910		* Stephenson, Thomas, D.Sc., F.C.S., Editor of the *Prescriber*, 6 South Charlotte Street, Edinburgh	
1925		* Stevens, Alexander, M.A., B.Sc., Lecturer in Geography in the University of Glasgow	
1886	C.	Stevenson, Charles A., B.Sc., M.Inst.C.E., 28 Douglas Crescent, Edinburgh	
1884		Stevenson, David Alan, B.Sc., M.Inst.C.E., Troqueer, Slateford, Midlothian	
1919		* Stevenson, David Alan, B.Sc., M.Inst.C.E., 22 Glencairn Crescent, Edinburgh 575	
1925		* Stewart, David Smith, B.Sc., Assoc.M.Inst.C.E., Lecturer on Structural Engineering Drawing in the University of Edinburgh. 21 Nelson Street, Edinburgh	
1921		* Stewart, Ian Struthers, M.D. (Edin.), Nordrach-on-Dee, Banchory	
1924		* Stiles, Sir Harold Jalland, K.B.E., M.B., F.R.C.S.E., LL.D., formerly Professor of Clinical Surgery in the University of Edinburgh. Whatton Lodge, Gullane, E. Lothian	
1902		* Stockdale, Herbert Fitton, LL.D., Director of the Royal Technical College, Glasgow, Clairinch, Upper Helensburgh, Dumbartonshire	
1889	C.	Stockman, Ralph, M.D., F.R.C.P.E., F.F.P S.G., Professor of Materia Medica and Therapeutics in the University of Glasgow 580	1903-05.
1906		Story, Fraser ,formerly Professor of Forestry, University College, Bangor, North Wales. 4K Artillery Mansions, Victoria Street, London, S.W. 1	
1907		* Strong, John, C.B.E., M.A., LL.D., Professor of Education in the University of Leeds. Devonshire Hall, Headingley, Leeds	
1903		Sutherland, David W., M.D., M.R.C.P., C.I.E., Lt.-Col. I.M.S., Principal and Professor of Medicine, Medical College, Lahore, India	
1925		Sutton, Richard L., M.D., LL.D., Professor of Diseases of the Skin in the University of Kansas School of Medicine, U.S.A.	
1905		† Swithinbank, Commander Harold William, Denham Court, Denham, Bucks 585	
1912		* Syme, William Smith, M.D. (Edin.), 11 Lynedoch Crescent, Glasgow	
1917	C. N.	* Tait, John, D.Sc., M.D., Professor of Physiology, M'Gill University, Montreal, Canada	
1904		* Tait, John W., B.Sc., formerly Rector of Leith Academy, Netherby, Pitkeathly, Bridge of Earn	
1898	C.	Tait, William Archer, D.Sc., M.Inst.C.E., 72A George Street, Edinburgh (Society's Representative on George Heriot's Trust)	1914-17, 1918-21. V-P 1921-24.
1895		† Talmage, James Edward, D.Sc., Ph.D., F.R.M.S., F.G.S., formerly Professor of Geology, University of Utah. 47 East S. Temple Street, Salt Lake City, Utah, U.S.A. 590	
1890	C.	† Tanakadate, Aikitu, Hon. Professor of Natural Philosophy in the Imperial University of Japan. Koisikawa, Zôsigayamati. 144, Tokyo, Japan	
1870		Tatlock, Robert R., F.C.S., City Analyst's Office, 156 Bath Street, Glasgow	
1899		Taylor, James, M.A., Mathematical Master in the Edinburgh Academy	
1917	C.	* Taylor, William White, M.A., D.Sc., Lecturer on Chemical Physiology, University, Edinburgh. Park Villa, Liberton, Edinburgh	
1892		Thackwell, J. B., M.B., C.M., D.P.H., Carlton House, 1 Prince of Wales Road, Battersea Park, London, S.W. 595	
1885	C.	Thompson, D'Arcy W., C B., D.Litt., F.R.S., Professor of Natural History, University, St Andrews. 44 South Street, St Andrews	1892-95, 1896-99, 1907-10, 1912-15, 1922-25. V-P 1916-19.
1917	C. N.	* Thompson, John M'Lean, M.A., D.Sc., F.L.S., Professor of Botany, University of Liverpool	
1905		* Thoms, Alexander, 7 Playfair Terrace, St Andrews	
1887		Thomson, Andrew, M.A., D.Sc., F.I.C., 145 Bruntsfield Place, Edinburgh	
1923	C.	* Thomson, George Paget, M.A. (Cantab.), Professor of Natural Philosophy in the University of Aberdeen. 600	
1896		Thomson, George Ritchie, C.M.G., M.B., C.M., Professor of Surgery, University College, Johannesburg, Transvaal	
1903		Thomson, George S., F.C.S., Ferma Albion, Marculesci, Roumania	
1906		* Thomson, Gilbert, M.Inst.C.E., 164 Bath Street, Glasgow	

Date of Election.			Service on Council, etc.
1887	C.	Thomson, J. Arthur, M.A., LL.D., Regius Professor of Natural History in the University of Aberdeen	1906–08, 1920–23.
1906	C.	Thomson, James Stuart, M.Sc., Ph.D., Zoological Department, University, Manchester 605	
1880		Thomson, John Millar, LL.D., F.R.S., Em.-Professor of Chemistry in King's College, London. 6 Douro Place, Kensington, London, W. 8	
1899		Thomson, R. Tatlock, F.C.S., 156 Bath Street, Glasgow	
1912	C.	Thomson, Robert Black, M.B. Edin., Aliwal North, Cape Province, S.A.	
1870		Thomson, Spencer C., Actuary, 10 Eglinton Crescent, Edinburgh	
1882		Thomson, Sir Wm., Kt., M.A., B.Sc., LL.D.. Principal, University of the Witwatersrand, P.O. Box 1176, Johannesburg, South Africa 610	
1917		* Thorneycroft, Wallace, J.P.. Coal and Iron Master, Plean House, Plean, Stirling	
1920		* Todd, John Barber, B.Sc., A.M.I.Mech.E., Lecturer in Engineering in the University of Edinburgh. 38 Upper Gray Street, Edinburgh	
1917		* Tovey, Donald Francis, B.A., Professor of Music, University, Edinburgh. 18 Buccleuch Place, Edinburgh	
1914		† Tredgold, Alfred Frank, M.D. (Durham), M.R.C.P., Lecturer on Mental Deficiency at London University, and Bethlem Royal Hospital, "St Martins," Guildford	
1915		* Trotter, George Clark, M.D., Ch.B. Edin., D.P.H. (Aberdeen), Medical Officer of Health, Metropolitan Borough, Islington. Braemar, 17 Haslemere Road, Crouch End, London, N. 8 615	
1922	C.	* Turnbull, Herbert Westren, M.A., Professor of Mathematics in the University of St. Andrews. Fereday Fellow of St. John's College, Oxford. 2 Queens Terrace, St Andrews	
1905		* Turner, Arthur Logan, M.D., F.R.C.S.E., 27 Walker Street, Edinburgh	
1906	C.	Turner, Dawson F. D., B.A., M.D., F.R.C.P.E., M.R.C.P., Lecturer on Medical Physics, Surgeons' Hall, Physician in charge of Radium Treatment, Royal Infirmary, Edinburgh. 37 George Square, Edinburgh	
1925		* Turner, Harry Moreton Stanley, M.B.E., M.R.C.S., L.R.C.P., D.T.M., Chevalier de l'Ordre Royale du Sauveur de Grèce. Wing-Commander, Royal Air Force Medical Service, Medical Department, Headquarters, Royal Air Force (Middle East). Cairo, Egypt	
1924		* Turner, Richard, O.B.E., M.B., C.M., Hotel Hydropathic, Peebles 620	
1895		Turton, Albert H., M.I.M.M., 233 George Road, Erdington, Birmingham	
1918	C.	* Tyrrell, G. W., A.R.C.Sc., Ph.D., F.G.S., Chief Assistant and Lecturer in Petrology, Geological Department, University, Glasgow	
1924	C.	* Vaidyanathaswamy Ramaswamy, M.A., 31 Ramaswamy Street, George Town, Madras	
1910		Vincent, Swale, M.D. Lond., D.Sc. Edin., etc., Professor of Physiology in the University of London, Physiological Laboratory, Middlesex Hospital Medical School, Berners Street, London, W. 1	
1891	C. B.	Walker, Sir James, Kt., D.Sc., Ph.D., LL.D., F.R.S., Professor of Chemistry in the University of Edinburgh. 5 Wester Coates Road, Edinburgh 625	1903–05, 1910–13 1922–25. V.P 1916–19.
1925	C.	* Walker, Frederick, B.A., Ph.D., 5 Wester Coates Road, Edinburgh	
1902		* Wallace, Alexander G., M.A., 56 Fonthill Road, Aberdeen	
1886	C.	Wallace. R., M.A., LL.D., F.L.S., Em.-Professor of Agriculture and Rural Economy in the University of Edinburgh. 45 E. Claremont Street, Edinburgh	
1898		Wallace, Wm., M.A., Belvedere, Alberta, Canada	
1920		* Walmsley, Thomas, M.D. (Glasgow), Professor of Anatomy in the University of Belfast. Netherwood, Dumfries 630	
1923		* Warren, John Alexander, M.Inst.C.E., M.Cons.E. (Westminster). 74 Balshagray Avenue, Partick	
1901	C.	Waterston, David, M.A., M.D., F.R.C.S.E., Professor of Anatomy, University, St Andrews	1916–19. 1925–
1923		* Watson, H. Ferguson, M.D. (Glasg.), Deputy Commissioner of the General Board of Control for Scotland. Northcote, Edinburgh Road, Perth	
1911		* Watson, James A. S., M.C., B.Sc., etc., Professor of Agriculture in the University of Edinburgh. 30 Mayfield Terrace, Edinburgh	
1900		Watson, Thomas P., M.A., B.Sc., Principal, Trade and Grammar School, Keighley 635	
1923	C.	* Watson, William, M.A. (Edin.), B.Sc. (Edin.), Lecturer in Physics, Heriot-Watt College, Edinburgh. 25 Warrender Park Terrace, Edinburgh	
1907		* Watt, Andrew, M.A., Meteorological Office, 6 Drumsheugh Gardens. 6 Woodburn Terrace, Edinburgh	1912–14.

Date of Election.			Service on Council, etc.
1911		† Watt, James, W.S., F.F.A., LL.D., Craiglockhart House, Slateford, Edinburgh	1924-
1911		* Watt, Rev. Lauchlan Maclean, M.A., D.D., Minister of Glasgow Cathedral. 1 Athole Gardens, Glasgow	
1896		† Webster, John Clarence, B.A., M.D., F.R.C.P.E., Professor of Obstetrics and Gynæcology, Rush Medical College, Shediac, N.B., Canada 640	
1907	B. C.	* Wedderburn, Ernest Maclagan, M.A., LL.B., W.S., D.Sc., Professor of Conveyancing in the University of Edinburgh. 6 Succoth Gardens, Edinburgh	1913-16, 1921-24.
1903	B. C.	* Wedderburn, J. H. Maclagan, M.A., D.Sc., P.O. Box 53, Princeton, N.J., U.S.A.	
1904		Wedderspoon, William Gibson, M.A., LL.D., Indian Educational Service, Senior Inspector of Schools, Burma. The Education Office, Rangoon, Burma	
1896		Wenley, Robert Mark, M.A., D.Sc., D.Phil., Litt.D., LL.D., D.C.L., Professor of Philosophy in the University of Michigan, Ann Arbor, U.S.A. Temp. Address, American University Union, 50 Russell Square, London, W.C. 1	1925-26.
1916		* White, J. Martin, Esq., of Balruddery, Balruddery, near Dundee 645	
1896	C.	White, Philip J., M.B., Professor of Zoology in University College, Bangor, North Wales	
1911		* Whittaker, Charles Richard, F.R.C.S. (Edin.), F.S.A. (Scot.) Lynwood, Hatton Place, Edinburgh	
1912	C.	* Whittaker, Edmund Taylor, Sc.D., F.R.S. (VICE-PRESIDENT), Foreign Member of the R. Accademia dei Lincei, Rome, Professor of Mathematics in the University of Edinburgh. 35 George Square, Edinburgh	1912-15, 1922-25. Sec. 1916-22. V-P 1925-
1918		* Whyte, Rev. Charles, M.A., LL.D., F.R.A.S., U.F. Church Manse, Kingswells, Aberdeen	
1918		* Wight, John Thomas, M.I.Mech.E., General Manager, Messrs MacTaggart, Scott & Co., Ltd., Station Iron Works, Loanhead. Calderwood Villa, Lasswade 650	
1925		* Wilkie, David Percival Dalbreck, O.B.E., M.D., Ch.M., F.R.C.S., Professor of Surgery in the University of Edinburgh. 56 Manor Place, Edinburgh	
1924		* Williams, William Arthur, F.I.C., 1 Lennox Street, Edinburgh	
1908		* Williamson, Henry Charles, M.A., D.Sc., formerly Naturalist to the Fishery Board for Scotland, Marine Laboratory, Aberdeen. Temp. address, 13 Couper Street, Dundee	
1910	C.	* Williamson, William, F.L.S., 7 Ventnor Terrace, Edinburgh	
1900		Wilson, Alfred C., F.C.S., Bloomfield House, Sadberge, near Darlington 655	
1911		* Wilson, Andrew, M.Inst.C.E., 66 Netherby Road, Trinity, Edinburgh	
1902	V. J.	* Wilson, Charles T. R., M.A., LL.D., F.R.S., Jacksonian Professor of Natural Philosophy in the University of Cambridge. Glencorse, Storey's Way, Cambridge	
1922		* Wilson, John, F.R.I.B.A., Fellow of the Inst. of Scottish Architects. Principal Architect, Scottish Board of Health. 20 Lomond Road, Edinburgh	
1895		Wilson-Barker, Sir David, Kt., R.N.R., F.R.G.S., formerly Captain-Superintendent Thames Nautical Training College, H.M.S. "Worcester," off Greenhithe, Kent. 207 Coleherne Court, London, S.W. 5	
1925		* Wilson, Harold A., M.A., D.Sc., F.R.S., Professor of Natural Philosophy. The Rice Institute, Houston, Texas, U.S.A. 660	
1920		* Wilson, Malcolm, D.Sc. (London), A.R.C.Sc., F.L.S., Reader in Mycology and Bacteriology in the University of Edinburgh. Brent Knoll, Kinnear Road, Edinburgh	
1924		* Wilson, William, M.A., LL.B., Advocate, Regius Professor of Public Law in the University of Edinburgh. Viewbank, Kirkbrae, Liberton	
1922	C.	* Wordie, James Mann, M.A. (Camb.), B.Sc. (Glasg.), Lecturer in Oceanography, University of Cambridge. 52 Montgomery Drive, Glasgow, and St John's College, Cambridge	
1911		* Wrigley, Ruric Whitehead, B.A. (Cantab.), Assistant Astronomer, Royal Observatory, Edinburgh	
1890		Wright, Johnstone Christie, Conservative Club, Edinburgh 665	
1896		Wright, Sir Robert Patrick, LL.D., formerly Chairman of the Board of Agriculture for Scotland. Wenallt House, Crosswood, Cardiganshire	
1882		Young, Frank W., F.C.S., Scottish Education Department (Ex-Service Student's Branch) (now retired). 25 Drumsheugh Gardens, Edinburgh	
1896	C.	Young, James Buchanan, M.B., D.Sc., Dalveen, Braeside, Liberton	
1904		Young, R. B., M.A., D.Sc., F.G.S., Professor of Geology and Mineralogy in the South African School of Mines and Technology, Johannesburg, Transvaal	

LIST OF HONORARY FELLOWS OF THE SOCIETY

(*At October* 26, 1925.)

HIS MOST EXCELLENT MAJESTY THE KING.
HIS ROYAL HIGHNESS THE PRINCE OF WALES.

Foreigners (limited to thirty-six by Law I).

Elected
1916 Charles Barrois, Professor of Geology and Mineralogy, Université, Lille, France: 37, rue Pascal, Lille.
1923 Henri Bergson, Honorary Professor, College of France, Paris.
1923 Marcellin Boule, Professor at the National Museum of Natural History, Laboratory of Palæontology, 3 Place Valhubert, Paris 5^{e}.
1905 Waldemar Christofer Brögger, Professor of Mineralogy and Geology, K. Frederiks Universitet, Oslo, Norway.
1916 Douglas Houghton Campbell, Professor of Botany, Leland Stanford Junior University, California, U.S.A.
1920 William Wallace Campbell, Director of the Lick Observatory, Mt. Hamilton, California, U.S.A.
1921 Reginald Aldworth Daly, Professor of Geology, Harvard University, Cambridge, Mass.
1910 Hugo de Vries, Professor of Plant Anatomy and Physiology, Lunteren, Holland.
1916 Marcel Eugène Emile Gley, Professor of Biology, Collège de France, Paris, Membre de l'Académie de Médecine, Paris: 14, rue Monsieur le Prince, Paris.
1910 Karl F. von Goebel, Professor of Botany, Universität, München, Germany.
1916 Camillo Golgi, Emer.-Professor of Pathology, Università, Pavia, Italy.
1905 Paul Heinrich Ritter von Groth, Professor of Mineralogy, Universität, München, Germany.
1913 George Ellery Hale, Director of Mount Wilson Solar Observatory (Carnegie Institution of Washington), Pasadena, California, U.S.A.
1921 Johan Hjort, formerly Director of Norwegian Fisheries, Bergen, Universitet, Oslo.
1923 Arnold Frederik Holleman, Professor of Organic Chemistry, University, Amsterdam.
1923 Tullio Levi-Civita, Professor of Mathematics (Higher Analysis), University, Rome.
1920 Hendrik Antoon Lorentz, Nobel Laureate, Physics, 1902, Professor of Physics, Leiden University.
1910 Albert Abraham Michelson, Nobel Laureate, Physics, 1907, Professor of Physics, University, Chicago, U.S.A.
1897 Fridtjof Nansen, Professor of Oceanography, K. Frederiks Universitet, Oslo, Norway.
1923 Arthur Amos Noyes, Institute of California, Pasadena, U.S.A.
1921 Heike Kamerlingh Onnes, Nobel Laureate, Physics, 1913, Universiteit, Leiden, Holland.
1908 Henry Fairfield Osborn, Professor of Zoology, Columbia University and American Museum of Natural History, New York, N.Y., U.S.A.
1908 Ivan Petrovitch Pavlov, Emeritus Professor of Physiology, Inst. Exper. Med., Leningrad, Nobel Laureate, Physiology and Medicine, 1904: 7, Linia, No. 2, Vassilievsky, Ostrov, Leningrad, Russia.
1920 Ch. Emile Picard, Perpetual Secretary, Academy of Sciences, Paris.
1921 Salvatore Pincherle, Professor of Mathematics in the University of Bologna.
1913 Santiago Ramón y Cajal, Nobel Laureate, Medicine, 1906, formerly Professor of Histology and Pathological Anatomy, Universidad, Madrid, Spain.
1923 Theodore William Richards, Nobel Laureate, Chemistry, 1914, Professor of Chemistry, Harvard University, Cambridge, Mass., U.S.A.
1920 Charles Richet, Professor of Physiology, Faculty of Medicine, Paris.
1920 Georg Ossian Sars, formerly Professor of Zoology, Oslo, and Director of Norwegian Fisheries.
1913 Vito Volterra, Professor of Mathematical Physics, Regia Università, Rome, Italy.
1923 Edmund Beecher Wilson, Professor of Zoology, Columbia University, New York, U.S.A.

Total, 31.

BRITISH SUBJECTS (LIMITED TO TWENTY BY LAW I).

Elected

1900 Sir David Ferrier, Kt., M.A., M.D., LL.D., F.R.S., Emer.-Professor of Neuro-Pathology, King's College, London. Athenæum Club, London.

1900 Andrew Russell Forsyth, M.A., Sc.D., LL.D., Hon. Math.D., F.R.S., formerly Chief Professor of Mathematics in the Imperial College of Science and Technology, London; formerly Sadleirian Professor of Pure Mathematics in the University of Cambridge. Imperial College of Science and Technology, London, S.W. 7.

1910 Sir James George Frazer, O.M., Kt., D.C.L., LL.D., Litt.D., F.R.S., Fellow of Trinity College, Cambridge.

1916 James Whitbread Lee Glaisher, M.A., Sc.D., F.R.S., Fellow of Trinity College, Cambridge.

1908 Sir Alexander B. W. Kennedy, Kt., LL.D., F.R.S., Past Pres. Inst. C.E., A7, Albany, Piccadilly, London, W.

1913 Horace Lamb, M.A., Sc.D., D.Sc., LL.D., F.R.S., formerly Professor of Mathematics in the University of Manchester. 6 Selwyn Gardens, Cambridge.

1908 Sir Edwin Ray Lankester, K.C.B., M.A., D.Sc., LL.D., F.R.S., 44 Oakley Street, Chelsea, London, S.W. 3.

1910 Sir Joseph Larmor, Kt., M.A., D.Sc., LL.D., D.C.L., F.R.S., Lucasian Professor of Mathematics in the University of Cambridge. St John's College, Cambridge.

1900 Archibald Liversidge, M.A., LL.D., F.R.S., Emer.-Professor of Chemistry in the University of Sydney, Fieldhead, Coombe Warren, Kingston, Surrey.

1921 William Henry Perkin, M.A., Ph.D., Sc.D., LL.D., F.R.S., Waynflete Professor of Chemistry in the University of Oxford.

1921 Sir Ronald Ross, K.C.B., K.C.M.G., F.R.S., Hon. Consultant in Malaria, Ministry of Pensions, London, 41 Buckingham Palace Mansions, S.W. 1.

1921 Sir Ernest Rutherford, O.M., Kt., M.A., D.Sc., B.A., LL.D., Pres.R.S., Nobel Laureate, Chemistry, 1908, Cavendish Professor of Experimental Physics in the University of Cambridge.

1916 Sir Arthur Schuster, Kt., Ph.D., D.Sc., LL.D., D. ès Sc. Geneva, Foreign Secretary and V.-P. of the Royal Society, Honorary Professor of Physics in the University of Manchester, Yeldall, Twyford, Berks.

1908 Sir Charles Scott Sherrington, O.M., G.B.E., M.A., D.Sc., M.D., LL.D., Past Pres. R.S., Waynflete Professor of Physiology in the University of Oxford. Physiological Laboratory, Oxford.

1913 Sir William Turner Thiselton-Dyer, K.C.M.G., C.I.E., M.A., LL.D., F.R.S., formerly Director of the Royal Botanic Gardens, Kew: The Ferns, Witcombe, Gloucester.

1905 Sir Joseph John Thomson, Kt., O.M., D.Sc., LL.D., Past Pres. R.S., Nobel Laureate, Physics, 1906, formerly Cavendish Professor of Experimental Physics, University of Cambridge, Master of Trinity College, Cambridge.

Total, 16.

CHANGES IN FELLOWSHIP DURING SESSION 1924–1925.

ORDINARY FELLOWS OF THE SOCIETY ELECTED.

ALEXANDER CRAIG AITKEN.
ARTHUR JAMES BALFOUR (The Rt. Hon. the Earl of Balfour).
THOMAS WILLIAM NAYLOR BARLOW.
ALBERT WILLIAM BORTHWICK.
SAHAY RAM BOSE.
GEORGE STUART CARTER.
JAMES CHUMLEY.
ARTHUR HENDERSON BRIGGS CONSTABLE (The Hon. Lord Constable).
ROBERT MELDRUM CRAIG.
ALEXANDER JOHN DEY.
ALEXANDER GRAHAM DONALD.
HENRY DRYERRE.
JOHN SAMUEL DUNKERLY.
GEORGE SAMUEL EASTWOOD.
WILLIAM LEONARD FERRAR.
WILLIAM KING GILLIES.
ARCHIBALD HAYMAN ROBERTSON GOLDIE.
DAVID MIDDLETON GREIG.
ALEXANDER MACMILLAN HERON.
CHRISTOPHER NICHOLSON JOHNSTON (The Hon. Lord Sands).
CHARLES NORMAN KEMP.
WILLIAM OGILVY KERMACK.
LEONARD AUGUSTUS LUCAS KING.
WILLIAM HENRY LANG.
JAMES MALLOCH.
RONALD CRAUFURD MUNRO-FERGUSON (The Rt. Hon. Viscount Novar).
WILLIAM GAMMIE OGG.
DAVID PENMAN.
HAROLD ROBINSON.
WILLIAM SADDLER.
ROBERT ARNOT STAIG.
ALEXANDER STEVENS.
DAVID SMITH STEWART.
RICHARD L. SUTTON.
HARRY MORETON STANLEY TURNER.
FREDERICK WALKER.
DAVID PERCIVAL DALBRECK WILKIE.
HAROLD WILSON.

ORDINARY FELLOWS DECEASED.

Sir WILLIAM F. BARRETT.
EDWIN H. BARTON.
FRANK E. BEDDARD.
FREDERICK H. BOWMAN.
J. J. GRAHAM BROWN.
JOHN YOUNG BUCHANAN.
THOMAS L. BUNTING.
Sir ARCHIBALD GEIKIE.
JAMES ANGUS CAMERON.
JOHN FRASER.
Professor A. GRAY.
JOHN ROBERTSON HENDERSON.
FRANCIS JONES.
Sir WILLIAM PECK.
JOSEPH M'GREGOR ROBERTSON.
DANIEL R. STEUART.
CHARLES TWEEDIE.
GEORGE YOUNG.

HONORARY FELLOWS DECEASED.

Sir FRANCIS DARWIN.
Sir THOMAS EDWARD THORPE.
GIO. BATTISTI GRASSI.

CEASED TO BE A MEMBER.

ALFRED OSWALD.

Additions to the Library—Presentations, etc.—1924–1925.

Abhandlungen der Heidelberger Akademie der Wissenschaften: Math. Nat. Kl. Jahrgang 1910– , Abhandlung 1– . 4to. Heidelberg, 1910. (*Exchange.*)

Astrographic Catalogue 1900·0. Hyderabad Section. Vol. V. Measures of Rectangular Co-ordinates and Diameters of 88,444 Star Images on Plates with Centres in Dec. −21°. (*Presented by Osmania University, Hyderabad.*)

British Antarctic ("Terra Nova") Expedition.
Zoology. Vol. VIII, No. 2, pp. 37–202. Crustacea. Part 9. Decapod Larvæ. By Robert Gurney. 4to. London, 1924.

Carnegie Institution of Washington Publications:—

228A. Studies on the Variation, Distribution, and Evolution of the Genus Partula. By H. E. Crampton. 4to. 1925.

310. Interferometer Experiments in Acoustics and Gravitation. By Carl Barus. 8vo. 1925.

322A. The Pleistocene of the Middle Region of North America and its Vertebrated Animals. By Oliver P. Hay. La. 8vo. 1924.

331. Cenozoic Gravigrade Edentates of Western N. America: with Special Reference to the Pleistocene Megalonychinæ and Mylodontidæ of Rancho la Brea. By Chester Stock. 4to. 1925.

340. Papers from the Department of Marine Biology. Vol. XIX. A Memorial to Alfred Goldsborough Mayer. 4to. 1924.

342. (Department of Marine Biology. Vol. XXI.) Samoan Foraminifera. By Joseph A. Cushman. La. 8vo. 1924.

343. Some Corals from American Samoa and the Fiji Islands. By J. E. Hoffmeister. 4to. 1925.

348. Additions to the Tertiary History of the Pelagic Mammals on the Pacific Coast of North America. By R. Kellog. 8vo. Washington, 1925.

351. Public Health Administration and the Natural History of Disease in Baltimore, Maryland, 1797–1920. By William Travis Howard, Jr. La. 8vo. 1924.

354. General and Physiological Features of the Vegetation of the more Arid Portions of Southern Africa, with Notes on the Climatic Environment. By William Austin Cannon. La. 8vo. 1924.

355. Experimental Vegetation: the Relation of Climaxes to Climates. By F. E. Clements and J. E. Weaver. La. 8vo. 1924.

356. The Phytometer Method in Ecology: the Plant and Community as Instruments. By F. E. Clements and G. W. Goldsmith. La. 8vo. 1924.

357. Root Behaviour and Crop Yield under Irrigation. By Frank C. Jean and John E. Weaver. La. 8vo. 1924.

358. History of Agriculture in the Northern United States, 1620–1860. By P. W. Bidwell and J. I. Falconer. 8vo. 1925.

359. List of Manuscripts concerning American History preserved in European Libraries. By D. M. Matteson. 8vo. 1925.

360. The Volcanic Activity and Hot Springs of Lassen Peak. By A. L. Day and E. T. Allen. 4to. 1925.

361. Contributions to Embryology. Vol. XVI., Nos. 78–84.

365. Reversible Variations in Volume, Pressure, and Movements of Sap in Trees. By D. T. MacDougal. 8vo. 1925.

366. Miocene Mollusks from Bowden, Jamaica (Paleoypods and Scaphopods). By W. P. Woodring. 8vo. 1925.
(*Exchange.*)

Catalogue of Indian Insects, Part 1. 8vo. Calcutta, 1921. (*Exchange.*)

Catalogue of Scientific Papers, Royal Society of London. Vol. XIX, T–Z. 4to. Cambridge, 1925. (*Purchased.*)

Ceylon Journal of Science:—
- Section G. Archæology, Ethnology, etc. Vol. I, Part 1– . 8vo. Ceylon and London, 1924.
- Section D. Medical Science. Vol. I, Part 1– . 8vo. Colombo and London, 1924. (*Exchange.*)

Collected Papers of the Rowett Research Institute. Vol. I. La. 8vo. Aberdeen, 1925. (*Presented by Dr J. B. Orr, Director.*)

Collet, L. W. Les Lacs. 8vo. Paris, 1925. (*Presented by the Author.*)

Contributions from the Museum of Geology, University of Michigan. Vol. I– . 8vo. New York, 1924. (*By Exchange.*)

Dainelli, Giotto. Le Condizione delle Genti. 4to. Bologna, 1924. (*Presented by Dr Du Riche Preller.*)

Denkschriften des Naturhistorischen Hofmuseums. Vol. I. Fol. Wien, 1917. (*Exchange.*)

Edinburgh University Library. Descriptive Catalogue of the Arabic and Persian MSS. 8vo. Edinburgh, 1925. (*Presented.*)

Engineer's Year-Book for 1925. 8vo. London, 1925. (*Presented by Mr W. Hanneford Smith.*)

Gunther, R. T. Early Science in Oxford. Vol IV– . The Philosophical Society. 8vo. Oxford, 1925. (*Purchased.*)

Hickson, Sydney J. An Introduction to the Study of Corals. (Manchester University. Vol. IV. Biological Series.) 8vo. Manchester, 1924. (*By Exchange.*)

Hydrographic Review. Vol. I, No. 2.—(International Hydrographic Bureau). 8vo. Monaco, 1924. (*Exchange.*)

Japanese Journal of Engineering. (Abstracts.) Vol. I. 4to. Tokyo, 1921. (*Exchange with the National Research Council of Japan.*)

Japanese Journal of Mathematics. (Transactions and Abstracts.) Vol. I. 4to. Tokyo, 1924. (*Exchange with the National Research Council of Japan.*)

Journal of the Optical Society of America, and Review of Scientific Instruments. Vol. X, No. 1. 8vo. Menasha, 1925. (*Exchange.*)

Journal of the Royal Technical College. No. 1– . 4to. Glasgow, 1924. (*Presented.*)

Levi-Civita, Tullio. Lezioni di calcolo differenziale assoluto. Raccolte e compilate dal Dott. Enrico Persico. 8vo. Rome, 1925. (*Presented by the Author.*)

Little, O. H. Geography and Geology of Makalla (South Arabia). Geological Survey Egypt. 8vo. Cairo, 1925.

M'Laren, Late Samuel Bruce. Scientific Papers, mainly on Electrodynamics and Natural Radiation. 8vo. Cambridge, 1925. (*Presented by University Press, Cambridge.*)

Namari, Ichitaro. The Electrolytic Separation of Magnesium from Magnesia. 4to. Sakai, 1924. (*Presented by the Author.*)

——— (Die) Naturwissenschaften, April 1925– . (*Exchange.*)

Norsk Ornithologisk Tidsskrift. No. 1, 1920. La. 8vo. Stavanger. (*Exchange Stavanger Museum.*)

Power Resources of the Commonwealth of Australia and the Mandated Territory of New Guinea: Report to the World Power Conference, London, 1924. 4to. Sydney, 1924. (*Presented by the Institution of Engineers, Australia.*)

Quarterly Journal of the Indian Chemical Society. Vol. I, No. 1– . 8vo. Calcutta, 1924. (*Exchange.*)

Scoresby Log-Book. Vols. I–VII, and Introductory Brochure. Fol. New York, 1916 and 1917. (*Presented by the Explorers' Club, New York.*)

Sewell, Robert. The Siddhantas and the Indian Calendar. 4to. Calcutta, 1924. *Presented by H.M. Government, India.*)

"Times" Survey Atlas of the World. By J. G. Bartholomew. Selfridge Edition. Fol. London, 1922. (*Purchased.*)

Wait, W. E. Manual of the Birds of Ceylon. 8vo. Ceylon and London, 1925. (*Exchange from Colombo Museum.*)

Wenley, R. M. Stoicism and its Influence. 8vo. Boston, Mass., U.S.A. (*Presented by the Author.*)

World List of Scientific Periodicals, 1900–1921. 4to. London, 1925. (*Purchased.*)

Presented by the late Charles Tweedie, Esq., M.A., B.Sc.

Barlow's Tables, etc. 8vo. London, 1886.

Chasles, M. Traité de Géométrie Supérieure. 2nd Edition. La. 8vo. Paris, 1880.

Humbert, G. Cours d'Analyse, Tomes 1 and 2. 4to. Paris, 1903 and 1904.

Maclaurin, Colin. Geometria Organica. 4to. London, 1720. Brankenridge's copy with MS. notes.

Picard, M. E. Cours d'Analyse par M. E. Picard. Rédigé par MM. L. Caron et Ch. Philippe. 4to. Paris, 1887.

Pryde, James. Mathematical Tables, consisting of Logarithms of Nos. 1 to 108,000. 8vo. London, 1892.

PORTRAITS, ETC.

Engraving of Men of Science, about the Period 1820. (*Presented by His Grace the Duke of Buccleuch.*)

Henri Poincaré Medallion. (*Presented by the Poincaré Memorial Committee.*)

Portrait of Professor F. O. Bower, F.R.S. (*Presented by Professor Bower.*)

LAWS OF THE SOCIETY.

Adopted July 3, 1916; amended December 18, 1916.

(Laws VIII, IX, and XIII amended May 3, 1920. Law VI amended February 7, 1921. Law XIX amended June 16, 1924.)

I.

THE ROYAL SOCIETY OF EDINBURGH, which was instituted by Royal Charter in 1783 for the promotion of Science and Literature, shall consist of Ordinary Fellows (hereinafter to be termed Fellows) and Honorary Fellows. The number of Honorary Fellows shall not exceed fifty-six, of whom not more than twenty may be British subjects, and not more than thirty-six subjects of Foreign States.

Fellows only shall be eligible to hold office or to vote at any Meeting of the Society.

ELECTION OF FELLOWS.

II.

Each Candidate for admission as a Fellow shall be proposed by at least four Fellows, two of whom must certify from personal knowledge. The Official Certificate shall specify the name, rank, profession, place of residence, and the qualifications of the Candidate. The Certificate shall be delivered to the General Secretary before the 30th of November, and, subject to the approval of the Council, shall be exhibited in the Society's House during the month of January following. All Certificates so exhibited shall be considered by the Council at its first meeting in February, and a list of the Candidates approved by the Council for election shall be issued to the Fellows not later than the 21st of February.

III.

The election of Fellows shall be by Ballot, and shall take place at the first Ordinary Meeting in March. Only Candidates approved by the Council shall be eligible for election. A Candidate shall be held not elected, unless he is supported by a majority of two-thirds of the Fellows present and voting.

IV.

On the day of election of Fellows two scrutineers, nominated by the President, shall examine the votes and hand their report to the President, who shall declare the result.

V.

Each Fellow, after his election, is expected to attend an Ordinary Meeting, and sign the Roll of Fellows, he having first made the payments required by Law VI. He shall be introduced to the President, who shall address him in these words:

> *In the name and by the authority of THE ROYAL SOCIETY OF EDINBURGH, I admit you a Fellow thereof.*

PAYMENTS BY FELLOWS.

VI.

Each Fellow shall, before he is admitted to the privileges of Fellowship, pay an admission fee of three guineas, and a subscription of three guineas for the year of election. He shall continue to pay a subscription of three guineas at the beginning of each session so long as he remains a Fellow.

Each Fellow who was elected subsequent to December 1916 and previous to December 1920 shall also pay a subscription of three guineas at the beginning of each session so long as he remains a Fellow.

Each Fellow who was elected previous to December 1916, and who has not completed his twenty-five annual payments,* shall, at the beginning of each session, pay three guineas or four guineas according as he has or has not made ten annual payments as reckoned from the year of election. Each Fellow who has completed or shall complete his payments shall be invited to contribute one guinea per annum or to pay a single sum of ten guineas.

A Fellow may compound for the annual subscriptions by a single payment of fifty guineas, or on such other terms as the Council may from time to time fix.

VII.

A Fellow who, after application made by the Treasurer, fails to pay any contribution due by him, shall be reported to the Council, and, if the Council see fit, shall be declared no longer a Fellow. Notwithstanding such declaration all arrears of contributions shall remain exigible.

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* The following is an extract from the previous law affecting Annual Subscribers elected prior to December 1916:—"Every Ordinary Fellow, within three months after his election, shall pay Two Guineas as the fee of admission, and Three Guineas as his contribution for the Session in which he has been elected; and annually at the commencement of every Session, Three Guineas into the hands of the Treasurer. This annual contribution shall continue for ten years after his admission, and it shall be limited to Two Guineas for fifteen years thereafter."

ELECTION OF HONORARY FELLOWS.

VIII.

Honorary Fellows shall be persons eminently distinguished in Science or Literature. They shall not be liable to contribute to the Society's Funds. Personages of the Blood Royal may be elected Honorary Fellows at any time on the nomination of the Council, and without regard to the limitation of numbers specified in Law I.

IX.

Honorary Fellows shall be proposed by the Council. The nominations shall be announced from the Chair at the First Ordinary Meeting after their selection. The names shall be printed in the circular for the last Ordinary Meeting of the Session, when the election shall be by Ballot, after the manner prescribed in Laws III and IV for the Election of Fellows.

EXPULSION OF FELLOWS.

X.

If, in the opinion of the Council, the conduct of any Fellow is injurious to the character or interests of the Society, the Council may, by registered letter, request him to resign. If he fail to do so within one month of such request, the Council shall call a Special Meeting of the Society to consider the matter. If a majority consisting of not less than two-thirds of the Fellows present and voting decide for expulsion, he shall be expelled by declaration from the Chair, his name shall be erased from the Roll, and he shall forfeit all right or claim in or to the property of the Society.

XI.

It shall be competent for the Council to remove any person from the Roll of Honorary Fellows if, in their opinion, his remaining on the Roll would be injurious to the character or interests of the Society. Reasonable notice of such proposal shall be given to each member of the Council, and, if possible, to the Honorary Fellow himself. Thereafter the decision on the question shall not be taken until the matter has been discussed at two Meetings of Council, separated by an interval of not less than fourteen days. A majority of two-thirds of the members present and voting shall be required for such removal.

MEETINGS OF THE SOCIETY.

XII.

A Statutory Meeting for the election of Council and Office-Bearers, for the presentation of the Annual Reports, and for such other business as may be arranged by the Council, shall be held on the fourth Monday of October. Each Session of the Society shall begin at the date of the Statutory Meeting.

XIII.

Meetings for reading and discussing communications and for general business, herein termed Ordinary Meetings, shall be held, when convenient, on the first and third Mondays of each month from November to July inclusive, with the exception that in January the meetings shall be held on the second and fourth Mondays.

XIV.

A Special Meeting of the Society may be called at any time by direction of the Council, or on a requisition to the Council signed by not fewer than six Fellows. The date and hour of such Meeting shall be determined by the Council, who shall give not less than seven days' notice of such Meeting. The notice shall state the purpose for which the Special Meeting is summoned; no other business shall be transacted.

PUBLICATION OF PAPERS.

XV.

The Society shall publish Transactions and Proceedings. The consideration of the acceptance, reading, and publication of papers is vested in the Council, whose decision shall be final. Acceptance for reading shall not necessarily imply acceptance for publication.

DISTRIBUTION OF PUBLICATIONS.

XVI.

Fellows who are not in arrear with their Annual Subscriptions and all Honorary Fellows shall be entitled gratis to copies of the Parts of the Transactions and the Proceedings published subsequently to their admission.

Copies of the Parts of the Proceedings shall be distributed by post or otherwise, as soon as may be convenient after publication; copies of the Transactions or Parts thereof shall be obtainable upon application, either personally or by an authorised agent, to the Librarian, provided the application is made within five years after the date of publication.

CONSTITUTION OF COUNCIL.

XVII.

The Council shall consist of a President, six Vice-Presidents, a Treasurer, a General Secretary, two Secretaries to the Ordinary Meetings (the one representing the Biological group and the other the Physical group of Sciences),* a Curator of the Library and Museum, and twelve ordinary members of Council.

ELECTION OF COUNCIL.

XVIII.

The election of the Council and Office-Bearers for the ensuing Session shall be held at the Statutory Meeting on the fourth Monday of October. The list of the names recommended by the Council shall be issued to the Fellows not less than one week before the Meeting. The election shall be by Ballot, and shall be determined by a majority of the Fellows present and voting. Scrutineers shall be nominated as in Law IV.

XIX.

The President may hold office for a period not exceeding five consecutive years; the Vice-Presidents, not exceeding three; the Secretaries to the Ordinary Meetings, not exceeding five; the General Secretary, the Treasurer, and the Curator of the Library and Museum, not exceeding ten; and ordinary members of Council, not exceeding three consecutive years; provided that the Treasurer may be re-elected for more than ten successive years in cases where the Council declares to the Society that an emergency exists.

XX.

In the event of a vacancy arising in the Council or in any of the offices enumerated in Law XVII, the Council shall proceed, as soon as convenient, to elect a Fellow to fill such vacancy for the period up to the next Statutory Meeting.

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* The Biological group includes Anatomy, Anthropology, Botany, Geology, Pathology, Physiology, Zoology; the Physical group includes Astronomy, Chemistry, Mathematics, Metallurgy, Meteorology, Physics.

POWERS OF THE COUNCIL.

XXI.

The Council shall have the following powers:—(1) To manage all business concerning the affairs of the Society. (2) To decide what papers shall be accepted for communication to the Society, and what papers shall be printed in whole or in part in the Transactions and Proceedings. (3) To appoint Committees. (4) To appoint employees and determine their remuneration. (5) To award the various prizes vested in the Society, in accordance with the terms of the respective deeds of gift, provided that no member of the existing Council shall be eligible for any such award. (6) To make from time to time Standing Orders for the regulation of the affairs of the Society. (7) To control the investment or expenditure of the Funds of the Society.

At Meetings of the Council the President or Chairman shall have a casting as well as a deliberative vote.

DUTIES OF PRESIDENT AND VICE-PRESIDENTS.

XXII.

The President shall take the Chair at Meetings of Council and of the Fellows. It shall be his duty to see that the business is conducted in accordance with the Charter and Laws of the Society. When unable to be present at any Meetings or attend to current business, he shall give notice to the General Secretary, in order that his place may be supplied. In the absence of the President his duties shall be discharged by one of the Vice-Presidents.

DUTIES OF THE TREASURER.

XXIII.

The Treasurer shall receive the monies due to the Society and shall make payments authorised by the Council. He shall lay before the Council a list of arrears in accordance with Rule VII. He shall keep accounts of all receipts and payments, and at the Statutory Meeting shall present the accounts for the preceding Session, balanced to the 30th of September, and audited by a professional accountant appointed annually by the Society.

DUTIES OF THE GENERAL SECRETARY.

XXIV.

The General Secretary shall be responsible to the Council for the conduct of the Society's correspondence, publications, and all other business except that which relates to finance. He shall keep Minutes of the Statutory and Special

Meetings of the Society and Minutes of the Meetings of Council. He shall superintend, with the aid of the Assistant Secretary, the publication of the Transactions and Proceedings. He shall supervise the employees in the discharge of their duties.

DUTIES OF SECRETARIES TO ORDINARY MEETINGS.

XXV.

The Secretaries to Ordinary Meetings shall keep Minutes of the Ordinary Meetings. They shall assist the General Secretary, when necessary, in superintending the publication of the Transactions and Proceedings. In his absence, one of them shall perform his duties.

DUTIES OF CURATOR OF LIBRARY AND MUSEUM.

XXVI.

The Curator of the Library and Museum shall have charge of the Books, Manuscripts, Maps, and other articles belonging to the Society. He shall keep the Card Catalogue up to date. He shall purchase Books sanctioned by the Council.

ASSISTANT-SECRETARY AND LIBRARIAN.

XXVII.

The Council shall appoint an Assistant-Secretary and Librarian, who shall hold office during the pleasure of the Council. He shall give all his time, during prescribed hours, to the work of the Society, and shall be paid according to the determination of the Council. When necessary he shall act under the Treasurer in receiving subscriptions, giving out receipts, and paying employees.

ALTERATION OF LAWS.

XXVIII.

Any proposed alteration in the Laws shall be considered by the Council, due notice having been given to each member of Council. Such alteration, if approved by the Council, shall be proposed from the Chair at the next Ordinary Meeting of the Society, and, in accordance with the Charter, shall be considered and voted upon at a Meeting held at least one month after that at which the motion for alteration shall have been proposed.

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